Feedback on Note 108.40

"A surprising coincidence between Pythagorean triples and an Euler-Cauchy differential equation" by
Allan J. Kroopnick, Mathematical Gazette, vol. 108, #573, 2024 Nov., pp 516-518.

In Note 108.40, Allan Kroopnick presents a pleasing connection between Pythagorean triples and Euler-Cauchy differential equations. The Euler-Cauchy ordinary differential equation is

$$x^2 \frac{d^2 y}{dx^2} - bx \frac{dy}{dx} + cy = 0$$

where b, c are constants.

Its solution (except in the case of equal roots) is

$y = Ax^r + Bx^s$, where $r = \dfrac{(b+1) - \sqrt{(b+1)^2 - 4c}}{2}$, $s = \dfrac{(b+1) + \sqrt{(b+1)^2 - 4c}}{2}$

Note 108.40 employs Pythagorean triples, where the two larger integers differ by one unit.

In this feedback, I extend the relationship to a more general set of Pythagorean triples.

From any pair of distinct positive values of the exponents r, s in the solution to the differential equation, one can construct a Pythagorean triple $u = |r - s|$, $v = 2\sqrt{rs}$, $w = r + s$. However, this does not guarantee that u, v, w will be integers (or even rational). Proceeding in the opposite direction, the exponents r, s in the solution to the differential equation are related to the positive integers in a Pythagorean triple $\{ u, v, w \}$, $\left(u, v, w \in \mathbb{N},\ w^2 = u^2 + v^2\right)$ by

$$r = \frac{w-u}{2}, \quad s = \frac{w+u}{2}, \qquad \text{with } b = w - 1, \quad c = \frac{w^2 - u^2}{4} = \frac{v^2}{4}$$

Setting $u = m^2 - n^2$, $v = 2mn$, $w = m^2 + n^2$, with positive integers $m > n$, the general solution of

$$x^2 \frac{d^2 y}{dx^2} - \left(m^2 + n^2 - 1\right) x \frac{dy}{dx} + m^2 n^2 y = 0$$

is $y = Ax^r + Bx^s$, where $r = n^2$, $s = m^2$. All seven values b, c, u, v, w, r, s are then positive integers. The Pythagorean triple is primitive if and only if m, n are of opposite parity and are co-prime.

Examples

1.

For m = 2, n = 1, the triple is (3, 4, 5) or, equivalently, (4, 3, 5): The associated ODE is $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4y = 0$ and its solution is $y = Ax^1 + Bx^4$. The choice $m = n + 1$ for any positive integer n reproduces the first three cases in Note 108.40.

2.
m = 10, n = 7 reproduces the triple (51, 140, 149) in the note.

3.
We can also obtain other cases, such as this "near-isosceles" case:
Choosing m = 12, n = 5, the triple is (119, 120, 169). The associated ODE is $x^2 \frac{d^2y}{dx^2} - 168x\frac{dy}{dx} + 3600y = 0$ and its solution is $y = Ax^{25} + Bx^{144}$.

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