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# EFFICIENT METHODS IN PLANAR LINKAGE KINEMATIC ANALYSIS

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Efficient Methods in Planar Linkage Kinematic Analysis



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## About the Book

This book is intended for practising engineers, researchers, graduate and undergraduate students who are interested in the kinematics of planar mechanisms.

The mateiral presented is the outcome of several years of research, teaching and industrial experience. The computational efficiency and accuracy have been very important in the field of kinematics of robotic manipulators which have inspired the authors to apply similar principles to the planar linkages also. There are plenty of discussions and explanations using the graphical technique also, without which the authors believe, any treatment of kinematics would be incomplete. Since the solution of problems require enormous computations, a special software has been developed for this purpose to be used on micro-computers. These can be obtained from the first author at a reasonable cost.

Numerous problems have been solved throughout the text to illustrate various concepts. The rotational transformation matrices have been used to transform velocities and accelerations from one coordinate system into another. The concept of guided and guiding links have been introduced for setting up of the secondary coordinate system in the calculation of the Coriolis acceleration which the authors recognize as one of the very difficult concepts to teach in the class.

Finally, the computer software has been organized in a modular form; thus it does not serve as a black box where the users only see the inputs and outputs. They do not see how a problem has been solved. In fact, in using this software, they are continuously involved in solving problems; only the number crunching work is left to the computer. This makes this software very versatile and especially suited to the students who must know how to solve problems. The software should not give them the solutions.

## About the First Author

Anand M. Sharan is a professor in the Faculty of Engineering and Applied Science at the Memorial University of Newfoundland, St. John's, Newfoundland, Canada. He received his B.Tech (Hons) degree from I.I.T. Bombay, India in 1966 and M.S. degrees in Materials Science and Mechanical Engineering from the Washington State University, U.S.A. in the years 1968 and 1970 respectively. He worked in the areas consulting, manufacturing and design for seven years in U.S.A. and Canada. He obtained his Ph.D. from the Concordia University, in the year 1982 and joined the Memorial University of Newfoundland. He has published approximately 70 papers in various national and international journals and conferences. He has carried out extensive research work in the areas of machine tool dynamics, CAE, and robotics. He is a reviewer of several leading international journals in his field. He is a member of American Society of Mechanical Engineers (ASME), Society for Industrial and Applied Mathematics (SIAM), and Association for Professional Engineers of Newfoundland (APEN), Canada.

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Rao V. Dukkipati is senior research officer in the Institute for Mechanical Engineering at the National Research Council of Canada, as well as being an Adjunct Professor of Mechanical Engineering at Concordia University and The University of Western Ontario, Canada. He received his Ph.D. in 1973 from Oklahoma State University, M.Sc. and M.E. degrees from The University of New Brunswick, Canada and Andhra University, India in 1971 and 1969 respectively. He received the B.E. degree from Sri Venkateswara University, India in 1966. During 1973 to 1978 he has worked as structural analyst in the advanced mechanical analysis division of the Pratt and Whitney Aircraft of Canada Limited. He has co-authored three books, Dynamics of Railway Vehicle System, Academic press, 1984, Computer Aided Simulation in Railway Dynamics, Marcel-Dekker Inc., 1988, and Mechanism and Machine Theory, Wiley Eastern Limited, 1989 and published over 100 articles in various national and international journals and conferences. He is a member of the American Society for Mechanical Engineers, and the Association for Professional Engineers of Ontario, Canada.

" In the beginning, there was neither nought nor aught.

Then there was neither sky nor atmosphere above.

What then enshrouded all this universe ?

In the receptacle of what was it contained?

Then was there neither death nor immortality,

Then was there neither day, nor night, nor light

nor darkness,

Only the Existent One breathed calmly, self Contained. "

( Rig Veda 10.121.1 )

Anand M. Sharan
dedicates this work
to his wife Veena,
son Rajat and
daughter Swati

Rao V. Dukkipati
dedicates this work
to his mother Annapurnamma
Dukkipati and his father
late Nagabhushanam Dukkipati

CONVERSION FROM U.S. CUSTOMARY UNITS TO SI UNITS

| TO CONVERT FROM | TO | MULTIPLY BY |
|---|---|---|
| Foot (ft) | Metre (m) | 3.048 E - 01 |
| Horsepower (hp) | Watt (W) | 7.457 E + 02 |
| Inch (in) | Metre (m) | 2.540 E - 02 |
| Pound force (lb) | Newton (N) | 4.448 E + 00 |
| Pound mass (lbm) | Kilogram (kg) | 4.536 E - 01 |
| Poundal (lbm ft/s$^2$) | Newton (N) | 1.383 E - 01 |
| Pound-foot (lb - ft) | Newton-metre (N - m)<br>Joule (J) | 1.356 E + 00<br>1.356 E + 00 |
| Pound-foot/second (lb - ft/s) | Watt (W) | 1.356 E + 00 |
| Pound-inch (lb - in) | Newton-metre (N - m)<br>Joule (J) | 1.128 E - 01<br>1.128 E - 01 |
| Pound-inch/second (lb - in/s) | Watt (W) | 1.128 E - 01 |
| Pound/foot$^2$ (lb/ft$^2$) | Pascal (Pa) | 4.788 E + 01 |
| Pound/inch$^2$ (lb/in$^2$), psi | Pascal (Pa) | 6.895 E + 03 |
| Revolutions/minute (rpm) | Radian/second (rad/s) | 1.047 E - 01 |
| Slug | Kilogram (kg) | 1.459 E + 01 |

## CONVERSION FROM SI UNITS TO U.S. CUSTOMARY UNITS

| TO CONVERT FROM | TO | MULTIPLY BY |
|---|---|---|
| Joule (J) | Pound-foot (lb - ft) | 7.376 E - 01 |
| Joule (J) | Pound-inch (lb - in) | 8.851 E + 00 |
| Kilogram (kg) | Pound mass (lbm)<br>Slug | 2.205 E + 00<br>6.852 E - 02 |
| Metre (m) | Foot (ft)<br>Inch (in) | 3.281 E + 00<br>3.937 E + 01 |
| Newton (N) | Pound (lb)<br>Poundal (lb - ft/s$^2$) | 2.248 E - 01<br>7.233 E + 00 |
| Newton - metre (N - m) | Pound-foot (lb - ft)<br>Pound-inch (lb - in) | 7.376 E - 01<br>8.851 E + 00 |
| Newton-metre/second (N - m/s) | Horsepower (hp) | 1.341 E - 03 |
| Pascal (Pa) | Pound/foot$^2$ (lb/ft$^2$)<br>Pound/inch$^2$ (lb/in$^2$), (psi) | 2.089 E - 02<br>1.450 E - 04 |
| Radian/second (rad/s) | Revolutions/minutes (rpm) | 9.549 E + 00 |
| Watt (W) | Horsepower (hp)<br>Pound-foot/second (lb - ft/s)<br>Pound-inch/second (lb - in/s) | 1.341 E - 03<br>7.376 E - 01<br>8.851 E + 00 |

# PREFACE

This book is being written at a time when computers and robots have made tremendous advancements in the manufacturing and design processes. Even though the design of robotic manipulators require the three-dimensional kinematic analysis, yet the concepts developed in the planar analysis are of immense help in designing these mechanisms. This book has been exclusively written about the kinematic analysis of linkages where analysis of cams are also included. It is felt that, with the principles discussed in this book, one can carry out the design of most of the planar machineries.

The authors having spent numbers of years in the industries as well as in the universities, felt the need for a computer software. This is because, a design process involves a number of feasible solutions, and in kinematics, because the parameters involved are vectorial in nature, require a large number of calculations. There are sophisticated programs available in the market but the beginners see these as black boxes, which is very undesirable. Therefore, with a great deal of effort, a software was developed and tried out in the classroom including in the final examinations; the students were allowed to use the software in these examinations. In this way, the painful task of number crunching was left to the computer to do. Actually, it is the concepts which are most important.

In Chapter 1, the basic concepts of mechanisms are discussed, followed by the displacement analysis in the Chapter 2. The solutions of planar vector equations are discussed in great details in this chapter and these solutions have been extensively used to solve problems in the velocity and acceleration analyses in the later chapters. The

Geometrical Method for solving these planar vector equations has been introduced in this chapter. The use of the computer software is also discussed in this chapter.

In the Chapter 3, a distinction has been made between the difference of velocities of two points which are, in the first case, located on the same rigid body and in the second case, not on the same body. The second one, relative velocity as is commonly known, is termed as the apparent velocity of one point with respect to the other in this book. This concept is later on helpful in determining the Coriolis acceleration. The graphical method is also quite extensively discussed in this chapter and throughout the book. We hold the opinion that the analytical calculations must be checked out graphically first. It has also been our experience that the concepts are grasped much easily by the graphical method by the students. The analytical method will yield better accuracy but it will come only after some experience. The velocity analysis of mechanisms has been discussed using, (a) link-by-link method and (b) the instantaneous center method.

Chapter 4 deals with the acceleration analysis. The equations of motion of a point are derived by a differentiation process relative to a fixed coordinate system. Next, the general equations i.e. acceleration of a point relative to a moving frame are also derived. The relationships between the apparent velocity discussed in the Chapter 3 and the Coriolis acceleration are established using a moving slider on a rotating link. Setting up of the secondary coordinate system, the path of a point and its curvature, are important aspects in the determination of the absolute acceleration of a point. The concept of a 'Guided Link' and a 'Guiding Link' are introduced to calculate the Coriolis term correctly.

In this book, wherever possible, the analytical expressions have been simplified and special effort has been made to reduce the computations involved. Even without the computer software, if the approach discussed in this book is followed, the calculations can be performed quite efficiently. The computer software adds to the efficiency and accuracy, a fact which the readers can verify by solving problems.

The authors wish to acknowledge the help with thanks of Narendra K. Sinha, Sudhir Kumar, Rajeeve Bahree, Sarah Prabhakaran, Jinesh Jain, Charles Dhanaraj, Parveen Kalra, Bhimavarapu S. Reddy, Manoj Tummala, Rama K.P. Koganti, and Rama K. Vallurupalli in the preparation of the software, drawings and the typing of the manuscript. The authors are specially thankful to the Dean of the Faculty of Engineering at Memorial University, Dr. G.R. Peters, for providing the excellent computing facilities which were necessary to complete this work. The authors feel thankful to the NSERC Canada which has provided research grants in the areas of robotics and machine tool dynamics. The concepts evolved through the research work were very useful in the development of the approach used in this book.

Anand Mohan Sharan

Rao V. Dukkipati

CONTENTS

# CHAPTER 1

## INTRODUCTION TO MECHANISMS

### 1.1 Basic Concepts in Mechanisms and Machines

The human civilization has used mechanisms for thousands of years. The purpose of a mechanism is to transfer motion whereas, a machine is an assembly of several members to do the work. An example of a mechanism is shown in Fig. 1.1 and it is called a crank-slider mechanism. This type of mechanism is used in the engine of automobiles. The slider is pushed forward in the power stroke of the engine. As the slider moves forward, it rotates the crank thus turning the crank shaft. In this way the reciprocating motion of the slider is converted into the rotary motion of the crank shaft. Fig 1.2 shows the application of four-bar mechanism

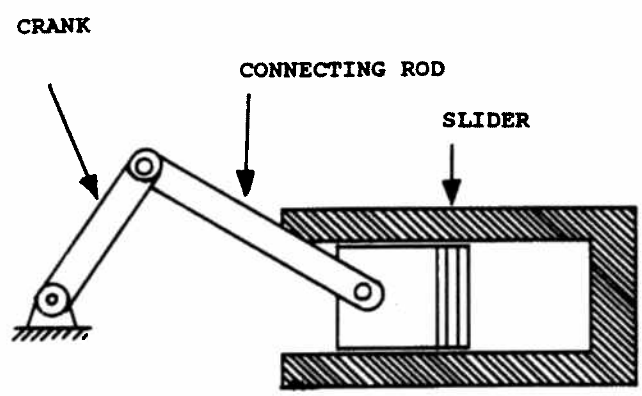



FIG. 1.1 CRANK SLIDER MECHANISM

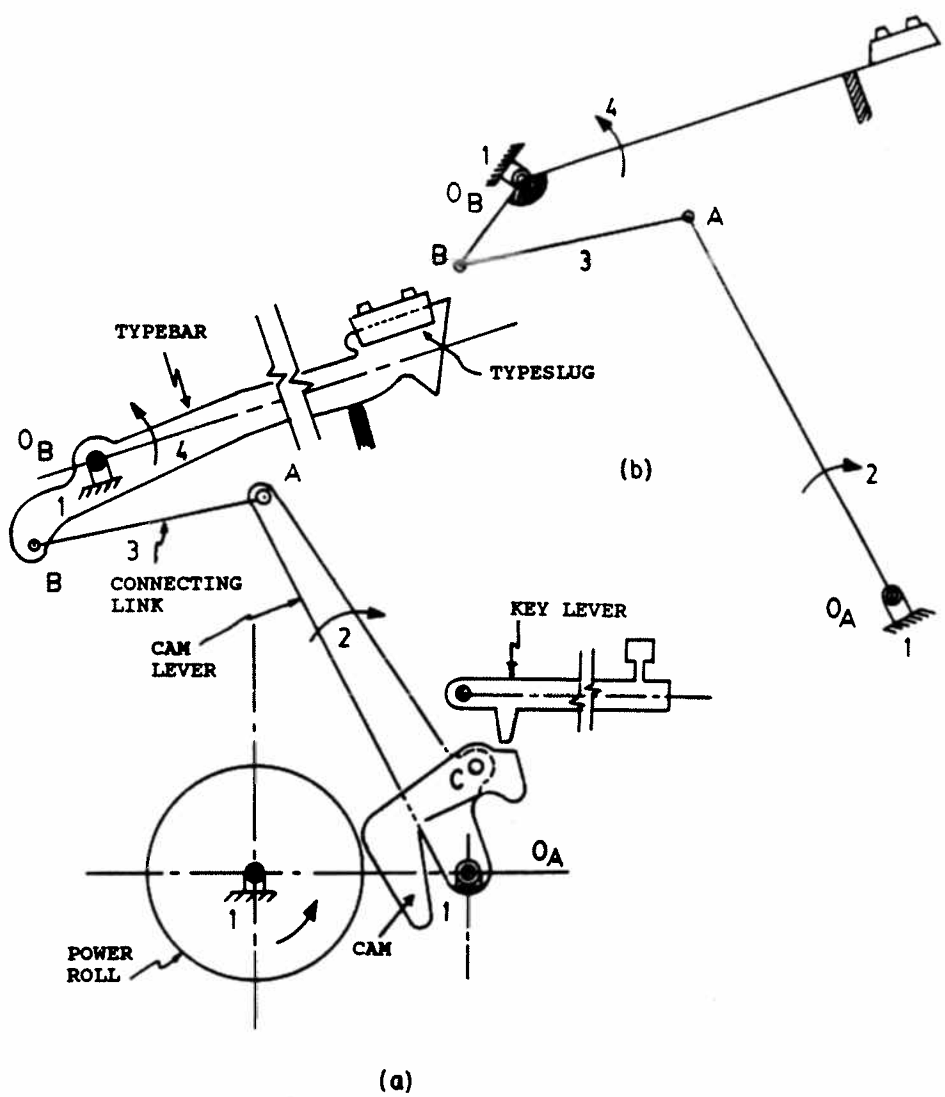



FIG. 1.2 A FOUR BAR MECHANISM USED AS PRINTING MECHANISM IN AN ELECTRIC TYPEWRITER

as a printing mechanism in an electric typewriter to transmit motion from one shaft to another. The typist presses a key lever which engages the cam with the power roll. The power roll is driven by an electric motor, and the cam is pivoted at point C on the cam lever. Consequently, the power roll drives the cam, and the cam in turn, drives the cam lever. The cam lever, marked as link 2 in Fig. 1.2b, drives the type bar, link 4, to print through link 3.

The crank slider mechanism shown in Fig. 1.1 can be thought of as made of a fixed link, a crank, a connecting rod and a slider. During its performance, except for the fixed link, all other links undergo either a translatory or a rotary motion. All mechanisms have at least one fixed link. If all the links are fixed then it would be called a structure 4 rather than a mechanism.

Mechanics is a science dealing with motion, forces, time, and is divided into two parts, statics and dynamics. Further sub-divisions of dynamics is shown in Fig. 1.3. Kinematics deals with motion such as velocity, acceleration etc. without any information about forces which cause such motion. In kinetics we study both the motion as well as the forces. It should be remembered that the sub-division of dynamics into kinematics and kinetics is based on the assumption that the body is rigid i.e. it does not deform. It implies that the distance between various points on the rigid body on the rigid body does not change during the motion of the link. We will investigate this fact further in the next chapter on the position analysis. If the body is flexible rather than rigid then the motions of various points will

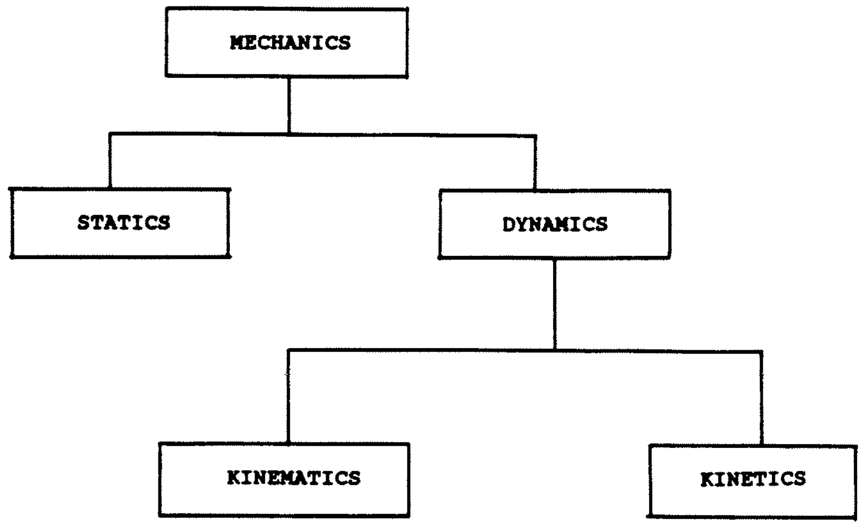



**FIG. 1.3 CLASSIFICATION OF MECHANISM**

depend on the forces applied. Therefore, this type of study is quite involved and is not a subject of study in this text. All real bodies are flexible to some degree but the deflections can be reduced while designing a machine or a mechanism component by varying geometrical parameters. In actual practice, one can carry out the kinematic analysis with the rigid body assumption and then perform the dynamic analysis in such a way that the deflections are kept to a minimum so that the rigid body assumption in the kinematic analysis is justified.

A mechanism consists of several linkages which are connected by joints such as pins or prismatic joints to form a closed or open-loop chain. Such chains are called mechanisms where, at least, one link is fixed called a frame and at least two other links are mobile. Clearly, in a mechanism relative motions between various links are possible and these motions depend upon the types of connections between various links. Some of these types of connections are shown in Fig. 1.4. The joint between crank and the fixed link is called a revolute or a pinned joint.

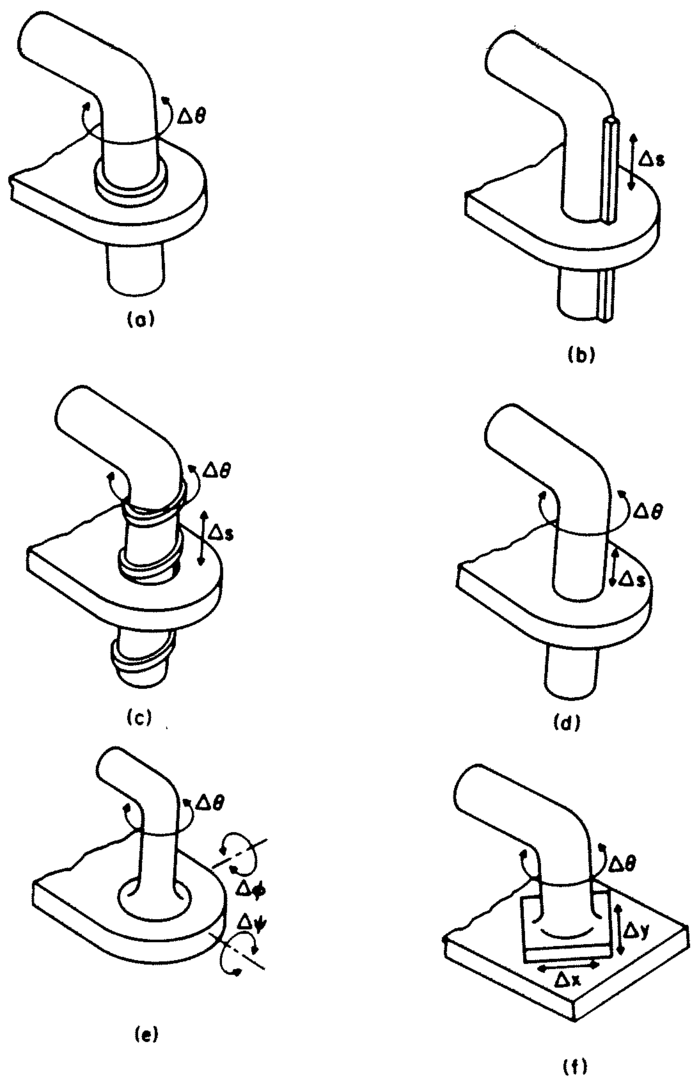



FIG 1.4 THE SIX LOWER PAIRS: (a) REVOLUTE OR PIN, (b) PRISM, (c) HELICAL, (d) CYLINDRIC, (e) SPHERIC, AND (f) PLANAR.

In a revolute joint, if one member is fixed then the other can rotate in a plane. The joints shown in Fig. 1.4 are idealized ones. The nut and bolt pair can be thought of as a helical joint. The type of a pair can be determined from their nature of the geometrical contact. The lower pairs have area contact whereas, the higher pairs have line or a point contact. An example of a lower pair is a shaft rotating in a bushing. Similarly, the rolling of a cylinder on a flat surface can be called as a higher pair. A mechanism where motions of various links occur in a given plane or in parallel planes is called a planar mechanism. In such cases the exact locations of various points can be seen from a direction normal to these planes. A mechanism where all the displacements can not be seen in their true shape from one view would fall into the category of a spatial mechanism. A few other concepts which are necessary to understand kinematics are inversion and mobility. The absolute motion of various links is dependent upon which of the link is fixed. The absolute motion of the links change if there is a change in the fixed link but the relative motion does not alter due to this change. The mechanisms obtained due to fixing of different links one at a time are called the kinematic inversions of each other. This is shown in Figs. 1.5 and 1.6.

In Fig. 1.5 the smallest link always rotates through 360 , and depending upon its location with respect to the frame, the four inversions are obtained. In all the four cases the Grashof's criteria: $s+L < p+q$ is satisfied where s and L are the shortest and longest links respectively; the other two are intermediate links. If $s+L > p+q$ four double-rocker mechanisms are obtained

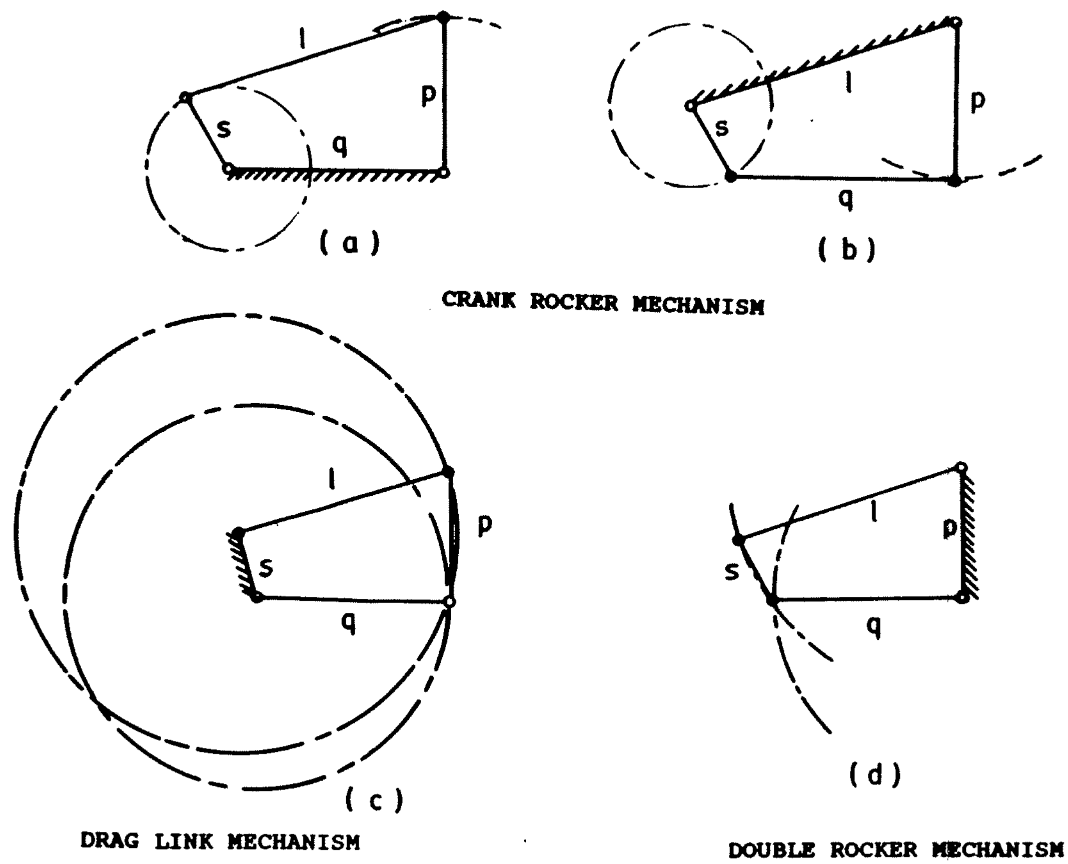



FIG. 1.5 VARIOUS INVERSIONS OF A FOUR BAR MECHANISM

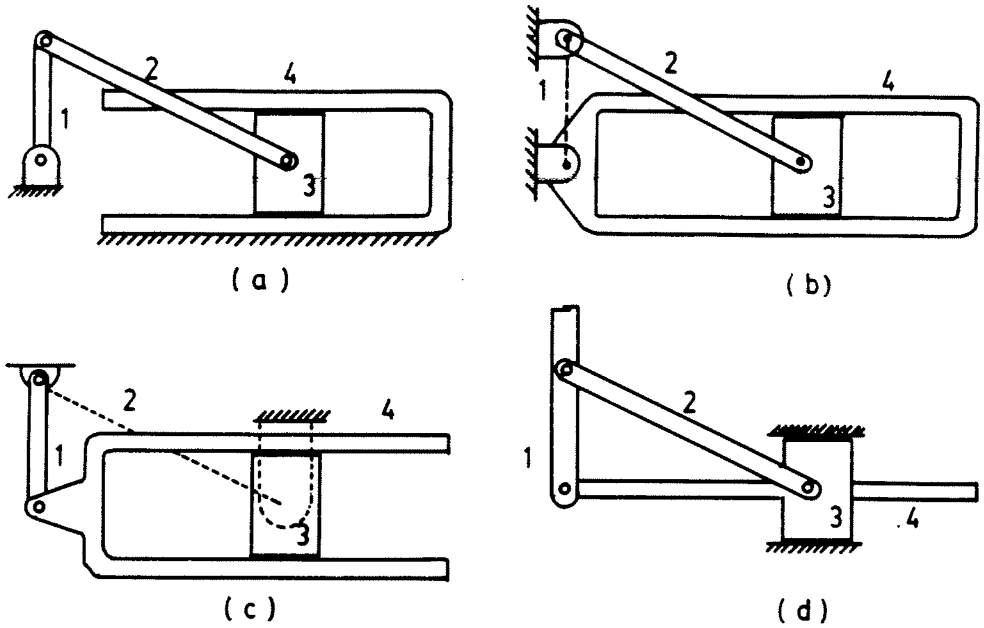



FIG. 1.6 VARIOUS INVERSIONS OF SLIDER CRANK MECHANISM

(a) CRANK SLIDER MECHANISM

(b) ROTATING SLIDER MECHANISM

(c) OSCILLATING SLIDER MECHAMISM

(d) STATIONARY SLIDER MECHANISM

based on various links being used as frames. If L+s = p+q then we would have mechanisms as shown in the Fig. 1.5 but the center lines of all links become collinear and the output link may change its direction of rotation unless an additional guidance is provided. Fig. 1.6 shows various mechanisms as a result of the inversion of the crank-slider mechanism. All of these mechanisms are quite useful. The number of degrees of freedom of a given mechanism are the number of independent parameters required to specify the given configuration of the mechanism. For example, Fig. 1.7a shows a four bar mechanism which has a fixed link or a frame (link 1). In the configuration shown if the angle between the crank and fixed link is given then, the other two links can be assembled in an alternate way also as shown in Fig. 1.7b i.e. the other two links can be assembled in two different configurations. The mobility, M of a mechanism can be arrived at by subtracting

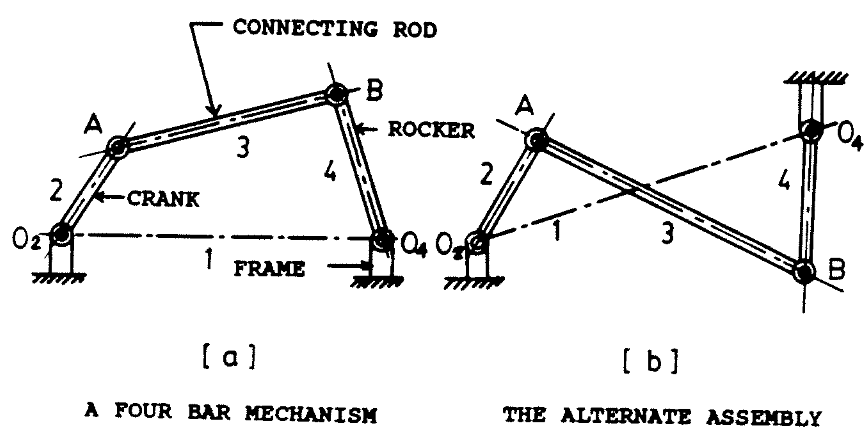



FIG. 1.7 AN ALTERNATE ASSEMBLY OF A FOUR BAR MECHANISM

the number of constraint equations (c) from the total degrees of freedom (N) which can be written as

$$M = N - C \tag{1.1}$$

N of a planar mechanism is equal to 3 x n where n is the total number of links in it. Here n has been multiplied by 3 because a link moving in a plane can have freedom to move along X, and Y axes and also rotate about the Z axis. Now let us consider the four bar mechanism which has 4 links out of which one link is fixed. N in this case is 4x3 = 12. Since one of the links is fixed therefore it loses 3 degrees of freedom. There are four pinned joints each of which allows only one degree of freedom i.e. it imposes two constraints per joint. Thus there are 8 constraints. Since we have already subtracted 3 constraints from the total degrees of freedom, therefore, in the case of four bar mechanism we would have

$$M = 3 \times 4 - (4 \times 2 + 3) = 1$$

The general formula for mobility of a planar mechanism can be written as

$$M = 3 * n - (n_1 * 1 + n_2 * 2 + 3) \tag{1.2}$$

where $n_1$ and $n_2$ are the number joints imposing 1 and 2 constraints respectively. The number of constraints imposed by a joint can be judged from Fig. 1.4. A rolling contact (without sliding) permits one degree of freedom due to the rotation whereas, if there is sliding also, then there will be two degrees of freedom. The additional one comes from the sliding action.

Fig. 1.8 shows various types of mechanisms and a structure. As discussed earlier the degrees of freedom of a joint in a planar

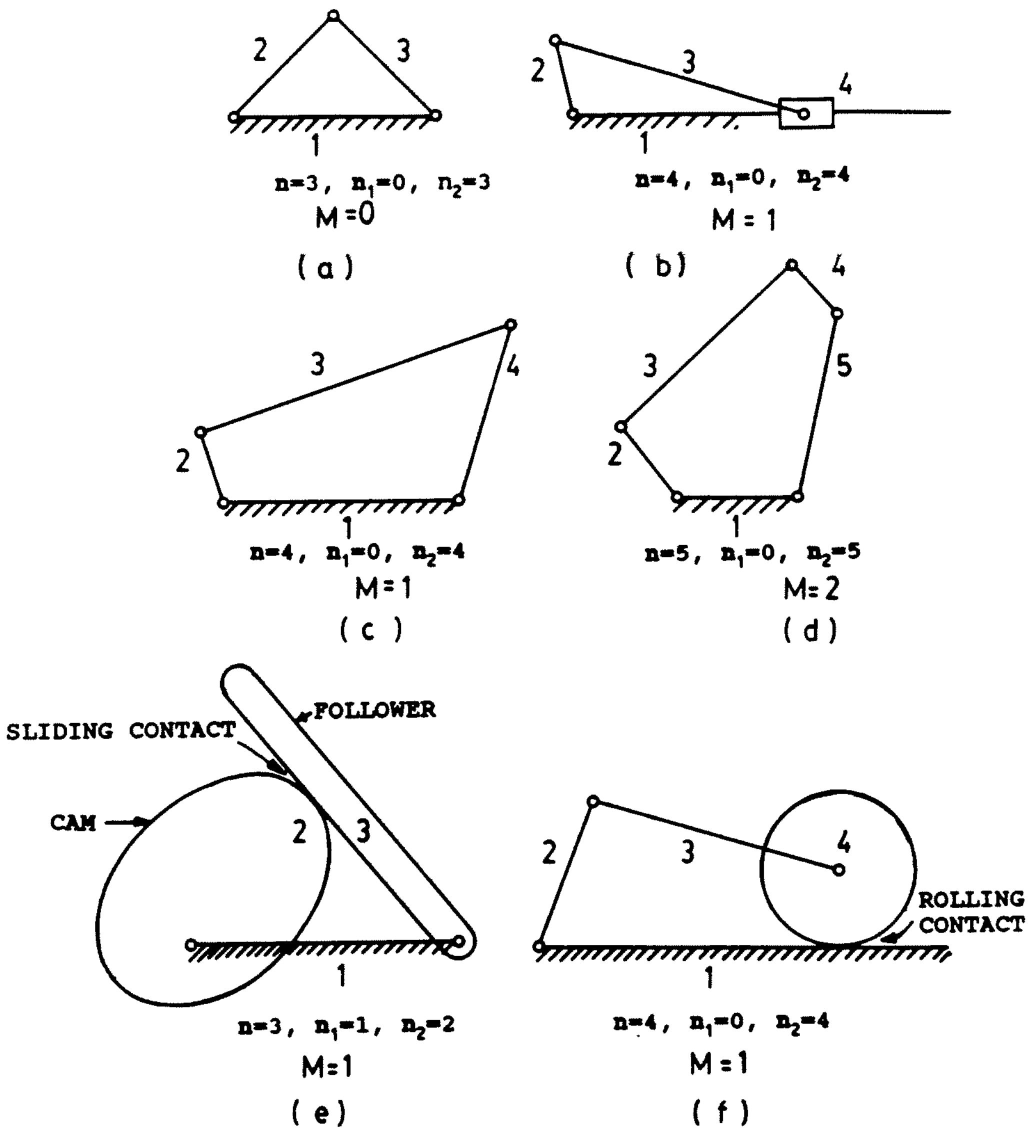


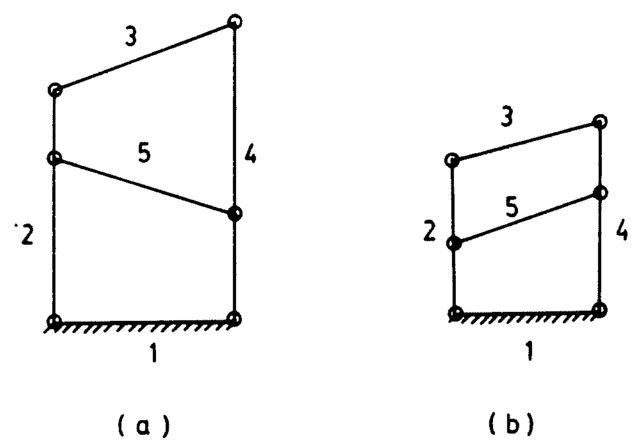

FIG. 1.8 MOBILITIES OF VARIOUS MECHANISMS

mechanism is equal to 3-n where n is the number of constraints imposed by the joint. A single degree of freedom joint will have n = 2 and a two-degree-of-freedom joint, equal to 1. To find out the mobility or the number of independent parameters to specify a mechanism one should, (a) first of all count the number of links and then inspect all the connections or joints one by one, (b) classify the joints by referring to Fig. 1.4 and, (c) use Eq. 1.2. For example, let us consider the mechanism shown in Fig. 1.8b. There are four links (n=4), three pin or revolute joints and one sliding joint. All are single-degree-of-freedom joints (n =4); there are no two-degree-of-freedom joints ($n_1$ =0).Now we use Eq. 1.2:

$$M = 3 \times 4 - (0 \times 1 + 4 \times 2 + 3) = 1$$

Let us examine the mechanism in the Fig. 1.8f. There are four links here also. There are 3 revolute joints and one rolling contact (it has one degree of freedom). If we use the Eq. 1.2, we will have M = 1. If we inspect Fig. 1.8e we would find that it too has M = 1. What happens if there is a rolling contact instead of sliding contact between links 2 and 3? If we reduce the mobility by one we will obtain a structure as in Fig. 1.8a. A rolling contact is like a pinned connection which allows one member to rotate about the other. Eq. 1.2 is also called the Kutzback criterion for the mobility of a planar mechanism. If M = 0, we have seen that it is a structure. If by using Eq. 1.2, we get M = -1 or less, then there are redundant constraints acting and the structure is called a statically indeterminate structure.

In the application of the Kutzback criteria one can notice that there is no consideration for the link dimensions or their relative orientations. Therefore, in certain situations, one does not obtain the correct number for the mobility. For example, Fig. 1.9 shows two mechanisms. In the mechanism in Fig. 1.9b, the link 5 is parallel to the link 3. It is a double parallelogram linkage with M = 1. The link 5 in Fig. 1.9a is not parallel to the link 3 thus making it immobile i.e. M = 0. Even though there are exceptions in determining the mobility using the Kutzback criteria yet, it is of immense value because in the early part of the design stage no one knows the exact dimensions of various links or their orientations.




(a) FIVE LINK MECHANISM

(b) DOUBLE PARALLELOGRAM

FIG. 1.9 FIVE LINK MECHANISM AND THE KUTZBACK CRITERIA

CHAPTER 2

# DISPLACEMENT ANALYSIS OF MECHANISMS

## 2.1 Position Vector and its Mathematical Representation

The first and foremost task in analyzing the motion of a particle or rigid bodies is to carry out the displacement analysis. To understand the motion of rigid bodies one has to first understand the motion of the various points on this body. In general, the motions of different points on a rigid body are different. Therefore,to analyze motions of various particles on a rigid body these could be thought of as points. The concept of motion of a particle arises from the displacement history of the particles i.e., to observe a motion, first of all one should observe the locations of the particles with respect to certain fixed coordinate system. To make the matter simpler, let us say the coordinate system is defined in a plane and we are assuming that the particle's motion is confined in this plane. The curve traced by the particle is shown in Fig. 2.1. If we join the

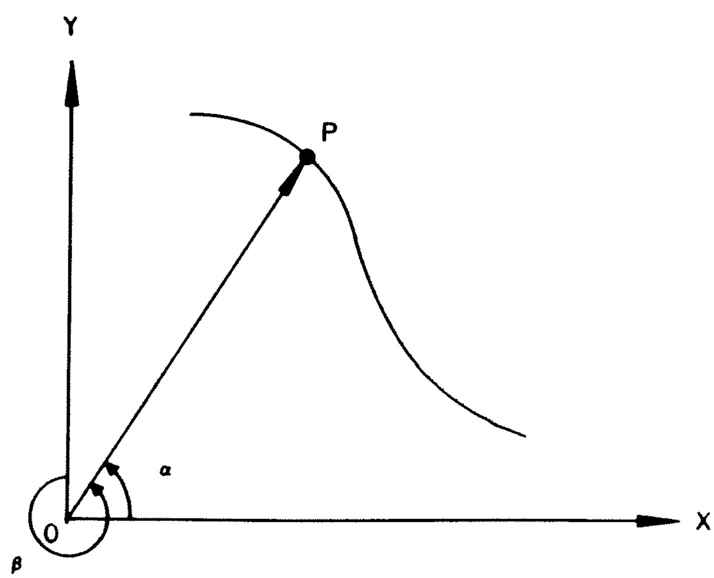



FIG. 2.1 POSITION VECTOR OF A POINT P

origin to a point P and consider it as a vector then we can call it a position vector of the particle at that instant of time. Therefore the position vector of a particle is a vector starting from the origin and terminating at the location of the particle at that instant of time.Just like any other vectors, the position vector has a magnitude and direction. The magnitude in this case will be the distance between 0 to P, and to represent the direction, we need to know the angles between the vector 0P and the X and Y axes as shown in Fig. 2.1. We have to use a convention to define the directions. The convention is that the angles must be measured counter-clockwise from the respective axes as shown in Fig.2.1. The position vector in three-dimensional space is shown in Fig. 2.2. The three angles in this case can be truly seen by observing perpendicular to three different planes. The first plane can be defined as that containing the X axis and 0P. Similarly, another plane will contain the Y axis and the line 0P. These are shown in Fig. 2.2. Besides these angles, one needs to define the unit vectors and specify the scale used. To explain the concepts, let us take an example of a force equal to 100 Newtons (N) acting on a particle of mass 10 kg. The example of a force is being used because we are familiar with it. It is a vector quantity and we want to represent it graphically. Suppose we also know that the force makes an angle of $\alpha = 45^0$ from the X axis. In the two dimensional space if we know the angle from the X axis then the angle $\beta$ from the Y axis can be calculated using the formula

$$\beta = (\alpha - 90) \quad \text{or} \quad (270^0 + \alpha) \qquad (2.1)$$
$$= 315^0$$

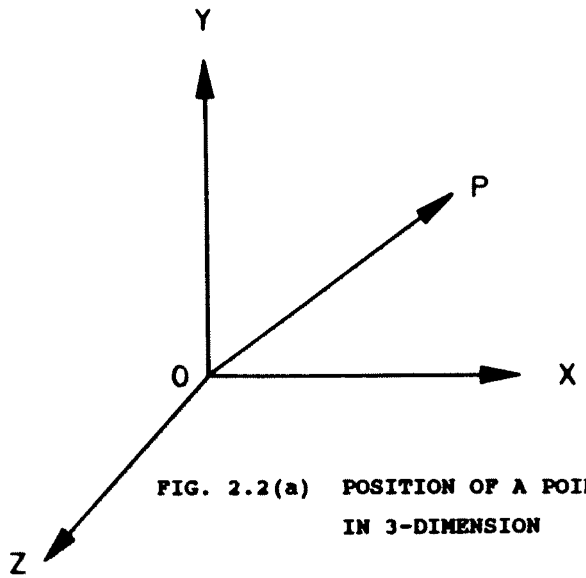



FIG. 2.2(a) POSITION OF A POINT IN 3-DIMENSION

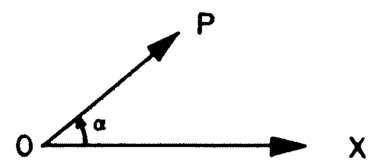



FIG. 2.2(b) DIRECTION COSINE WITH RESPECT TO X-AXIS

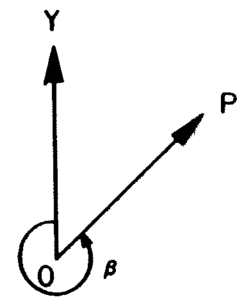



FIG. 2.2(c) DIRECTION COSINE WITH RESPECT TO Y-AXIS

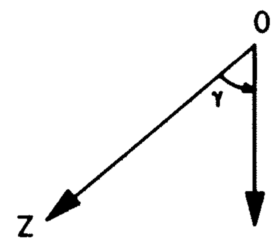



FIG. 2.2(d) DIRECTION COSINE WITH RESPECT TO Z-AXIS

In other words, out of $\alpha$ and $\beta$, only one of them is an independent parameter. To represent the magnitude we need to define the scale. Suppose we represent 10 N by 1 mm then this force will be represented by a line 10 mm long which is at $\alpha^0$ from the X axis as shown in Fig. 2.3. The unit vector, 1N, will be represented by a line 0.1 mm along the X or Y axes. The unit force vectors along the X and Y directions are symbolically represented by i and j respectively. Both of these vectors have the same magnitude equal to 1 N.

The position vector at P will be represented by $\mathbf{R}_{PO}$ where it is understood that the tail of this vector is at 0 and the tip at P (refer to Fig. 2.1). We can also have a unit vector in the direction of $\mathbf{R}_{PO}$ and it will be represented as $\hat{R}_{PO}$. The carat symbol above a letter will always represent a unit vector. Since it is a unit vector its magnitude will be equal to unity. Let

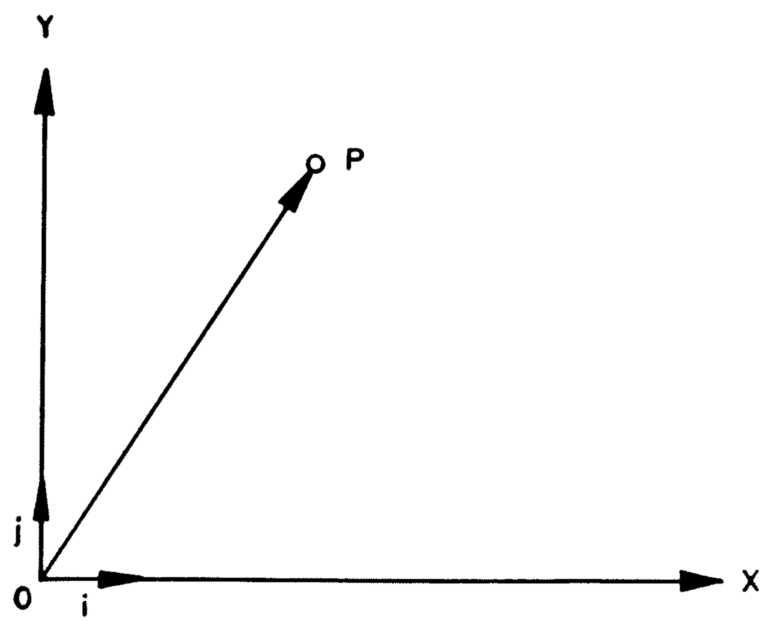



FIG. 2.3 POSITION OF A POINT P

$|\mathbf{R}_{PO}|$ or $R_{PO}$ represent the magnitude of the vector $\mathbf{R}_{PO}$, then $R_{PO}$ can be obtained by

$$R_{PO} = \frac{\mathbf{R}_{PO}}{|\mathbf{R}_{PO}|} = \frac{\mathbf{R}_{PO}}{R_{PO}} \tag{2.2}$$

$\mathbf{R}_{PO}$ can also be expressed in terms of its components along the X and Y directions as

$$\mathbf{R}_{PO} = \mathbf{R}^{X}_{PO}\ i + \mathbf{R}^{Y}_{PO}\ j$$

This vector can also be expressed in a polar form as $\mathbf{R}_{PO} = R_{PO}\angle\alpha$ where the angle $\alpha$ is measured in the counter-clockwise manner from the X axis. The mathematical representation of the vector in terms of $\alpha$ will be

$$\begin{aligned} \mathbf{R}_{PO} &= R_{PO}\ \mathrm{Cos}\alpha\ i + R_{PO}\ \mathrm{Sin}\alpha\ j \\ &= R_{PO}\ \mathrm{Cos}\alpha\ i + R_{PO}\ \mathrm{Cos}(90^{0}-\alpha)\ j \\ &= R_{PO}\ \mathrm{Cos}\alpha\ i + R_{PO}\ \mathrm{Cos}\beta\ j \end{aligned} \tag{2.3}$$

It has been shown here that if we know the magnitude of a vector and the angles made by this vector from the coordinate axes measured in the counter-clockwise manner then we can represent the vector shown in Eq. (2.3). Thus the unit vector $\hat{R}_{PO}$ using Eqs. (2.2) and (2.3) will be given by

$$\hat{R}_{PO} = \mathrm{Cos}\alpha\ i + \mathrm{Cos}\beta\ j \tag{2.4}$$

This relationship shows that once we know the magnitude and direction of a vector then the expression for its unit vector is automatically known. Sometimes $\mathrm{Cos}\alpha$ and $\mathrm{Cos}\beta$ are also referred to as the direction cosines of the vector. In the two dimensional space, only one direction cosine is needed, the other can be calculated using it, and in the three dimensional space there are three direction cosines. Suppose these are called $\mathrm{Cos}\alpha$, $\mathrm{Cos}\beta$, and

$\cos\gamma$, then only two of them are independent and the third can be calculated using the relationship

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{2.5}$$

For the two dimensional space the corresponding equation will be

$$\cos^2\alpha + \cos^2\beta = 1 \tag{2.6}$$

It should be added here that between $0^0$ to $360^0$, the arc cosine function is a double valued function, therefore, it requires an additional effort to know the correct value of the unknown angle if we use Eq. (2.6). For example, let us assume that for a vector we have $\alpha = 30^0$, then we can write

$$\cos^2 30 + \cos^2\beta = 1$$

Therefore,

$$\beta = \cos^{-1}\left(\pm\sqrt{1 - \cos^2 30}\right)$$

$$= \cos^{-1}\left(\pm\sqrt{1 - (0.866)^2}\right)$$

$$= \cos^{-1}(\pm 0.5)$$

We will obtain two values which are $60^0$ or $300^0$ corresponding to 0.5 because the cosine function is positive in the first and fourth quadrants. If we also check the value of $\beta$ using Eq (2.1), we will find that $\beta = 300^0$ is the correct solution. There is no ambiguity in using Eq (2.1). Similarly there are two other solutions corresponding to the negative value which we are not seeking.

Fig. 2.4 shows the path of a particle at various instants of time. At any instant of time we must know the X and Y components of this vector to define it completely. Alternately, we must know the magnitude and the angle of this vector with respect to the X axis. During its motion the X and Y components of the position