vector will be a function of time. To specify the position vector we must know each component as a function of time i.e., either

$$\left.\begin{aligned} R^X_{PO} &= f_1(t) \text{ , and} \\ R^Y_{PO} &= f_2(t) \end{aligned}\right\} \quad (2.7)$$

or

$$\left.\begin{aligned} R_{PO} &= f_3(t) \text{ , and} \\ \alpha &= f_4(t). \end{aligned}\right\} \quad (2.8)$$

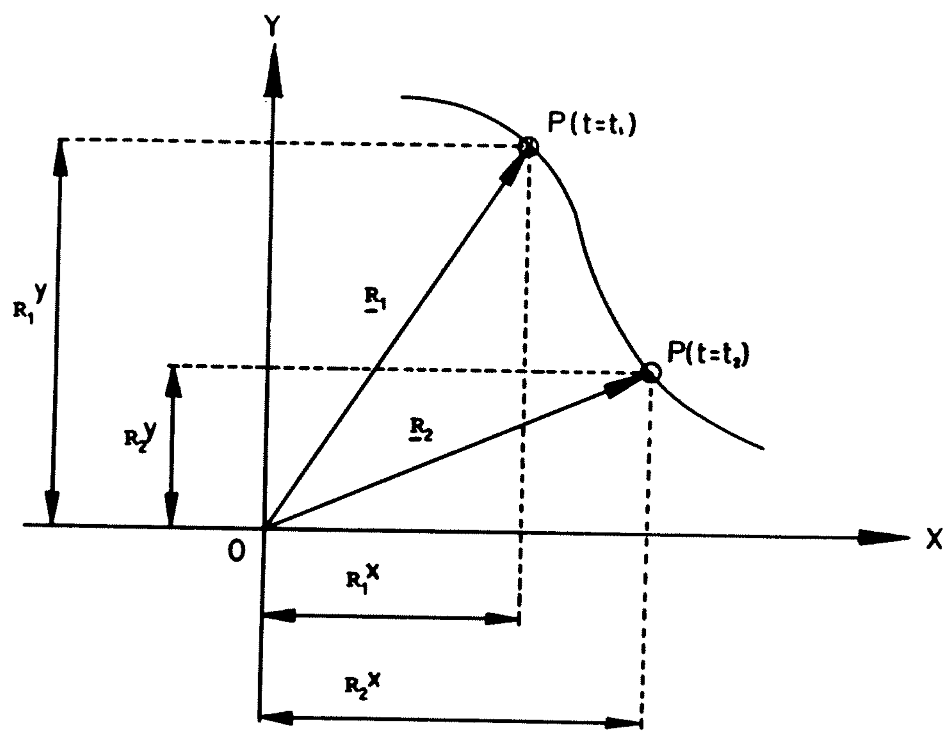

FIG. 2.4 POSITION OF A POINT P AT TWO INSTANTS OF TIME

2.2 The Position Difference Between Two Points, Apparent Position, and Absolute Position of a Point.

The position vectors of two points P_1 and P_2 are shown in Fig. 2.5. The position difference vector $R_{P_2P_1}$ is the vector difference of the vectors R_{P_2O} and R_{P_1O}. It is represented by the equation

$$R_{P_2P_1} = R_{P_2O} - R_{P_1O} \tag{2.9}$$

Remember that this is a vectorial equation which can be solved either graphically or analytically. In Fig. 2.5 the sum of the vectors R_{P_1O} and $R_{P_2P_1}$ will be equal to R_{P_2O}. Therefore, $R_{P_2P_1}$ is a vector difference vector. R_{P_2O} and R_{P_1O} represent the position vectors of two points P_2 and P_1 which indicate the positions of these points with respect to the origin whereas the position

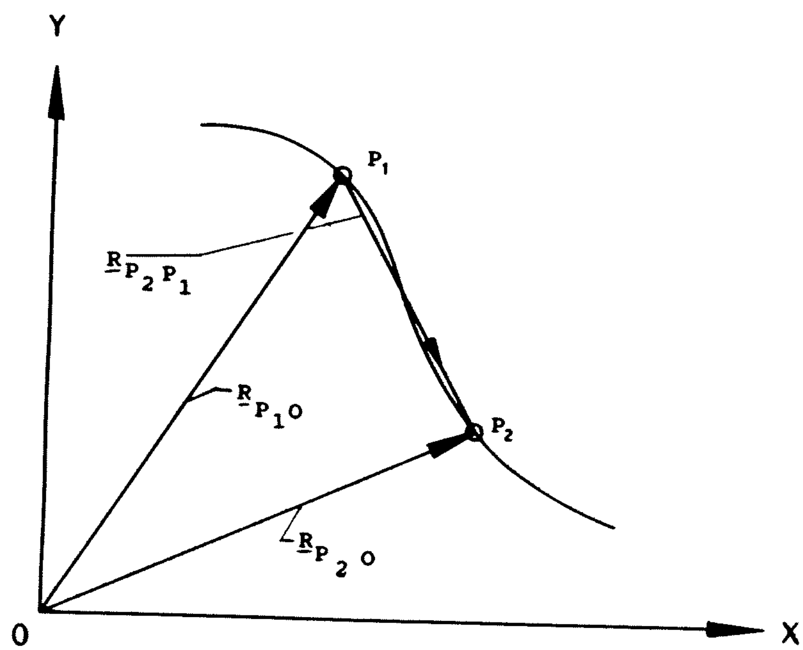

FIG. 2.5 POSITION DIFFERENCE VECTOR

difference vector $\mathbf{R}_{P_2P_1}$ specifies the position P_2 relative to P_1. This is a useful concept because, quite often, one is interested in knowing the relative position only rather than the position with respect to the origin. As an example, suppose P_1 and P_2 represent the locations of the down'towns of two cities. A man wants to go from P_1 to P_2 by car. If he knows, (a) the distance between these two points and (b) the angular direction with respect to whatever coordinate system he chooses to define his orientation, then he can reach his destination in a precise manner.

Let us say, we have vectors $\mathbf{R}_{P_20} = 20i + 15j$ and $\mathbf{R}_{P_10} = 10i + 4j$ then

$$\mathbf{R}_{P_2P_1} = \mathbf{R}_{P_20} - \mathbf{R}_{P_10} = (20i + 15j) - (10i + 4j)$$
$$= 10i + 11j$$
$$= 14.866 \angle(\alpha = \tan^{-1}(11/10))$$
$$= 14.866 \angle(\alpha = 47.726^0)$$

A word of caution is in order here regarding the angle 47.726^0. Actually, there will be another value 180^0 apart from this solution which will correspond to $\tan^{-1}(-11/-10)$. Some of the calculators can not distinguish between the two. There are two ways to avoid this difficulty. In the first method, calculate the magnitude of $\mathbf{R}_{P_2P_1}$ which in this case will be given by

$$|\ \mathbf{R}_{P_2P_1}| = \sqrt{10^2 + 11^2} \quad = 14.866$$

Only the positive sign should be used because the magnitude can not be a negative quantity. Consider 14.866 as the hypotenuse of a right-angled triangle as shown in the Fig. 2.6. The Sinα and

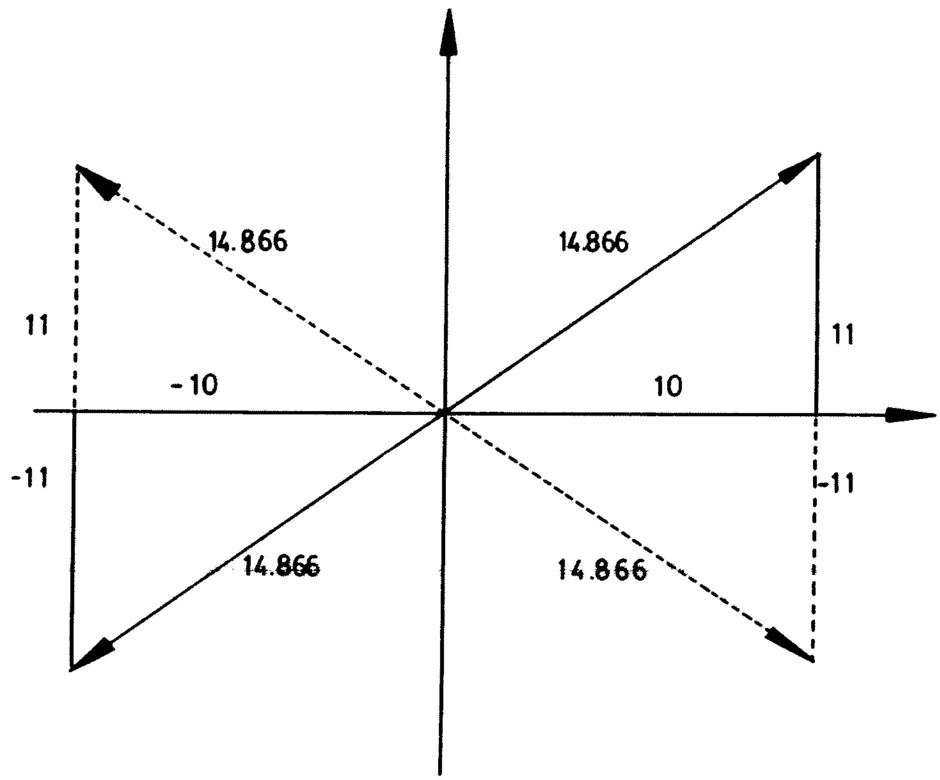

FIG. 2.6 DETERMINING THE DIRECTION OF A VECTOR

Cosα values will be 11/14.866 and 10/14.866 respectively. Both of these values are positive; therefore, α has to be less than 90^0. Now let us take the case of $R_{P_2P_1} = -10i - 11j$. $R_{P_2P_1}$ in this case also will be equal to 14.866 but Sinα and Cosα correspondingly will be -11/14.866 and -10/14.866 respectively, both of which are negative. The angle α in this case will be in the third quadrant.

Therefore, the procedure to obtain the correct α would be divided into two steps. In the first step, calculate α assuming the coefficients of i and j as positive. The magnitude $|R_{P_2P_1}|$ is always positive. This will yield α^*. In the next step check the signs of the ratios for Sinα and Cosα. Four possibilities exist and the solutions corresponding to each of these possibilities are given below:

(a) If Sinα and Cosα both are positive then α remains unchanged.Use $\alpha = \alpha^*$ (2.9a)

(b) If Sinα is positive and Cosα is negative then

$\alpha = 180^0 - \alpha^*$ (2.9b)

(c) If Sinα and Cosα are both negative then

$\alpha = 180^0 + \alpha^*$ (2.9c)

(d) If Sinα is negative but Cosα is positive then

$\alpha = 360^0 - \alpha^*$ (2.9d)

The procedure is illustrated in Fig. 2.6.

In the second method we calculate sinα and Cosα first. For example, if $R_{P_2P_1}$ = 10i -11j then Sinα = -0.740 and Cosα = 0.673. Then calculate Tan $\alpha/2$ using the formula

$$\text{Tan } \alpha/2 = \frac{1 - \text{Cos}\alpha}{\text{Sin}\alpha} \qquad (2.10)$$

$$= (1-0.673)/(-0.740)$$

$$= -0.44189$$

The angle $\alpha/2$ in this case is equal to -23.840^0, so α will be equal to -47.680^0, i.e., the angle has been measured clockwise because of the negative sign. Whenever we get negative angles we should add 360^0 to it to make it counter-clockwise. Therefore, it becomes 312.32^0 counter clockwise. The principle in this method is that if α is in the first or the fourth quadrants then these calculators will give correct results. The problem of α occurring in the second or the third quadrant is resolved by taking half their values so that $\alpha/2$ will be in the first or fourth quadrants. The quadrants can be seen in Fig. 2.6. In these two quadrants, the calculators or computers yield correct angles without any ambiguity. For example, if α is 120^0 then using this procedure, $\alpha/2$ value will be 60^0 and it can be correctly calculated.

Similarly if α is equal to 260^0 i.e., this angle is the same as -100^0, therefore, $\alpha/2$ will be -50^0 which is in the fourth quadrant. In this way one can resolve these types of problems while using calculators or computers. The angle α of vectors, not passing through the origin, can be easily seen by drawing another set of coordinate axes (X′-Y′) parallel to the reference coordinate axes as shown in Fig. 2.7. If the scale in this new set of axes is the same, then the unit vectors i′, j′, and k′ will be equal to i, j, and k respectively, because the magnitude of any of these vectors of the two sets are equal to 1. The conversion of vectors defined in two different set of coordinate axes which are not parallel requires a bit of explanation. First we define another set of axes parallel to (X-Y) which is shown as (X_1 -Y_1)

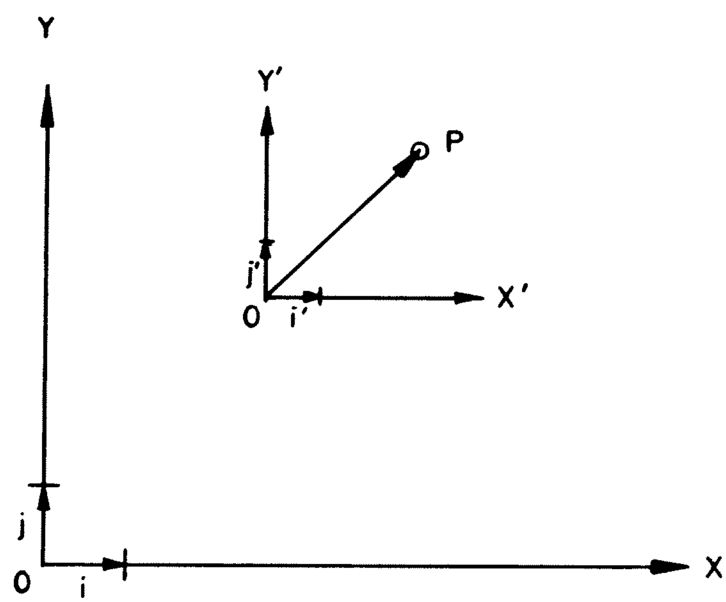

FIG. 2.7 DIRECTION OF A VECTOR **P** WITH RESPECT TO ANOTHER SET OF PARALLEL AXES

in Fig. 2.8. The coordinate of point P in the $(X_1 - Y_1)$ system having coordinates (10,4) in the (X-Y) systems, will be X_1^P = (10-2), and Y_1^P = (4-3) where the point 0_1 has coordinates (2,3) with respect to (X-Y). In the second step we find out X_2^P and Y_2^P using equations

$$\left.\begin{aligned} X_2^P &= \text{Cos}\theta\ X_1^P - \text{Sin}\theta\ Y_1^P \\ Y_2^P &= \text{Sin}\theta\ X_1^P + \text{Cos}\theta\ Y_1^P \end{aligned}\right\} \tag{2.11}$$

The two equations can be written in a matrix form as

$$\begin{vmatrix} X_2^P \\ Y_2^P \end{vmatrix} = \begin{vmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{vmatrix} \begin{vmatrix} X_1^P \\ Y_1^P \end{vmatrix} \tag{2.12}$$

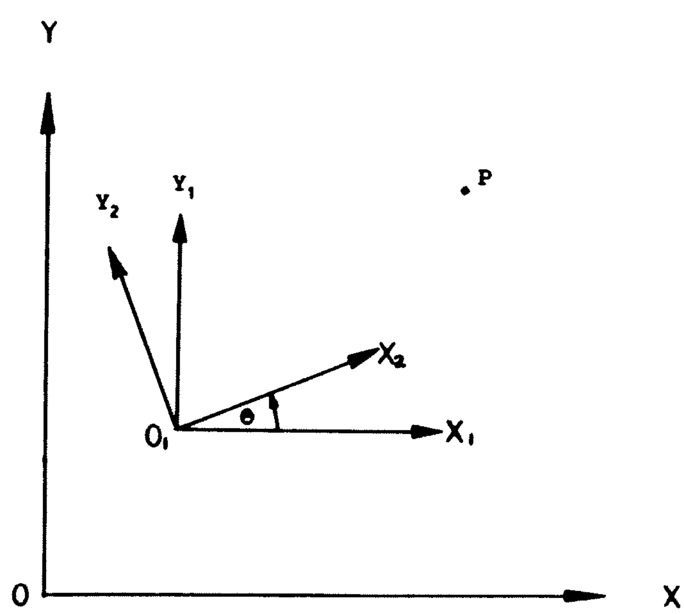

FIG. 2.8 POSITION VECTOR IN X-Y AND X_1-Y_1 COORDINATE SYSTEMS

In Eq. (2.12), the coordinates have been written in a vector form and the matrix on the right hand side is called a rotational transformation matrix because it relates two vectors due to the rotation of the coordinate system. To use such rotational matrices the origins of the two sets must be coincident. Substituting the numerical values corresponding to $\theta = 20^0$ we get

$$\begin{vmatrix} X_2^P \\ Y_2^P \end{vmatrix} = \begin{vmatrix} \text{Cos } 20 & -\text{Sin } 20 \\ \text{Sin } 20 & \text{Cos } 20 \end{vmatrix} \begin{vmatrix} 8 \\ 1 \end{vmatrix} = \begin{vmatrix} 7.176 \\ 3.676 \end{vmatrix}$$

If we know a vector in the $(X_2 - Y_2)$ coordinate system and would like to know it in the $(X - Y)$ system then we can use the same procedure as before. In this case we will have

$$\begin{vmatrix} X_1^P \\ Y_1^P \end{vmatrix} = \begin{vmatrix} \text{Cos } 340 & -\text{Sin } 340 \\ \text{Sin } 340 & \text{Cos } 340 \end{vmatrix} \begin{vmatrix} 7.176 \\ 3.676 \end{vmatrix} = \begin{vmatrix} 8.0 \\ 1.0 \end{vmatrix}$$

and then we can get $X^P = (8+2)$ and $Y^P = (1+3)$ by shifting the origin from O_1 to 0. The angle 340^0 is used above because the axis X_1 is obtained by rotating X_2 by 340^0 in the ccw direction.

So far we discussed how to calculate the position vectors in two different coordinate systems which can have same or different orientations and their origins may or may not coincide. Once we obtain the position vectors then the position difference vector can be calculated by taking the difference of the two. If we consider one of the coordinate systems as absolute then the vectors defined with respect to all other coordinate systems will be called apparent vectors. It is a matter of arbitrary selection which one we call the absolute coordinate system. But once one of them is selected as an absolute, then the rest of them are referred to as apparent systems.

If an apparent position vector (refer to Fig. 2.8) is known then the absolute position vector can be calculated using the equation

$$\mathbf{R}_{PO} = \mathbf{R}_{O_1O} + \mathbf{R}_{PO_1} \qquad (2.13)$$

In the example we discussed just earlier, this would be

$$\begin{vmatrix} X^P \\ \\ Y^P \end{vmatrix} = \begin{vmatrix} 2.0 \\ \\ 6.0 \end{vmatrix} + \begin{vmatrix} \text{Cos } 340 & -\text{Sin } 340 \\ \\ \text{Sin } 340 & \text{Cos } 340 \end{vmatrix} \begin{vmatrix} 7.176 \\ \\ 3.676 \end{vmatrix}$$

Here, we were just interested in calculating the X and Y coordinates of the point P in the absolute coordinate system. Once we know them then we can represent as $\mathbf{R}_{PO}$

$$\mathbf{R}_{PO} = X^P i + Y^P j$$

In Eq. (2.13), we know the apparent position vector $\mathbf{R}_{PO_1}$, the coordinates of the origin of (X_2- Y_2) system and the orientation of (X_2- Y_2) system. Once we have all this information then we can convert all apparent vectors to absolute vectors. The usefulness of this concept can be understood if we take an example of three satellites in the equatorial plane at an angle of 120^0 apart. If they move in an orbit sufficiently away from the earth then they can locate any object on the earth and relay the information to each other or back to the control station on the earth. For communications, they must be in the line of sight with each other. The coordinate system on the earth (control station) can be considered as absolute and the rest as apparent. This concept becomes quite useful if the objects can not be seen from the absolute coordinate system then one can use an apparent system provided the location and the orientation of the apparent system relative to the absolute system is known.

If the position vectors are defined with respect to the absolute coordinate system then we will denote $\mathbf{R}_{P0}$ as $\mathbf{R}_P$ for a point P. In general, we will use (X-Y) system as the absolute system and the number 0 (zero) will be used with it and all other systems as apparent and denoted by higher numbers. Eq. (2.13), in the new notation, will be

$$\mathbf{R}_{P/0} = \mathbf{R}_{0_1 0/0} + \mathbf{R}_{P/2}$$

All the position vectors will have only one subscript and the position difference vectors, two. For example $\mathbf{R}_{P_2 P_1/0}$ will denote the position difference vector from P_1 to P_2 and defined in the absolute coordinate system. If no number is typed, for example $\mathbf{R}_P$,then it is understood that it is a vector in the absolute coordinate system. Thus $\mathbf{R}_{P0/0}$, $\mathbf{R}_{P0}$, and $\mathbf{R}_P$ would mean the same vector.

2.3 Position Analysis of Mechanisms

There are several methods available to find the position vectors of various points in a mechanism. The easiest and conceptually the simplest method is the graphical method. The points in the planar mechanism move in a plane or parallel planes. Therefore, the correct relative locations can be observed in a view which is observed from the normal direction to these planes. Depending on the problem, the position analysis of mechanism may require a solution of a vector equation which may have two solutions. A two-dimensional vector equation can be solved for two unknowns. We have seen earlier that it requires two quantities to define a vector which can be either its magnitude and direction or its X and Y components. Suppose **A** is a vector whose magnitude is known but the direction is unknown. We will use a Symbol $\checkmark$ for known and * for unknown; thus we can represent **A** as $\overset{\checkmark}{A}^{*}$. The analytical method of addition and subtraction of two vectors has already been discussed earlier and the graphical representation is shown in Fig. 2.9. There are three vectors **A**, **B**, and **C** is this figure. If there are more than three vectors in an equation then we can always combine them into three. For example, we are given the equation

$$\overset{\checkmark}{A}^{*} = \overset{\checkmark}{L}^{\checkmark} + \overset{*}{M}^{\checkmark} + \overset{\checkmark}{N}^{\checkmark} + \overset{\checkmark}{P}^{\checkmark} \tag{2.14}$$

to be solved. Here, the vectors **L**, **N** and **P** are completely known and can be replaced by a vector $\overset{\checkmark}{C}^{\checkmark}$. Then the equations will be

$$\overset{\checkmark}{C}^{\checkmark} = \overset{\checkmark}{L}^{\checkmark} + \overset{\checkmark}{N}^{\checkmark} + \overset{\checkmark}{P}^{\checkmark} \quad \text{and}$$

$$\overset{\checkmark}{A}^{*} = \overset{\checkmark}{C}^{\checkmark} + \overset{*}{M}^{\checkmark} \tag{2.15}$$

$$\text{Let } \overset{*}{B}^{\checkmark} = -\overset{*}{M}^{\checkmark} ,$$

then we can write

$$\overset{\checkmark}{C}^{\checkmark} = \overset{\checkmark}{A}^{*} + \overset{*}{B}^{\checkmark} \tag{2.16}$$

If we can solve Eq. (2.16) to obtain **B** then the magnitude of **M** will be equal to the magnitude of **B** but it will be opposite in direction. The Eq. (2.16) is in a standard form and depending upon the types of unknowns, it has been classified into four cases by Chace.[+] The four cases are summarized below:

Case 1 Magnitude and direction of the same vector are the unknowns. For example

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[+]M. A. Chace, Vector Analysis of Linkages, J. Eng. Ind. Ser. B., Vol. 55, No.3, pp. 289-297, Aug. 1963.

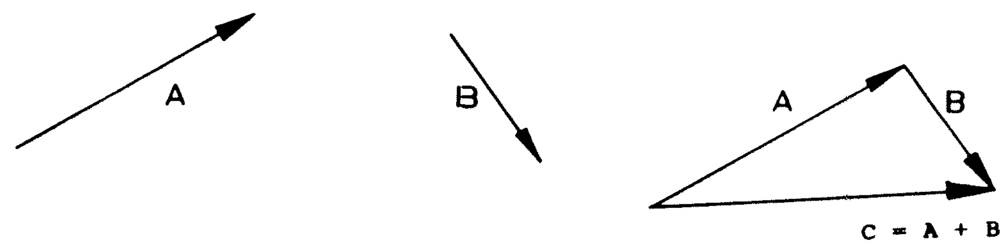

FIG. 2.9(a) ADDITION OF TWO VECTORS

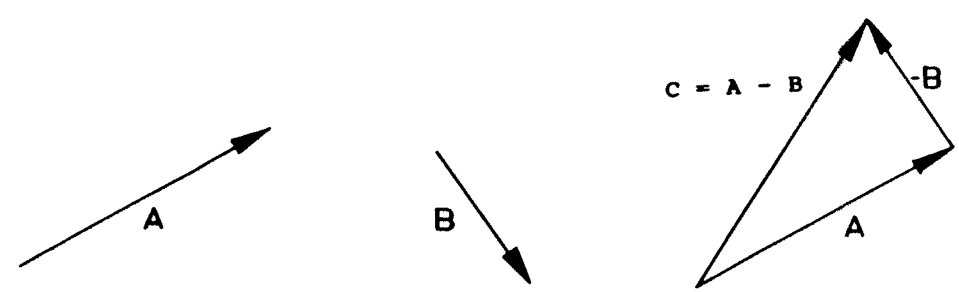

FIG. 2.9(b) SUBTRACTION OF TWO VECTORS

$$\overset{*}{C}^{*} = \overset{\surd}{A}^{\surd} + \overset{\surd}{B}^{\surd} \qquad (2.17)$$

Case 2a The magnitudes of two vectors are unknowns as represented by

$$\overset{\surd}{C}^{\surd} = \overset{*}{A}^{\surd} + \overset{*}{B}^{\surd} \qquad (2.18)$$

Case 2b The magnitude of one and the direction of the other are the unknowns. The equation for this case will be

$$\overset{\surd}{C}^{\surd} = \overset{\surd}{A}^{\bullet} + \overset{\bullet}{B}^{\surd} \qquad (2.19)$$

Case 2c The directions of two vectors are the unknowns, and the corresponding equation will be

$$\overset{\surd}{C}^{\surd} = \overset{\surd}{A}^{\bullet} + \overset{\surd}{B}^{\bullet} \qquad (2.20)$$

To solve the case 1 graphically, the procedure is the same as in Fig. 2.9. It is a simple case of vector addition. The case 2a can be solved using the following steps as shown in Fig. 2.10:

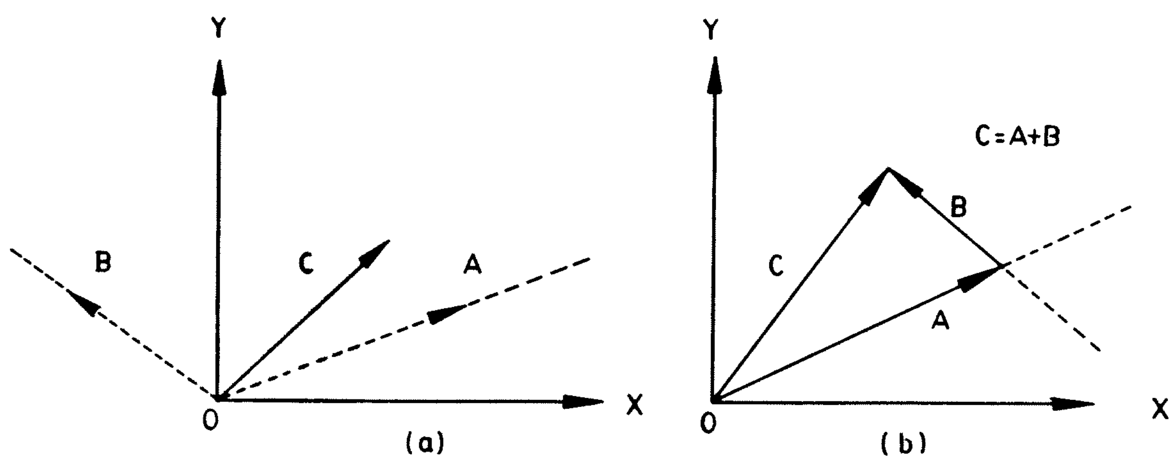

FIG. 2.10(a) VECTOR C AND DIRECTIONS OF A AND B GIVEN

(b) GRAPHICAL SOLUTION OF CASE 2a

1. Choose a coordinate system and a scale to represent the unit vectors in the X and Y directions respectively.
2. Draw the vector **C** at a convenient point; it is completely defined.
3. Construct two lines parallel to the two known directions such that the first line passes through the tail and the other through the tip or the terminus of **C** as shown in the figure. At the point of intersection of these lines, the vectors **A** and **B** meet.
4. Label the vectors **A** and **B** remembering that the two lines were drawn parallel to the respective directions of these two vectors.

In case 2b, the unknowns are the magnitude of one vector and the direction of the other as shown in Fig. 2.11a. The steps in this cases are:

1. Draw the vector **C** as before.
2. Draw a line parallel to the known direction ($\overset{*}{B}^{\checkmark}$) from the tail of the vector **C**.
3. Construct a circular arc of radius equal to the magnitude of **A** from the tip of the vector **C**. It will intersect the line which was drawn parallel to $\overset{*}{B}^{\checkmark}$ at two points.
4. Label the vectors **A** and **B** as shown in Fig. 2.11. Remember that when the magnitude of a vector is given, we have to draw a circular arc of radius equal to the magnitude.

In case 2c, the magnitudes of both vectors **A** and **B** are known and we have to determine their directions. The steps in this case would be:

1. Draw the vector **C** as before.

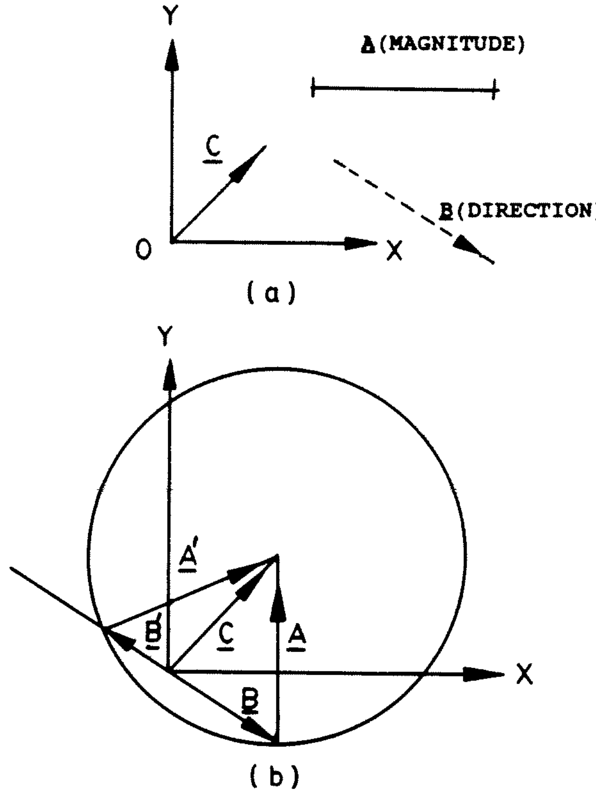

FIG. 2.11(a) VECTORS C, A, AND B AS GIVEN

(b) GRAPHICAL SOLUTION OF CASE 2b

2. From the tail of the vector **C** draw an arc of radius equal to |**A**|.

3. Similarily draw another arc from the tip of **C** of radius equal to |**B**|. The arcs will intersect at two points.

4. Label the vectors **A** and **B** in both the solutions. One has to be careful while labelling the vectors; they should be labelled in accordance with their respective magnitudes.

As a point of clarification, it should be stated here that the direction of any vector lying in a plane is completely defined if the angle between it and X or Y axis is known. Suppose the angle is known with respect to the X axis then as discused earlier, the angle from the Y axis can be calculated using Eq.

(2.1). In the graphical analysis there is no need to calculate the angle from the Y axis. Figs. 2.9 to 2.12 show all the four cases If the magnitude of a vector is unknown then it can be scaled from the drawing. On the other hand, if the direction is unknown then the first step is to draw another set of axes at the tail of the vector whose angle is to be found out and then in the second step measure the angle counter-clockwise between the positive X axis and the vector. If someone measures the angle counter-clockwise

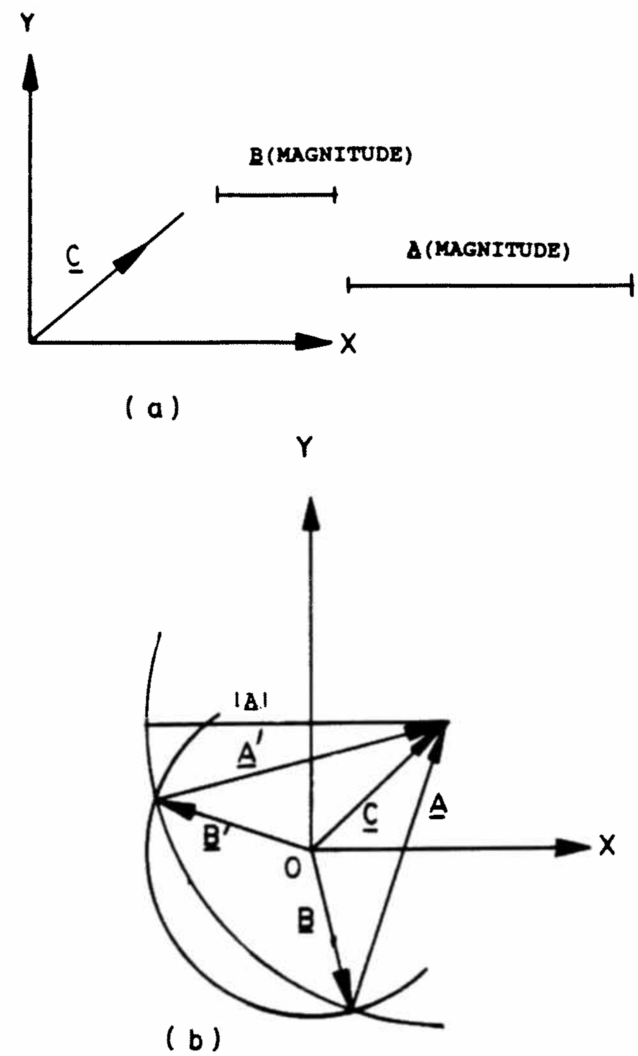

FIG. 2.12(a) VECTORS C, A, AND B AS GIVEN

(b) GRAPHICAL SOLUTION OF CASE 2c

from the Y axis then he or she is not wrong. For him or her, the Y axis is the reference axis. But most commonly used axis is the X axis. The representation of the vectors in the polar form would amount to measuring the angles from the X axis because the reference axis for the angular measurement normally coincides with the X axis. If the unit vector is to be represented analytically, then we need to know the angles with respect to the X and Y axes as expressed in Eq. (2.4). It should be noticed that in the graphical method of solution of vector equations, we are obtaining the unknown quantities such as the magnitude or direction in the polar form. This is because when we wrote the equations for all the four cases, these equations were expressed in the polar form. Once we know these unknown quantities, then we can express these vectors in terms of their cartesian components, if desired. In all the last three cases, C could be given to us either in terms of the cartesian components or in the polar form. Even if it was given in terms of the cartesian components still we could draw each of the components and find the resultant of the two as the summation of the two vectors. When the vectors are added analytically we always add the components along the X and Y directions. The graphical technique will be used very extensively in this text and the awareness of the fact that the solutions are performed in the polar form will make the task a bit easier when we carry out the velocity and acceleration analyses. In these analyses, there are addition or subtraction of vectors involved and these vectors may be present in one (polar) form or the other (cartesian).

In the position analysis of mechanisms one has to determine the position vectors of various points on the link. While specifying the problem, the link lengths and joint angles are given in such a way that there are maximum of two unknowns per equation. There may be several links involved and it is always possible to divide the problem of solving the entire mechanism into sub-problems and solve these.

2.4 Solution of Planar Vector Equations Using Complex-Algebra

It is possible to solve the vector equations which are planar only, using the complex numbers. If these vectors, i.e., position, velocity, and acceleration, exist in the three-dimensional space then the complex algebra method can not be used. We are using these vectors defined in two-dimensions only. A complex vector has a real and an imaginary part as shown in Fig. 2.13. One can express complex numbers in terms of the rectangular

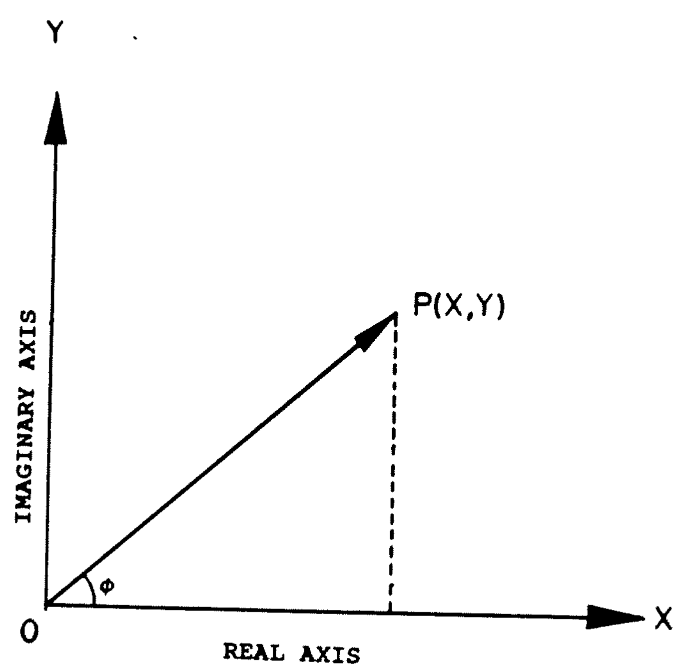

FIG. 2.13 COMPLEX VECTOR R_{PO}

components, if they are given in the polar form. In the polar form, the angle is measured counter-clockwise from the real axis. The addition or subtraction of complex numbers is done just like ordinary vectors in terms of the rectangular components. The division or multiplication of complex vectors is very simple in the polar form, whereas the division of ordinary vectors is not defined. We will see in the next chapter that the differentation of complex vector is very simple and leads to considerable simplifications in the mathematical analysis of velocities and accelerations. Example 2.1 shows various mathematical operations using the complex numbers. Just like the graphical method of solving vector equations, one can also use complex numbers to solve these for all the four cases.

Example 2.1

Express the two complex vectors **A** = 3 + 2j and **B** = 5 + 7j in polar forms and find (a) **A** + **B** (b) **A** - **B** (c) **A** × **B** (d) **A** / **B**.

Solution:

$$\mathbf{A} = (3^2 + 2^2)^{1/2} \underline{/\ \tan^{-1}(2/3)} = 3.606 \underline{/\ 33.69}$$

$$\mathbf{B} = (5^2 + 7^2)^{1/2} \underline{/\ \tan^{-1}(7/5)} = 8.602 \underline{/\ 54.462}$$

(a) $\mathbf{A} + \mathbf{B} = (3 + 2j) + (5 + 7j) = 8 + 9j = 12.042 \underline{/\ 48.366}$

(b) $\mathbf{A} - \mathbf{B} = (3 + 2j) - (5 + 7j) = -2 - 5j = 5.385 \underline{/248.199}$

(c) $\mathbf{A} \times \mathbf{B} = (3.606)(8.602)\underline{/(33.69+54.462)} = 31.019\underline{/88.152}$

(d) $\mathbf{A} / \mathbf{B} = \frac{3.606}{8.602} \underline{/(33.69-54.462)} = 0.419\underline{/339.228}$

In case 1, we are given the vectors **A** and **B**, and we have to find **C**. The derivation of equations is straight forward because we can write

$$C = C e^{j\theta_c} = A e^{j\theta_A} + B e^{j\theta_B} \tag{2.21}$$

$$= (A \cos\theta_A + j \sin\theta_A) + B(\cos\theta_B + j \sin\theta_B)$$

$$= X + Y j$$

$$= \sqrt{(X^2 + Y^2)}\; e^{j\psi} \tag{2.22}$$

where $\psi = \tan^{-1}(Y/X)$ and the correct quadrant of ψ can be known using Eqs. (2.9a) to (2.9d). Comparing the left and right hand sides, we get $C = \sqrt{(X^2 + Y^2)}$ and $\theta_C = \psi$. There is only one solution in this case.

In the case 2a, we start with $\mathbf{C} = \overset{*}{\mathbf{A}}^{\surd} + \overset{*}{\mathbf{B}}^{\surd}$ where the unknowns are A and B. We multiply both sides of this equation by $e^{-j\theta_B}$ after expressing them in the polar form. Thus we will obtain

$$C e^{j(\theta_c - \theta_B)} = A e^{j(\theta_A - \theta_B)} + B \tag{2.23}$$

Writing this equation in terms of the rectangular coordinates we get

$$C \cos(\theta_c - \theta_B) + j \sin(\theta_c - \theta_B) = A \cos(\theta_A - \theta_B) + Aj \sin(\theta_A - \theta_B) + B \tag{2.24}$$

Equating the real and imaginary parts on both sides we will have

$$C \cos(\theta_c - \theta_B) = A \cos(\theta_A - \theta_B) + B, \text{ and} \tag{2.25}$$

$$C \sin(\theta_c - \theta_B) = A \sin(\theta_A - \theta_B) \tag{2.26}$$

Therefore, A can be obtained from Eq. (2.26) as

$$A = \frac{C \sin(\theta_c - \theta_B)}{\sin(\theta_A - \theta_B)} \tag{2.27}$$

We can carry out a similar process by multiplying Eq. (2.21) by $e^{-j\theta_A}$ and obtain the value of B as

$$B = \frac{C \sin(\theta_c - \theta_A)}{\sin(\theta_B - \theta_A)} \tag{2.28}$$

In this case also there would be only one solution. In the case

2b, the unknowns are θ_A and B. We can obtain the solution for this case by following a procedure similar to case 2a. The solutions for unknowns are:

$$\theta_A = \theta_B + \sin^{-1}\left\{ \frac{C \sin(\theta_c - \theta_B)}{A} \right\} \tag{2.29}$$

$$B = C \cos(\theta_c - \theta_B) - A \cos(\theta_A - \theta_B) \tag{2.30}$$

The arc Sine term is double valued, so we will get two solutions θ_A, and θ_A. When we substitute these values in Eq. (2.30), we will obtain two values of B. The solution sets will be θ_A , B and $\theta_{A'}$, B'.

The unknowns in case 2c are θ_A and θ_B. The procedure to obtain these two angles in terms of known parameters is again same as before. We start with the equation

$$C\, e^{j\theta_C} = A\, e^{j\theta_A} + B\, e^{j\theta_B} \tag{2.21}$$

Multiplying both sides by $e^{-j\theta_B}$ and then writing the real and imaginary parts separately one obtains

$$A \cos(\theta_A - \theta_C) = C - B \cos(\theta_B - \theta_C) \tag{2.31}$$

$$A \sin(\theta_A - \theta_C) = - B \sin(\theta_B - \theta_C) \tag{2.32}$$

Squaring and adding these two equations we get

$$A^2 = C^2 + B^2 - 2\, B\, C \cos(\theta_B - \theta_C) \tag{2.33}$$

The only unknown in this equation is θ_B . Thus θ_B can be obtained by

$$\theta_B = \theta_C \mp \cos^{-1}\left[\frac{C^2 + B^2 - A^2}{2\, C\, B} \right] \tag{2.34}$$

Similarly one can obtain an equation for θ_A as

$$\theta_A = \theta_B \pm \cos^{-1}\left[\frac{C^2 + A^2 - B^2}{2\, C\, A} \right] \tag{2.35}$$

Since arc cosine function is double valued, the correct pair of θ_A and θ_B can be checked by substituting these pairs in any one of Eqs. (2.31) and (2.32).

Example 2.2

Case 1 : Given the vectors **A** = 8.247 $\angle 255.969$ and **B** = 6.404 $\angle 128.659$. Find **C**.

Solution

In case 1, the two unknowns are C and θ_C. We begin the solution by separating the real and imaginary parts of the equation (2.21)

$$C(\text{Cos}\ \theta_C + j\ \text{Sin}\ \theta_C) = A(\text{Cos}\ \theta_A + j\ \text{Sin}\ \theta_A) + B(\text{Cos}\ \theta_B + j\ \text{Sin}\ \theta_B) \qquad \text{(a)}$$

Equating the real and imaginary terms in Eqn. (a), we obtain two real equations corresponding to the horizontal and vertical components of the vector equation.

$$C\ \text{Cos}\ \theta_C = A\ \text{Cos}\ \theta_A + B\ \text{Cos}\ \theta_B \qquad \text{(b)}$$

$$C\ \text{Sin}\ \theta_C = A\ \text{Sin}\ \theta_A + B\ \text{Sin}\ \theta_B \qquad \text{(c)}$$

By squaring and adding these two equations (b) and (c), θ_C is eliminated and a solution is found for C.

$$C = \sqrt{A^2 + B^2 + 2AB\ \text{Cos}(\theta_B - \theta_A)} \qquad \text{(d)}$$

Therefore

$$C = \sqrt{(8.247)^2+(6.404)^2+2(8.247)(6.404)\ \text{Cos}(128.659 - 255.969)}$$

$$= 6.708$$

The angle θ_C is found from

$$\theta_C = \tan^{-1}\left[\frac{A\ \text{Sin}\theta_A + B\ \text{Sin}\theta_B}{A\ \text{Cos}\theta_A + B\ \text{Cos}\theta_B}\right]$$

$$= \tan^{-1}\left[\frac{8.247\ \mathrm{Sin}(255.969)\ +\ 6.404\ \mathrm{Sin}(128.659)}{8.247\ \mathrm{Cos}(255.969)\ +\ 6.404\ \mathrm{Cos}(128.659)}\right]$$

$$= 206.565^{0}$$

Case 2a : Given C = $6.708\underline{/206.565}$, θ_A = 255.969 and θ_B = 128.659.

Find A and B.

Solution

A can be obtained from Eq. (2.27)

$$A = \frac{C\ \mathrm{Sin}\ (\theta_C - \theta_B)}{\mathrm{Sin}\ (\theta_A - \theta_B)}$$

$$= \frac{6.708\ \mathrm{Sin}\ (206.565 - 128.659)}{\mathrm{Sin}\ (255.969 - 128.659)} = 8.247$$

Similarly, B can be obtained from Eq. (2.28)

$$B = \frac{C\ \mathrm{Sin}\ (\theta_C - \theta_A)}{\mathrm{Sin}\ (\theta_B - \theta_A)}$$

$$= \frac{6.708\ \mathrm{Sin}\ (206.565 - 255.969)}{\mathrm{Sin}\ (128.659 - 255.969)} = 6.387$$

Case 2b : Given C = $6.708\underline{/206.565}$, B = 8.246 and θ_A = 128.659.

Find θ_B and A.

Solution

We can obtain the solution for this case by following a procedure similar to case 2a. The solutions for the unknowns are:

$$\theta_B = \theta_A + \mathrm{Sin}^{-1}\left\{\frac{C\ \mathrm{Sin}\ (\theta_C - \theta_A)}{B}\right\} \qquad \text{(e)}$$

$$= 128.659 + \mathrm{Sin}^{-1}\left\{\frac{6.708\ \mathrm{Sin}(206.565 - 128.659)}{8.246}\right\}$$

The arc Sine term in Eq. (e) is double valued, so we will get two solutions θ_B and $\theta_{B'}$. They are

$$\theta_B = 128.659 + \begin{pmatrix} 52.659 \\ 127.304 \end{pmatrix}$$

Therefore $\theta_B = 181.355^0$ and $\theta_{B'} = 255.963^0$.

The solution for A is given by the following equation which is similar to Eq. (2.30):

$A = C\ \text{Cos}(\theta_C - \theta_A) - B\ \text{Cos}(\theta_B - \theta_A)$

$= 6.708\ \text{Cos}(206.565 - 128.659) - 8.246\ \text{Cos}(181.355 - 128.659)$

$= -3.592$

Hence the two values of A are

$A = -3.592\underline{/128.659} = 3.592\underline{/(128.659+180)} = 3.592\underline{/308.659}$

and $A' = 6.708\ \text{Cos}(206.565 - 128.659) - 8.2460\ \text{Cos}(255.963 - 128.659)$

$= 6.403$

Case 2c : Given $C = 6.708\underline{/206.565}$, $A = 8.246$, $B = 6.403$. Find θ_A and θ_B.

Solution

θ_B is obtained by using Eq. (2.34)

$$\theta_B = \theta_C \mp \text{Cos}^{-1}\left[\frac{C^2 + B^2 - A^2}{2CB}\right]$$

$$= 206.565 \mp \text{Cos}^{-1}\left[\frac{6.708^2 + 6.403^2 - 8.246^2}{2(6.708)(6.403)}\right]$$

$$= 206.565 \mp 77.905$$

Hence $\theta_B = 128.659^0$ and $\theta'_B = 284.47^0$

Similarly one can obtain θ_A from Eq. (2.35) i.e.,

$$\theta_A = \theta_B \pm \text{Cos}^{-1}\left[\frac{C^2 - B^2 + A^2}{2CA}\right]$$

$$= 206.565 \pm \text{Cos}^{-1}\left[\frac{6.708^2 - 6.403^2 + 8.246^2}{2(6.708)(8.246)}\right]$$

$$= 206.565 \pm 49.399$$

Hence $\theta_A = 255.964^0$ and $\theta'_A = 157.166^0$

2.5 The Solutions of Planar Vector Equations Using the Chace Method

Before discussing the actual solutions, let us review some of the formulas of the vector dot and cross products. Here we assume that the position, velocity, and acceleration vectors lie in the (X-Y) plane, so their Z components will be zero. First we will discuss some of the basic rules about the vectors which are:

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative law for addition),

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associative law for addition),

$m\,\mathbf{A} = \mathbf{A}\,m$ (commutative law for multiplication by a scalar m), and

$m(\mathbf{A} + \mathbf{B}) = m\,\mathbf{A} + m\,\mathbf{B}$ (distributive law for multiplication by a scalar).

The cross products do not follow the commutative law. We must follow the following rules:

$i \times j = k$	$j \times i = -k$	$i \times i = 0$
$j \times k = i$	$k \times j = -i$	$j \times j = 0$
$k \times i = j$	$i \times k = -j$	$k \times k = 0$

Let us define two vectors **A** and **B** as

$$\mathbf{A} = i\,A_1 + j\,A_2 + k\,A_3 \qquad (2.36)$$

$$\mathbf{B} = i\,B_1 + j\,B_2 + k\,B_3 \qquad (2.37)$$

then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= i\ (A_2B_3 - A_3B_2) + j\ (A_3B_1 - A_1B_3) + k\ (A_1B_2 - A_2B_1) \qquad (2.38)$$

The dot product of the vectors **A** and **B** is a scalar quantity. If θ is the angle between these two vectors then

$$\mathbf{A}.\mathbf{B} = |\mathbf{A}|\ |\mathbf{B}|\ \text{Cos}\theta. \qquad (2.39)$$

In the case of unit vectors, we have

$$i.i = j.j = k.k = 1, \text{ and}$$

$$i.j = j.k = k.i = 0.$$

Therefore, we can write

$$\mathbf{A}.\mathbf{A} = A_1^2 + A_2^2 + A_3^2\ , \qquad (2.40)$$

$$\mathbf{A}.\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3, \text{ and} \qquad (2.41)$$

$$\mathbf{A}.\ (\mathbf{A} \times \mathbf{B}) = \mathbf{B}.\ (\mathbf{A} \times \mathbf{B}) = 0$$

If **C** is a third vector expressed as

$$\mathbf{C} = C_1\ i + C_2\ j + C_3\ k$$

then

$$\mathbf{A}.\ (\mathbf{B} \times \mathbf{C}) = \mathbf{B}.\ (\mathbf{C} \times \mathbf{A}) = \mathbf{C}.\ (\mathbf{A} \times \mathbf{B}).$$

Any of these equalities will numerically be equal to

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1(B_2C_3 - B_3C_2) + A_2\ (B_3C_1 - C_3B_1) + A_3\ (B_1C_2 - B_2C_1) \qquad (2.42)$$

We have already discussed the unit vectors, and suppose a vector **A** is completely defined or known i.e. we know its magnitude and direction with respect to the X axis then, one way to obtain the unit vector A is to divide **A** by its magnitude A i.e.,

$$A = \frac{\underline{A}}{A} = \frac{1}{A}\left(A_1 i + A_2 j + A_3 k\right)$$

$$= \frac{A_1}{A} i + \frac{A_2}{A} j + \frac{A_3}{A} k$$

If $A_3 = 0$, we would have

$$A = \frac{A_1}{A} i + \frac{A_2}{A} j + 0\ k$$

$$= \text{Cos}\ \theta_A\ i + \text{Cos}\ \beta\ j + \text{Cos}\ 90\ k \qquad (2.43)$$

where $\beta = (270^0 + \theta_A)$ or $(90^0 - \theta_A)$ (2.44)

The angle between the planar vector and the Z axis is always equal to 90^0. Thus the knowledge of θ_A is sufficient to define A. Now we will discuss all the four cases.

Case 1 The known parameters are A, $\hat{A}$, B, and $\hat{B}$. We can obtain **C** as

$$\mathbf{C} = A\ \hat{A} + B\ \hat{B} \qquad (2.45)$$

All the parameters on the right hand side are known. Therefore, **C** can be calculated in this equation.

Example 2.3

Add and subtract the vectors **A** and **B** given by **A** = 6i + 3j, and **B** = 3i + 4j.

Solution

Let **C** = **A** + **B**

= (6i + 3j) + (3i + 4j)

= (9i + 7j)

Similarly, we can write **D** = **A** - **B**

= (6i + 3j) - (3i + 4j)

= 3i - j

Example 2.4

Find the dot and the cross products of the vectors **A** and **B** given in the Example 2.3.

Solution

Let **C** = **A** . **B**

$= (6i + 3j) \cdot (3i + 4j)$

If we use Eq. (2.41) we will have

$C = A \cdot B = (6 \times 3 + 3 \times 4) = 30$

Similarly, if we have a vector

$D = A \times B = (6i + 3j) \times (3i + 4j)$

We can use Eq. (2.38) and we can write

$C = i(3 \times 0 - 0.4) + j(0 \times 3 - 6 \times 0) + k(6 \times 4 - 3 \times 3)$

$= 0i + 0j + 15k$

Example 2.5

Find the sum of vectors **A** and **B** given by

$A = 10 \angle 30^0$, $B = 5 \angle 60^0$

Solution

We can find the solution by using Eqs. (2.44) and (2.45). The expression for unit vectors A and B will be

$A = i \cos 30^0 + j \cos(270 + 30)$

$= i \cos 30 + j \cos 300$

Similarly, we can write

$B = i \cos 60 + j \cos(270 + 60)$

$= i \cos 60 + j \cos 330$

Now we can use Eq. (2.45) and write

$C = 10(i \cos 30 + j \cos 300) + 5(i \cos 60 + j \cos 330)$

$= 11.162\, i + 9.333\, j = 14.55 \angle 39.896^0$

Case 2a In this case we know $\hat{A}$, $\hat{B}$ and **C** but the unknowns are A and B.

The equations for A and B are

$$A = \frac{C.(\hat{B} \times k)}{A.(B \times k)} \quad \text{, and} \tag{2.46}$$

$$B = \frac{C.(\hat{A} \times k)}{B.(A \times k)} \tag{2.47}$$

It will require a lot of computations if we use Eqs. (2.46) and (2.47) as given above. Since the vectors **C**, **A**, and **B** are planar, the coefficients of k for the vectors **C**, **A**, and **B** are zero. We can use Eq. (2.38) to reduce our computations. To do this, let

$$\begin{aligned} \mathbf{C} &= C_1 i + C_2 j + 0\,k, \\ B &= b_1 i + b_2 j + 0\,k, \\ \hat{A} &= a_1 i + a_2 j + 0k, \text{ and} \\ k &= 0i + 0j + 1k, \end{aligned} \tag{2.48}$$

then

$$\mathbf{C} \cdot (\mathbf{B} \times \mathbf{k}) = C_1 (b_2) + C_2 (-b_1) = C_1 b_2 - b_1 C_2, \tag{2.49}$$

$$A \cdot (B \times k) = a_1 b_2 - a_2 b_1, \quad \text{and} \tag{2.50}$$

$$B \cdot (A \times k) = b_1 a_2 - b_2 a_1 \tag{2.51}$$

Using Eqs. (2.48) to (2.50) in Eqs. (2.46) and (2.47) we get

$$A = \frac{C_1 b_2 - b_1 C_2}{a_1 b_2 - a_2 b_1} = \frac{C_1 b_2 - b_1 C_2}{D_1} \tag{2.52}$$

$$B = \frac{C_1 a_2 - a_1 C_2}{b_1 a_2 - b_2 a_1} = \frac{C_1 a_2 - a_1 C_2}{-D_1} \tag{2.53}$$

where $D_1 = a_1 b_2 - a_2 b_1$

Example 2.6

Find the magnitudes of vectors **A** and **B**, given **C** = 6.71 $\angle 206^0$, θ_A = 255.96^0, and $\theta_B = 128.66^0$.

Solution

First we will express C, A and B in terms of their cartesian components which are:

$$C = -6.027\ i - 2.950\ j + 0\ k,$$

$$A = i\ \cos(255.96^0) + j\ \cos(270 + 255.96) + 0\ k$$

$$B = i\ \cos(128.66) + j\ \cos(270 + 128.66) + 0\ k$$

Substituting the components of each of these vectors in Eqs. (2.52) and(2.53) we get

$$A = \frac{\{(-6.027) \times 0.781\} - \{(-2.950)(-0.625)\}}{(-0.242)(0.781) - (-0.970)(-0.625)}$$

$$= \frac{-4.707 - 1.844}{-0.189 - 0.606} = \frac{-6.551}{-0.795} = 8.230$$

$$B = \frac{\{(-6.027) \times 0.970\} - \{(-2.950)(-0.242)\}}{0.795}$$

$$= \frac{5.846 - 0.714}{0.795} = 6.458$$

Note that the denominators in Eqs. (2.52) and (2.53) are of equal magnitude but opposite in sign; this results in reduced computations. The solution is shown in Fig. 2.14.

For case 2b the unknowns are $\hat{A}$ and $\hat{B}$. The expressions for the vectors **A** and **B** can be derived as

$$\mathbf{B} = \{C\ .\ (\mathbf{A} \times \mathbf{k})\}\ (\mathbf{A} \times \mathbf{k}) \pm \left(\sqrt{B^2 - \{C\ .\ (\mathbf{A} \times \mathbf{k})\}^2}\ A\right) \quad (2.54a)$$

$$\mathbf{A} = \mathbf{C} - \mathbf{B} \quad (2.54b)$$

Using Eq. (2.38), (**A** x **k**) can be expressed as

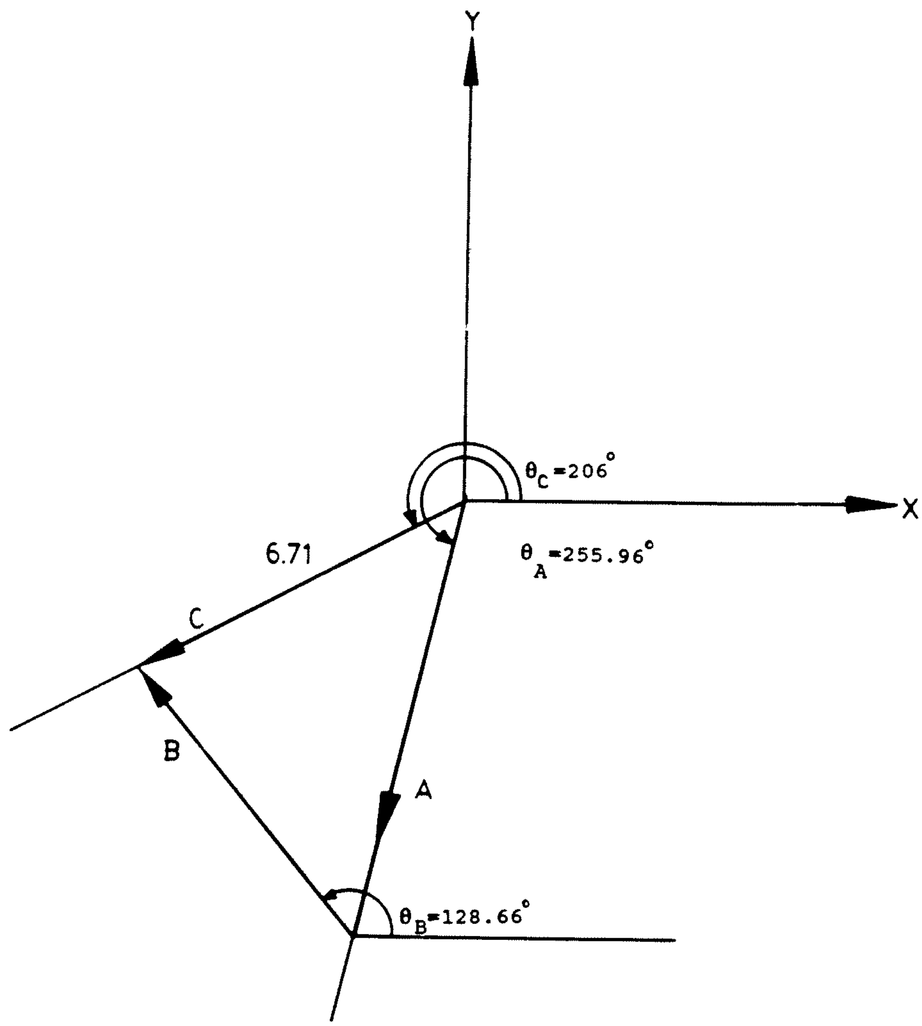

FIG. 2.14 SOLUTION OF CASE 2A USING THE VECTOR METHOD

$$(\mathbf{A} \times \mathbf{k}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i(a_2) + j(-a_1) + k\ 0 \qquad (2.55)$$

We have already seen that

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{k}) = C_1 a_2 - C_2 a_1 = x_1 \text{ (say)} \qquad (2.56)$$

$$\text{and let } \sqrt{B^2 - x_1^2} = y_1 \qquad (2.57)$$

Thus we can express Eq. (2.54a) as

$$\mathbf{B} = x_1(a_2 i - a_1 j + 0\ k) \pm y_1(a_1 i + a_2 j + 0\ k) \qquad (2.58)$$

The steps therefore are:

1. Define the known vectors C and the unit vector **A** as

   $\mathbf{C} = C_1 i + C_2 j + 0\ k$

   $A = a_1\ i + a_2\ j + 0\ k$

2. Calculate $(\mathbf{A} \times \mathbf{k})$ using Eq. (2.55), x_1 using Eq. (2.56) and y_1 using Eq. (2.57)

3. Obtain the two solutions for **B** using Eq. (2.54)

4. Obtain the two solutions for **A** using Eq. (2.54b)

Example 2.7

Find the solutions of Case 2b, given $\mathbf{C} = 6.71\angle 206^0$, B = 8.250 and $\theta_A = 128.66^0$

Solution

We can write

$$\mathbf{C} = 6.71\angle 206 = i\ (-6.027) + j\ (-2.950) + k$$

$$A = i \cos 128.66 + j \cos(270 + 128.66)$$

$$= i\ (-0.625) + j\ (0.781)$$

Using Eqs. (2.56) and (2.57) we have

$$x_1 = (-6.027)(0.781) - (-2.950)(-0.625)$$

$$= -4.707 - 1.844 = -6.551$$

and $$y_1 = \sqrt{(8.250)^2 - (-6.551)^2}$$

$$= \sqrt{68.063 - 42.916}$$

$$= \sqrt{25.147} = 5.015$$

Now we can obtain the vector **B** using Eq. (2.58) as

$$\mathbf{B} = (-6.551)\{i(0.781) - j((-0.625) + 0\ k\}$$

$$\pm 5.015\ \{i(-0.625) + j(0.781) + 0\ k\}$$

$$= \{i(-5.116) - j\ 4.094 + 0\ k\}$$

$$\pm \{i(-3.134) + j\ 3.917 + 0\ k\}$$

Thus we obtain

$$\mathbf{B}_1 = -8.25\ i - 0.177\ j + 0\ k,$$

$$\mathbf{B}_2 = -1.982\ i - 8.01\ j + 0\ k,$$

$$\mathbf{A}_1 = \mathbf{C} - \mathbf{B}_1 = \{i\ (-6.027) + j\ (-2.950) + 0\ k\}$$

$$- \{i(-8.25) - j\ 0.177 + 0\ k\}$$

$$= 2.223\ i - 2.773\ j + 0\ k$$

$$\mathbf{A}_2 = \mathbf{C} - \mathbf{B}_2$$

$$= \{i\ (-6.027) + j\ (-2.950) + 0\ k\}$$

$$- \{i\ (-1.982) + j\ (-8.011) + 0\ k\}$$

$$= -4.045\ i + 5.061\ j + 0\ k$$

The solution obtained is also shown in Fig. 2.15.

In the case 2c, the known parameters are C, A, and B, and the unknowns are A and B. The solution for **A** and **B** are given as

$$\mathbf{A} = \left(\pm \sqrt{A^2 - \left(\frac{A^2 - B^2 + C^2}{2C}\right)^2}\right) (\hat{C} \times k) + \left(\frac{A^2 - B^2 + C^2}{2C}\right)^2 \hat{C} \qquad (2.59)$$

$$\mathbf{B} = \left(\mp \sqrt{A^2 - \left(\frac{A^2 - B^2 + C^2}{2C}\right)^2}\right) (\hat{C} \times k) + \left(\frac{B^2 - A^2 + C^2}{2C}\right)^2 \hat{C} \qquad (2.59)$$

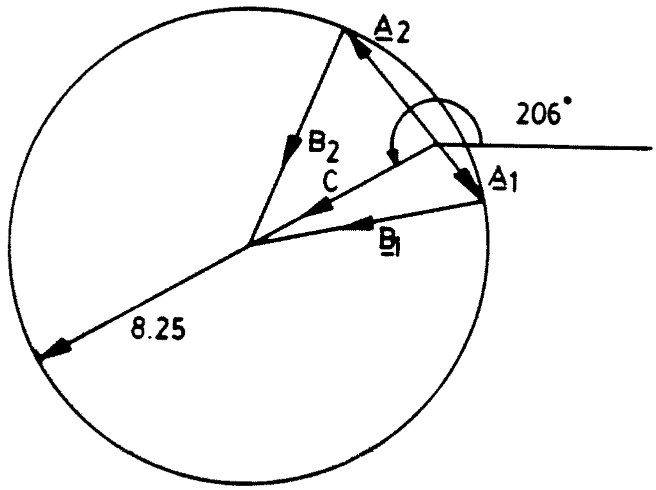

FIG. 2.15 SOLUTION OF CASE 2B USING THE VECTOR METHOD

To obtain these vectors in the simpler form for computation purposes, let

$$\frac{A^2 - B^2 + C^2}{2C} = x_2$$
$$\sqrt{A^2 - x_2^2} = y_2, \tag{2.62}$$
$$\left[\frac{B^2 - A^2 + C^2}{2C}\right] = z_2$$

and we can write

$$(C \times k) = \begin{vmatrix} i & j & k \\ c_1 & c_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i\,c_2 + j\,(-c_1) + 0\,k$$
$$= c_2 i - c_1 j + 0\,k \tag{2.65}$$

Using these, we can rewrite Eqs. (2.59) and (2.60) as

$$\mathbf{A} = \pm\, y_2(c_2 i - c_1 j + 0\,k) + x_2(c_1 i + c_2 j + 0\,k) \tag{2.66}$$
$$\mathbf{B} = \mp\, y_2(c_2 i - c_1 j + 0\,k) + z_2(c_1 i + c_2 j + 0\,k) \tag{2.67}$$

Example 2.8

Find the vectors **A** and **B** if C = $6.71\underline{/206.5}$, A = 8.2, and B = 6.4.

Solution

We will compute x_2, y_2, and z_2 as given in Eqs. (2.61) to (2.63). Thus we will have

$$x_2 = \frac{8.2^2 - 6.4^2 + 6.71^2}{2 \times 6.71} = 5.313$$

$$y_2 = \sqrt{8.2^2 - 5.313^2} = 6.246, \text{ and}$$

$$z_2 = \frac{6.4^2 - 8.2^2 + 6.71^2}{2 \times 6.71} = 1.396$$

Using Eqs. (2.66) and (2.67) we will have

$$\mathbf{A} = \pm\, 6.246(-0.448\ i + 0.894\ j + 0\ k) + 5.313(-0.894 i - 0.448\ j + 0\ k)$$

$$= \pm\, (-2.798\ i + 5.584\ j + 0\ k) + (-4.750\ i - 2.380\ j + 0\ k)$$

$$\mathbf{A}_1 = -7.548\ i + 3.204\ j + 0\ k$$

$$\mathbf{A}_2 = -1.952\ i - 7.964\ j + 0\ k$$

Now, rather than using Eq. (2.67) we will calculate $\mathbf{B}_1$, and $\mathbf{B}_2$ from the known values of $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively. Thus, we can write

$$\mathbf{B}_1 = \mathbf{C} - \mathbf{A}_1 = (-6.001\ i - 3.003\ j) - (-7.548\ i + 3.204\ j)$$

$$= 1.547\ i - 6.207\ j$$

$$\mathbf{B}_2 = \mathbf{C} - \mathbf{A}_2 = (-6.001\ i - 3.003\ j) - (-1.952\ i - 7.964\ j)$$

$$= -4.049\ i + 4.961\ j$$

The solution procedure is also shown in Fig.2.16.

Example 2.9

In a four-bar mechanism shown in Fig. 2.17, the following data are given:

$\mathbf{R}_{AD} = 120\underline{/180}$; $\mathbf{R}_{BA} = 30\underline{/60}$; $\mathbf{R}_{CB} = 100$;

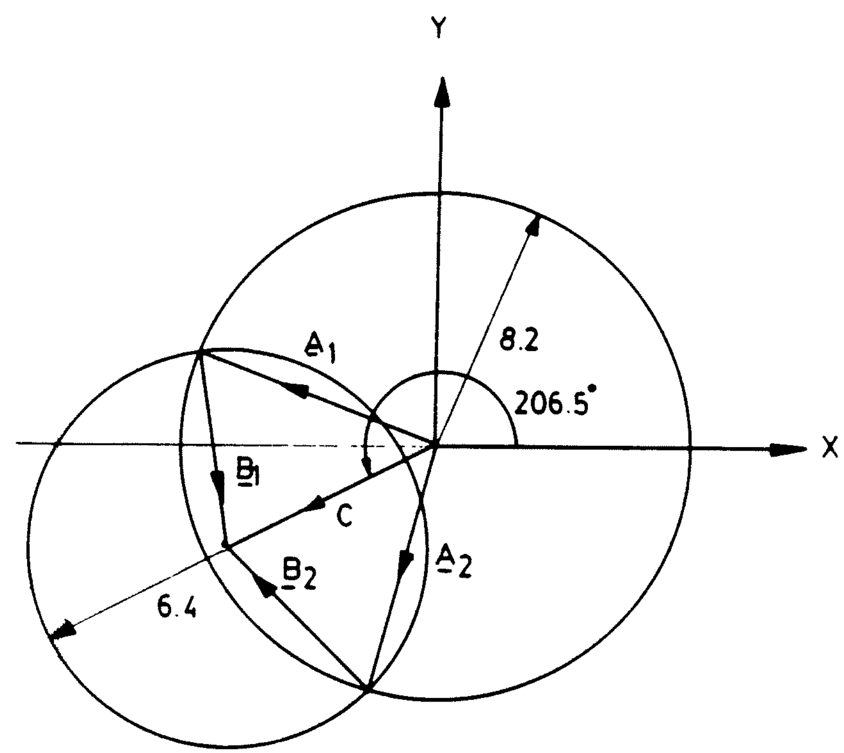

FIG.2.16 CASE 2C USING THE VECTOR METHOD

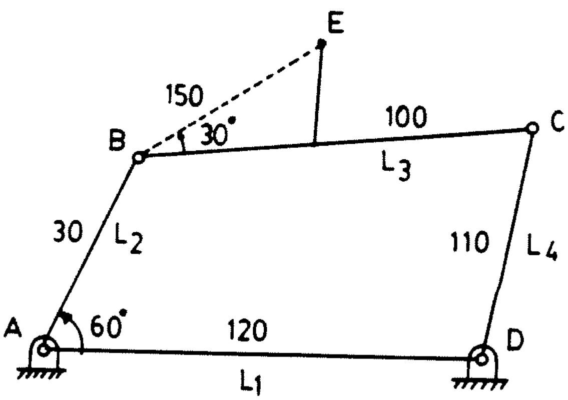

FIG. 2.17 A FOUR BAR MECHANISM

$R_{DC} = 110$; $\angle EBC = 30$;and $R_{EB} = 150$.

Find (a) the orientation of links L_3 and L_4 (both solutions) and (b) the position vector of the coupler point E.

In the first step we obtain $\mathbf{R}_{BD}$ using case 1 by writing

$$\mathbf{R}_{BD} = \underline{R}_{AD} + \underline{R}_{BA} = 120\angle 180 + 30\angle 60$$
$$= 108.167\angle 166.102$$

Next, we obtain both solutions of vectors $\mathbf{R}_{CB}$ and $\mathbf{R}_{DC}$ using case 2c using $\mathbf{R}_{DB} = \mathbf{R}_{CB} + \mathbf{R}_{DC}$

$$108.42\angle -13.8^0 = 100\angle \theta_{CB} + 110\angle \theta_{DC}$$

The two solutions are

$$\mathbf{R}_{C_1B} = 100\angle 49.76$$

$$\mathbf{R}_{C_2B} = 100\angle 282.640$$

$$\mathbf{R}_{DC_1} = 110\angle 291.711$$

$$\mathbf{R}_{DC_2} = 110\angle 40.690$$

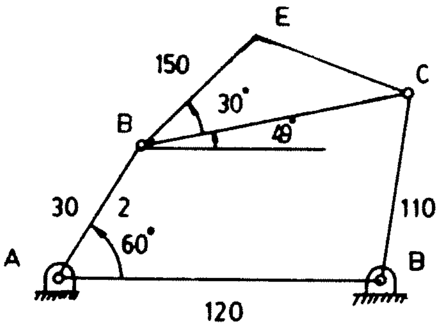

FIG. 2.18(a)

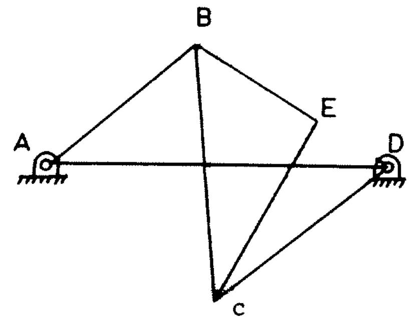

FIG. 2.18(b)

These solutions are schematically shown in Figs. 2.18a and 2.18b. The position vector of the coupler point E can be obtained by writing

$$R_{E_1A} = R_{BA} + R_{E_1B} = 30\angle 60 + 150\angle(49.76+30) = 178.522\angle 76.503$$

$$R_{E_2A} = R_{BA} + R_{E_2B} = 30\angle 60 + 150\angle(282.640+30) = 143.926\angle 324.115$$

2.6 The Geometrical Method

All the four cases discussed are shown in Figs. 2.19 and 2.20. The angles between **A**, **B**, and **C**, in all these figures are γ, δ, and θ respectively. In case 1 and case 2a, there is only one solution, whereas in cases 2b and 2c, there are two solutions. The diagrams in the first set of solutions in all the four cases look similar but important points to note are that the angular relationships between the vectors can be determined by producing the vectors, if necessary, beyond the point of intersection as shown in Fig. 2.19(a). At any of the apexes, they must point away from the apex.

(a) when the vectors at a given apex, point in the direction away from it then by rotating one of the vectors clockwise or

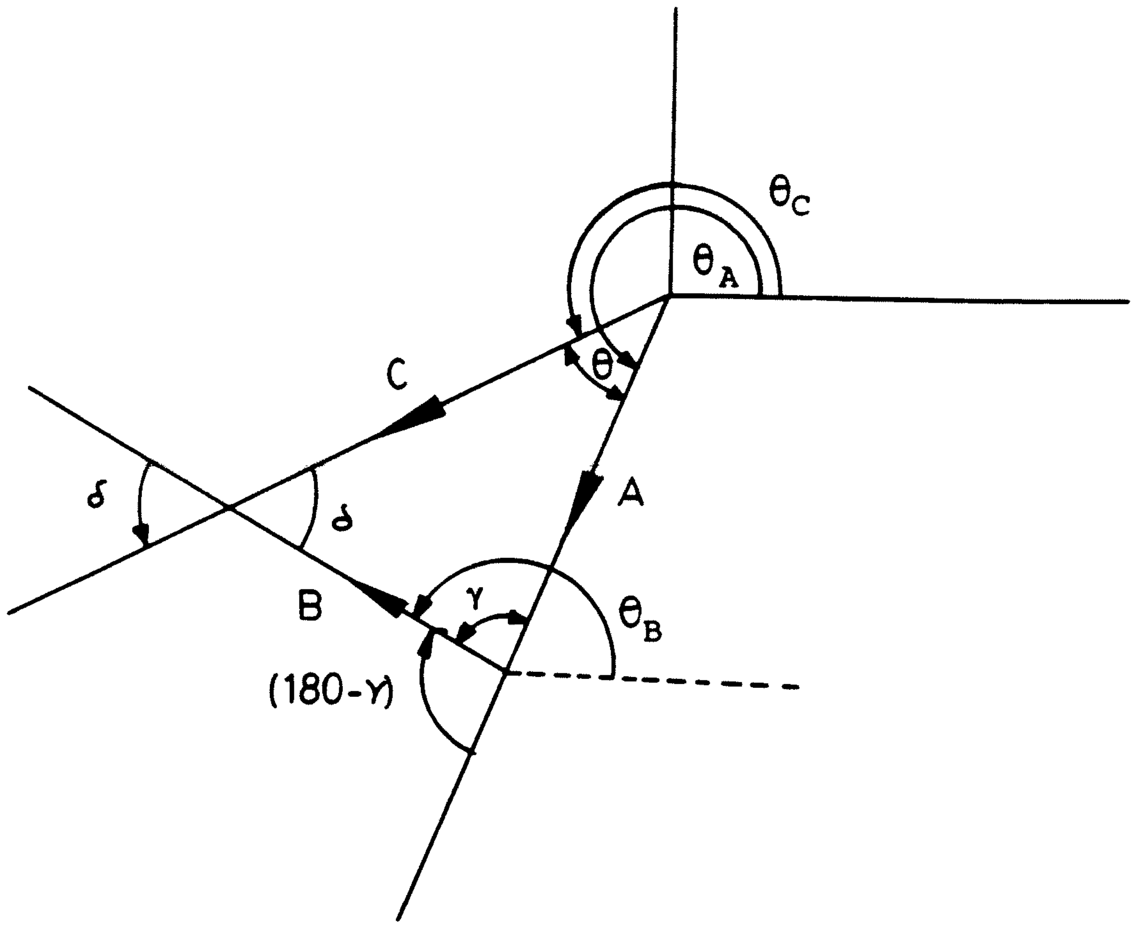

FIG.2.19(a)

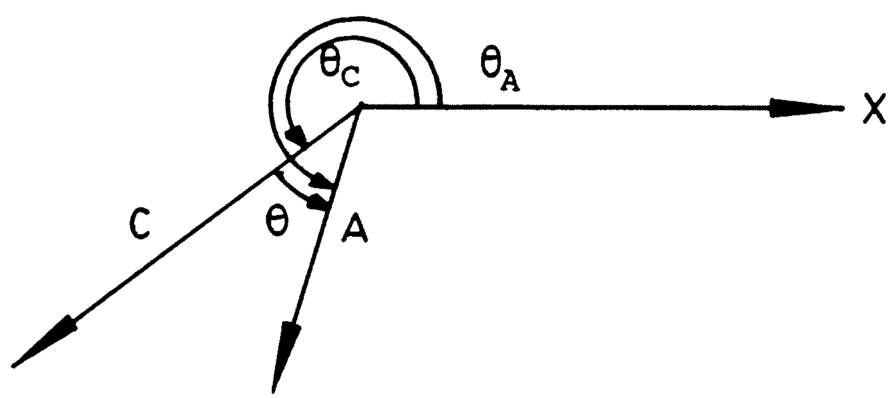

FIG. 2.19(b)

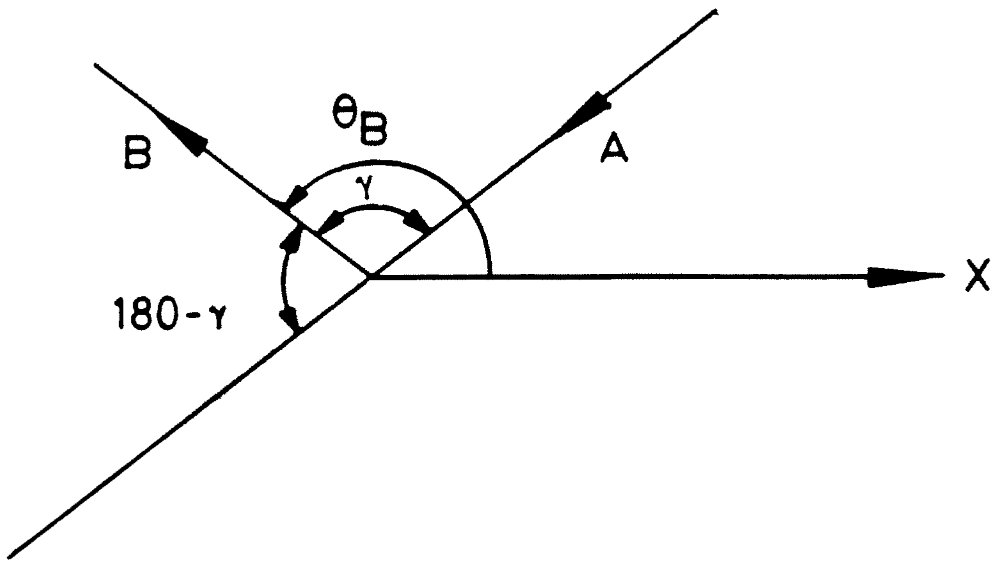

FIG. 2.19(c)

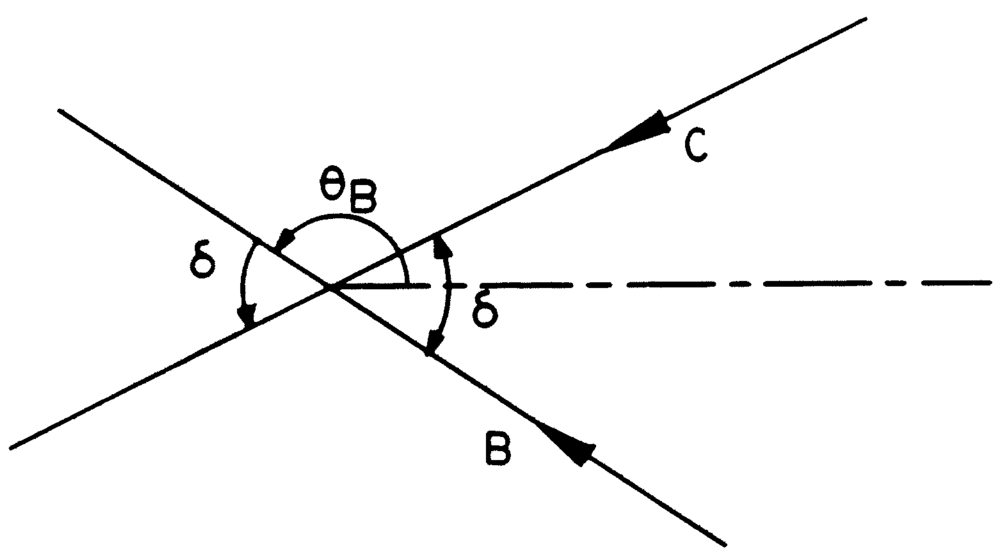

FIG. 2.19(d)

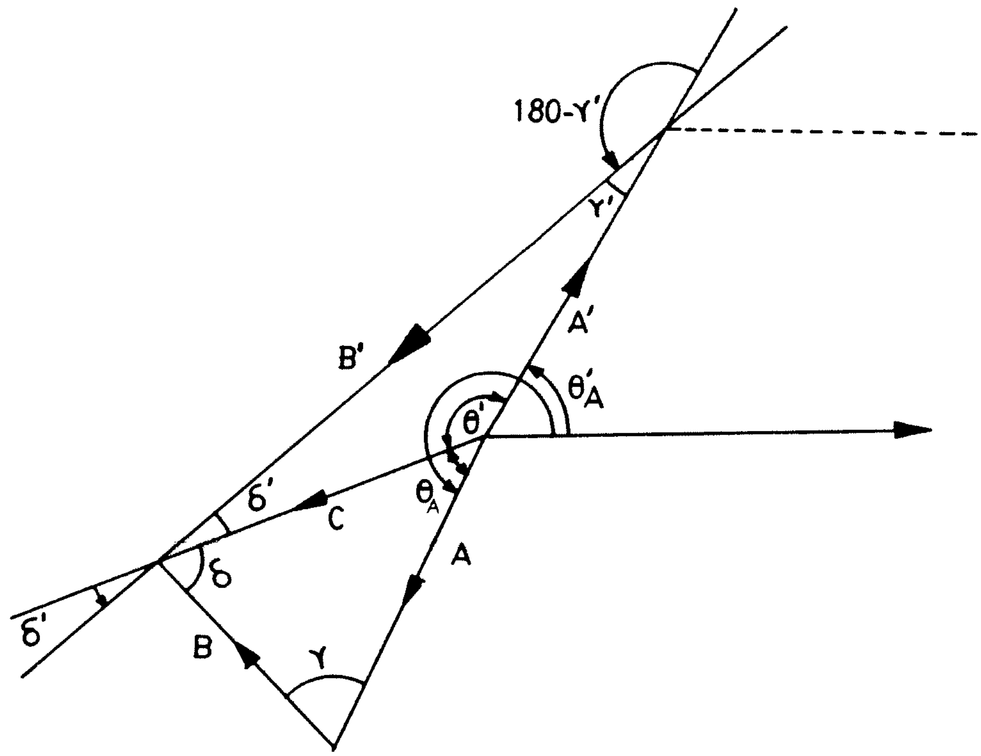

FIG. 2.20(a)

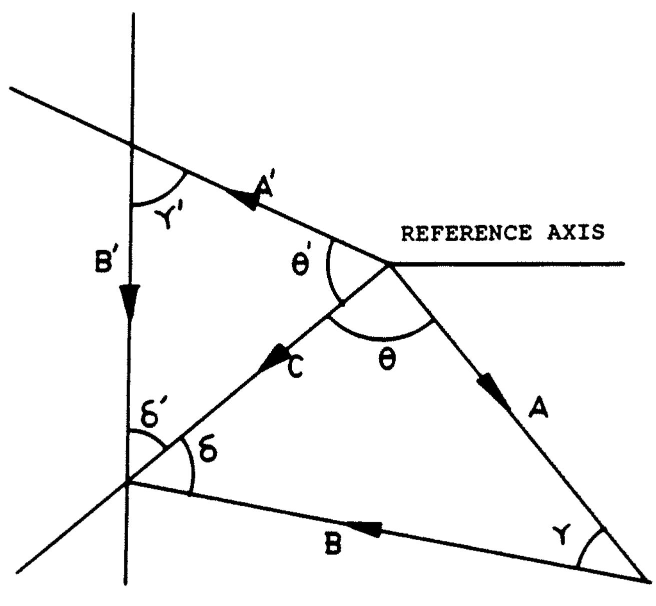

FIG. 2.20(b)

counter-clockwise about the apex, we can make this vector coincide with the other vector. Mathematically, we can write this relationship, for example, in the case of **C** and **A** as (here both vectors are pointing away from the apex c)

$$\theta_A - \theta = \theta_C \qquad (2.68)$$

The -ve sign indicates that **C** can be rotated by θ degrees in counter-clockwise direction to make it coincident with **A**. On the other hand, at the apex where **A** and **B** intersect, one of the vectors is pointing towards and the other away from the apex. Therefore, the angular relationship between the two can be written as

$$\theta_A - (180^0 - \gamma) = \theta_B \qquad (2.69)$$

Here the vector **A** (the vector which points towards the apex) has to be extended beyond the apex to see the rotation clearly. In case of **B** and **C**, both are pointing towards the apex, so both of these vectors have to be extended first and then rotations can be seen. Clearly here, it is **B** which has to be rotated counter - clockwise. Thus we will have

$$\theta_C = \theta_B + \delta \qquad (2.70)$$

In the second set of solutions, as in case 2b (refer to 2.20a), we will have the following relationships:

$$\theta_{A'} = \theta_A - 180 \qquad (2.71)$$

$$\theta_{B'} = \theta_{A'} + (180 - \gamma') \qquad (2.72)$$

$$\theta_{B'} = \theta_C + \delta' \qquad (2.73)$$

Using the principles discussed above the set of solutions for case 2c as shown in Fig. 2.20b,will be

$$\theta_{A'} = \theta_C - \theta \qquad (2.74)$$

$$\theta_{B'} = \theta_{A'} + (180 - \gamma') \qquad (2.75)$$

$$\theta_{B'} = \theta_C + \delta' \tag{2.76}$$

Besides all these relationships, one always has to remember that

$$\theta + \gamma + \delta = 180 \quad \text{and} \tag{2.77a}$$

$$\theta' + \gamma' + \delta' = 180. \tag{2.77b}$$

Therefore, $|\theta|$, $|\delta|$, and $|\gamma|$ should be individually $\leq 180^0$. This becomes important when we use the relationships

$$\frac{A}{\text{Sin } \delta} = \frac{B}{\text{Sin } \theta} = \frac{C}{\text{Sin } \gamma} \tag{2.79}$$

and

$$\text{Cos } \gamma = \frac{A^2 + B^2 - C^2}{2\ A\ B} \tag{2.80}$$

While using Eq. (2.79), sine of any of the angles will be positive quantity, so there will be two solutions corresponding to θ and $(180 - \theta)$. Then one has to select the correct angle using Eq. (2.77). Equations similar to Eq. (2.80) do not pose such ambiguities because Cos γ will be a positive quantity, so the solutions will be in the first and fourth quadrants. The fourth quadrant solution can not be accepted because $|\gamma|$ has to be less than 180^0. Before we finish the discussion let us summarize case-by-case the methods of solutions.

Case 1

Here A, θ_A, B, and θ_B are given and we have to find C and θ_C. First, we should make a neat sketch and determine γ using an equation similar to Eq. (2.69). If θ_B is greater than θ_A then the equation will be

$$\theta_B - (180 - \gamma) = \theta_A$$

otherwise it will be Eq. (2.69). It is the vector which has greater angle, is rotated in clockwise direction which results in subtracting $(180 - \gamma)$ from its angle. In the next step, one can calculate C using Eq. (2.80), and then use the second of the equalities in Eq. (2.79) to calculate θ. Finally, θ_C can be calculated by using Eq. (2.68).

Case 2a

Here we know C, θ_C , θ_B , and θ_A . We have to find A and B. The unknowns can be calculated quite easily in this case. The angles θ, γ, and δ can be calculated using Eqs. (2.68), (2.69) and (2.70). In the next step, one can calculate the magnitudes using Eq. (2.79).

Case 2b

Here, we are given C, θ_C , θ_A , and B. The vector whose angle is given, we will call it **A**. There are two sets of solutions in this case. An important point to remember is that we are not only given θ_A but also $\theta_{A'} = \theta_A - 180$. We have to find solutions for each of these two cases.

In the first step, we calculate θ using Eq. (2.68). After this γ and δ can be determined using Eq.(2.79), and θ_B can be calculated from either Eq. (2.68) or Eq. (2.69). In the second set of solutions, a similar procedure can be adopted.

Case 2c

Here, we are given C, θ_C , A, and B. We have to find out θ_A and θ_B. We have to remember that in the second set of solution, $\theta_{A'} = \theta_C - \theta$. i.e., vectors **A** and **A′** are at $\pm\, \theta^0$ from the vector **C**.

In this case all the magnitudes of the triangle are known. Therefore, one can calculate each of the angles θ, γ, and δ using Eq. (2.80) in the first step and in the second step, one can calculate θ_A and θ_B from Eqs. (2.68) and (2.69).

Example 2.10 (Case 1)

Find the resultant of the vectors **A** = $10\angle 30$ and **B** = $20\angle 120$

Solution

In Fig. 2.21, we have $\theta_B = \theta_A + (180 - \gamma)$. Substituting for θ_B and θ_A we get $\gamma = 90^0$. Now we can use Eq. (2.80) i.e.,

$$C = \sqrt{10^2 + 20^2 - 2 \times 10 \times 20 \text{ Cos } 90}, \qquad \text{(a)}$$

and Eq. (2.79) as

$$\frac{10}{\text{Sin } \delta} = \frac{20}{\text{Sin } \theta} = \frac{C}{\text{Sin } 90} \qquad \text{(b)}$$

The value for C using Eq. (a) is 22.361. Now we can use the second equality in the Eq. (b) as

$$\theta = \text{Sin}^{-1}\left(\frac{20 \text{ Sin } 90}{22.361}\right) = 63.433^0$$

$$\delta = \text{Sin}^{-1}\left(\frac{10 \text{ Sin } 90}{22.361}\right) = 26.560^0$$

An important point to note here is that the second solution for $\theta = 180 - 63.433 = 116.567$ is ruled out because $(\alpha + \gamma + \delta)$ has to be equal to 180^0. Since we know that γ is 90^0, therefore this value of θ is not acceptable. Similar reasoning is applicable in the case of δ. θ_C can be calculated using either θ or δ. For example,

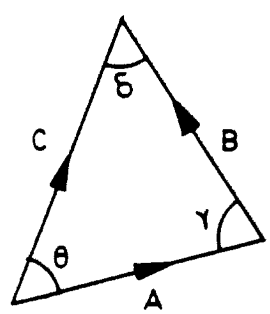

FIG. 2.21

$\theta_C = \theta_A + \theta = 30 + 63.433 = 93.433^0$

$\theta_C = \theta_B - \delta = 93.44^0$

Example 2.11 (Case 2a)

Find the magnitudes of vectors A and B given $\theta_A = 240$, $\theta_B = 70^0$, and $C = 60\angle 120$.

Solution

Referring to the Fig. 2.22, we can write

$\theta_A = \theta_B + (180 - \gamma)$ or

$\gamma = \theta_B - (\theta_A - 180) = \theta_B - \theta_A + 180$

$= 70 - (240 - 180) = 10$, and

$\theta = \theta_A - \theta_C = 240 - 120 = 120$

Therefore, $\delta = 180 - (\gamma + \theta) = 180 - (10 + 120) = 50$

Now we can use Eq. 2.79 as

$$\frac{A}{\text{Sin } 50} = \frac{B}{\text{Sin } 120} = \frac{60}{\text{Sin } 10} \quad \text{(a)}$$

Thus we have

$$A = \frac{\text{Sin } 50}{\text{Sin } 10} \times 60 = 264.688, \text{ and}$$

$$B = \frac{\text{Sin } 120}{\text{Sin } 10} \cdot 60 = 299.234$$

There is only one solution in this case because there were no inverses of any of the trignometric functions involved.

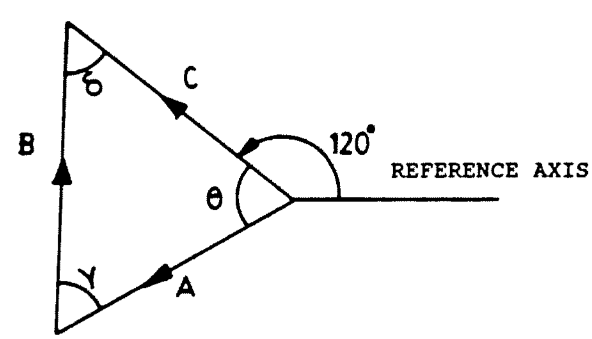

FIG. 2.22

Example 2.12 (Case 2b)

Using C = $120\angle 240^0$, θ_A = 70^0 and B = 170, find A and θ_B .

Solution

As stated earlier in the geometrical method the sketch has to be neatly drawn so that the correct value of θ_A is chosen. In the problem specified, θ_{A_1} = 70^0 will have one solution and θ_{A_2} = (70 + 180) will have another solution. There are two solutions for case 2b. As per the information specified, the two solutions are shown in Fig. 2.23.

In the first solution let us use, θ_2 = 250 - 240 = 10 and

$$\frac{120}{\sin \gamma_2} = \frac{A_2}{\sin \delta_2} = \frac{170}{\sin 10} \qquad (a)$$

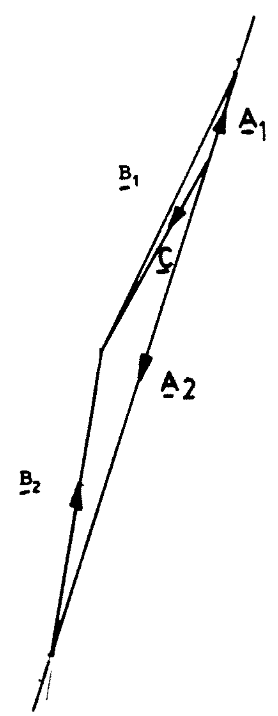

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FIG. 2.23

From this, we get

$$\gamma_2 = \mathrm{Sin}^{-1}\left(\frac{\mathrm{Sin}\ 10 \times 120}{170}\right) = 7.04^0, \text{ and}$$

$$\delta_2 = 180 - (10 + 7.04) = 162.96^0$$

Thus
$$\theta_{B_2} = \theta_{A_2} - (180 - \gamma_2)$$

$$\theta_{B_2} = 7.04 + 70 = 77.04^0$$

Using Eq. (a) we have

$$A_2 = \frac{\mathrm{Sin}(162.96) \times 170}{\mathrm{Sin}\ 10} = 286.882$$

To obtain the second solution we have

$$\theta_1 = \theta_C - \theta_{A_1} = 240 - 70 = 170$$

If we use this in Eq. (a), we will have

$$\gamma_1 = \mathrm{Sin}^{-1}\left(\mathrm{Sin}\ 170 \times \frac{120}{170}\right) = 7.04^0,$$

$$\delta_1 = 180 - (170 + 7.04) = 2.96^0,$$

$$\theta_{B_1} = \theta_{A_1} - (180 - \gamma_1)$$

$$= 70 + (180 - \gamma_1)$$

$$\theta_{B_1} = 250 - \gamma_1 = 250 - 7.04^0 = 242.96 \text{ and}$$

$$A_1 = \frac{\mathrm{Sin}\ 2.96^0 \times 170}{\mathrm{Sin}\ 170} = 50.554$$

Example 2.13 (Case 2c)

Find the two solutions corresponding to $C = 90\angle 210$, $A = 70$, $B = 80$.

Solution

We first find θ using Eq. (2.80) where

$$\theta = \cos^{-1}\left(\frac{90^2 + 70^2 - 80^2}{2 \times 90 \times 70}\right) = \pm 58.411^0$$

Therefore, $\theta_A = \theta_C \pm \theta = 210 \pm 58.40$. Thus, we have $\theta_{A_1} = 268.411^0$ and $\theta_{A_2} = 151.589$. We can calculate γ using Eq. (2.86) i.e.,

$$\gamma = \cos^{-1}\left(\frac{80^2 + 70^2 - 90^2}{2 \times 80 \times 70}\right) = \pm 73.398^0$$

$\gamma = -73.398 = 360 - 73.398 = 286.602$ is not acceptable because it is greater than 180^0. From inspection of Fig. 2.24 we can write $\theta_{B_2} = 151.589 + (180 - 73.398) = 258.191^0$ and $\theta_{B_1} = 268.411 + (180 - 73.398) = 161.809^0$

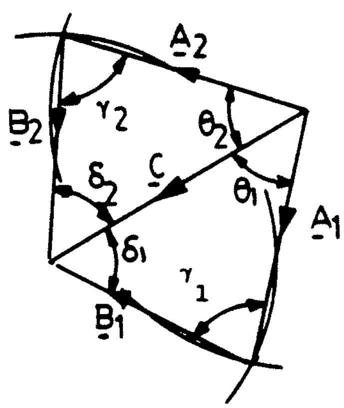

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FIG. 2.24

2.7 The Computer Programs

There are 18 computer programs which will reduce the labor required in performing a large amount of calculations using calculators. The advantages of using such programs are that, (a) the repeated calculations are done accurately, and (b) very quickly. In this way the tedious task of number crunching is left for the computer; the enigineers only have to make decisions about using particular programmes while solving a given problem.

In all of these programs, the input can be given either in the polar coordinates or in the Cartesian coordinates but the output comes out in both system of coordinates. The input and output angles are in degrees. The main menu is shown in Fig. 2.25. The data can be typed in with minimum of one blank space between them (Free Format).

Case 1 can be used for the summation of vectors which can also be done using programs 5 or 15. Programs 7 and 8 are for vector cross and dot products respectively, whereas conversion of rectangular (Cartesian) to polar form and vice versa can be done using programs 8 and 9 respectively. Program 11 is used where we have to solve

$$\mathbf{C}^{\surd\surd} + \mathbf{B}^{\times\surd} = \mathbf{A}^{\times\surd}$$

This type of equation can be converted into

$$\mathbf{C}^{\surd\surd} = \mathbf{A}^{\times\surd} + \mathbf{D}^{\times\surd} \quad .....\text{(case 2a)}$$

where

$$\theta_D = (\theta_B \pm 180)$$

If one wishes, one could solve for **D** using case 2a (program 2).Program 12 is useful when a vector is defined with respect to a set of coordinates which are rotated at an angle θ degrees with