COURSE 7944 ROBOTICS AND AUTOMATION

TERM: SPRING 2010

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OFFICE HOURS FOR CONSULTATIONS: 1:15 P.M. to 3:15 P.M. Wednesdays (EN-3068)
TEXTBOOK - INTRODUCTION TO ROBOTICS , MECHANICS, AND CONTROL BY JOHN J. CRAIG , **THIRD EDITION - 2005**

COURSE CONTENTS

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CH. 2 SPATIAL DESCRIPTIONS AND TRANSFORMATIONS

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CH. 9 LINEAR CONTROL OF MANIPULATORS

AN INTRODUCTION TO ROBOT VISION

	BETTER OF THE TWO OPTIONS	
	OPTION 1	OPTION 2
ASSIGNMENT & LAB	10	10
TEST (JUNE 16, 2010)	30	35
FINAL	60	55
TOTAL	100	100

PLEASE NOTE:

TEST AND FINAL -- **OPEN TEXT BOOK, AND SUPPLIED NOTES (WITHOUT SOLVED PROBLEMS)**

CHAPTER 1

1.1 BACKGROUND

Application of Robots

(a) Automation - {Repeated jobs, Repetitive jobs}

(b) Hazardous or Inaccessible Environment

Hazardous - Nuclear Reactors

- Furnaces operations

- Mines

Inaccessible Environment - Deep sea

- Outer space

What is Robot?

If a mechanical device can be programmed to perform a wide variety of works then it is a robot.

The mechanical devices which perform to produce one class of task are called machines. In general, robotics involves more sophisticated knowledge of kinematics, dynamics and controls than machines.

It is quite difficult to very precisely define these terms. Mechanical Engineers are mainly involved in the statics and dynamic analysis of Robots. Control Theory provides tools for designing and evaluating algorithms to REALIZE OR ACHIEVE DESIRED MOTIONS OR FORCES.


CAUTION
DO NOT MOVE AXIS 1 or AXIS 2 MANUALLY
Must Be Powered by Controller to Move.
PEGASUS
DANGER
THIS MACHINE STARTS AUTOMATICALLY

1.2 SOME TERMINOLOGIES IN MECHANICS AND CONTROL

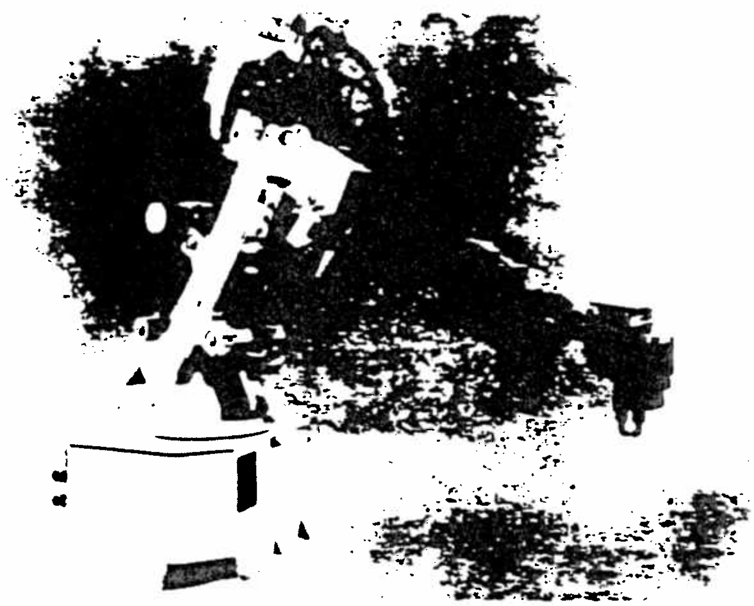


FIGURE 1.4 The Cincinnati Milacron 776 manipulator has six rotational joints and is popular in spot welding applications. Courtesy of Cincinnati Milacron.

(a) POSITION AND ORIENTATION

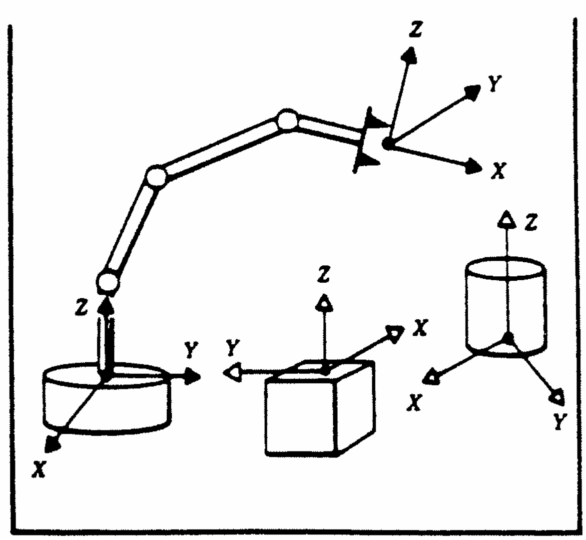


FIGURE 1.5 Coordinate systems or "frames" are attached to the

In the use of robots, we are constantly concerned about the location and orientation of objects in three dimensional space.

This requires a systematic analysis where different coordinate systems, one each, on every link, are set up. There is a separate coordinate system attached to the ground (stationary) and is called the REFERENCE OR GLOBAL COORDINATE SYSTEM OR FRAME. Any coordinate system can be described either with reference to another moving coordinate system or the REFERENCE COORDINATE SYSTEM.

The description of any coordinate system requires the knowledge of

(a) the position vector of its origin

(b) the orientations (direction cosines) of each of its x, y, and z axes with respect to the FRAME OR THE REFERENCE SYSTEM.

(a) POSITION VECTOR OF ITS ORIGIN

It is a vector, in three dimensions joining the origin of the FRAME to the origin of the coordinate system. This vector will have three components along the REFERENCE AXES.

(b) ORIENTATION

The orientation of any of the axes is represented by its direction cosine which is nothing but the cosines of the angle between this axis and the reference axis. The angle has to be measured in a plane containing this axis and the reference axis - it is an unique plane. Suppose α_1, β_1, and γ_1 are the angles the O_1x_1 axis makes respectively with the X, Y, and Z axes, then we can represent the $x_1 - O_1 - y_1 - z_1$ system by a (4x4) matrix as

$$
{}^{o}_{o1}[\ T\ ] = \begin{array}{c}
\begin{array}{cccc} \quad x_1 & \quad y_1 & \quad z_1 & \quad R_{o,o} \end{array} \\
\begin{bmatrix}
\mathrm{Cos}\alpha_1 & \mathrm{Cos}\alpha_2 & \mathrm{Cos}A_3 & (R_{o,o})_x \\
\cos\beta_1 & \mathrm{Cos}\beta_2 & \mathrm{Cos}\beta_3 & (R_{o,o})_y \\
\mathrm{Cos}\gamma_1 & \mathrm{Cos}\gamma_2 & \mathrm{Cos}\gamma_3 & (R_{o,o})_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\tag{1}
$$

The first three columns show the direction cosines of each of the axes of the O_1 system. The fourth column contains the three components of the position vector of the origin, O_1. The FOURTH ROW contains three zeros and a one and the reason for it will be made clear later on.

If we want to grip certain objects in the Cartesian space using the manipulator then we have to have our end effector oriented in the space in a certain way. To do this, all the connecting links have to have certain angular relationships with each other. If we attach one frame to each of these links then we can (a) either write relationships between each of these frames with the Global Frame or (b) relationship between the consecutive frames.

In Chapter 2, it will be shown that given the relationships in either forms (a) and (b), one can obtain the other. In other words, it is sufficient to provide information in one of the two forms.

FORWARD KINEMATICS OF MANIPULATORS

Kinematics is a science of motion where parameters (velocity, displacement, acceleration etc.) are determined without taking into consideration, the forces. All these parameters are time dependent.

The links are considered rigid i.e., they are not stretchable or bendable. The connections between the links are called JOINTS and there is only one degree of freedom at each joint in this course.

These joints are usually instrumented with position sensors which enable us to know the relative position of each link at a given instant of time. If we obtain the continuous time histories of the relative positions, and if we use the data set where the each of the link lengths are given, then we can calculate the KINEMATICS PARAMETERS at any instant of time.

In this course we would study two types of joints:

(a) Revolute Joints.

(b) Sliding Joints.

END EFFECTOR - At the free end of the chain of links which make up the manipulator is the END EFFECTOR. Depending on the intended application of the robot, the end effector may be a gripper, welding torch, electromagnet or any other device.

We generally describe the position of the manipulator by giving the description of the tool frame which is attached to the gripper relative to the base frame which is attached to the nonmoving base of the manipulator.

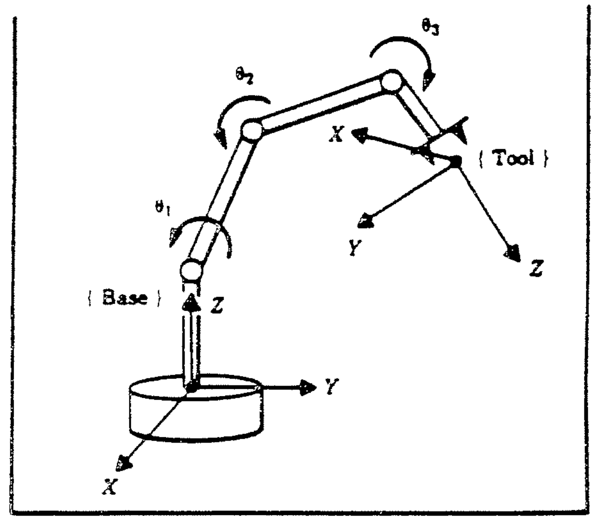


FIGURE 1.6 Kinematic equations describe the tool frame relative to the base frame as a function of the joint variables.

In FORWARD KINEMATICS we compute the position and orientation of the end effector given the joint angles. In other words, we are given the JOINT SPACE DATA as well as Link Lengths etc. and then we compute the orientation and location (x,y,z) of the end effector coordinate system. Here the calculations are done with respect to the base frame. It is the option (a) on page 7.

INVERSE KINEMATICS OF MANIPULATOR

In the INVERSE KINEMATICS problems we have: Given the position and orientation of the end effector, calculate ALL POSSIBLE sets of joint angles.

The Inverse solutions are:

1) Difficult to obtain (Equations are non-linear).

2) Multiple solutions exist.

3) Closed form solutions may not be possible to obtain.

4) Sometimes the solutions may not exist.

VELOCITIES, STATIC FORCES, SINGULARITIES

The vector containing the joint angles

$$\underline{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{2}$$

represents the JOINT SPACE. The vector involving the coordinates of the point or points represents the Cartesian Space.

In Equation

$$^0\{ V \} = {}^0[J]\{ \dot{\theta} \} \tag{4}$$

$$\underline{V} = \begin{Bmatrix} X_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{Bmatrix} \qquad (3)$$

The vector $\{V\}$ is related to the vector $\{\dot{\theta}\}$ by a matrix [J].

If $\{V\}$ represents the various components of the tip velocity in the Cartesian space and $\{\dot{\theta}\}$, the joint velocities of the manipulator then the matrix [J] is called a JACOBIAN. If this matrix [J] is SINGULAR i.e., if its determinant for a given $\theta_i = 0$ then $[J]^{-1}$ will not exist. This particular combination of $\theta_i = 0$ defines the singularity of the manipulator. In these configurations, infinite torques are required to impart given tip velocities. Therefore in planning tip paths, singular configurations are avoided.

DYNAMICS

It is a field of study where forces which cause the motion are also included in the analysis. To give certain type of motion, say a constant speed or Velocity in the

Cartesian space would require a complex or intricate time history of motor torques to each of the arms. These torques are calculated using either Lagrange's Equations of Motion or Newton's Equations of Motion. These are actually called Newton-Euler Dynamic Equations.

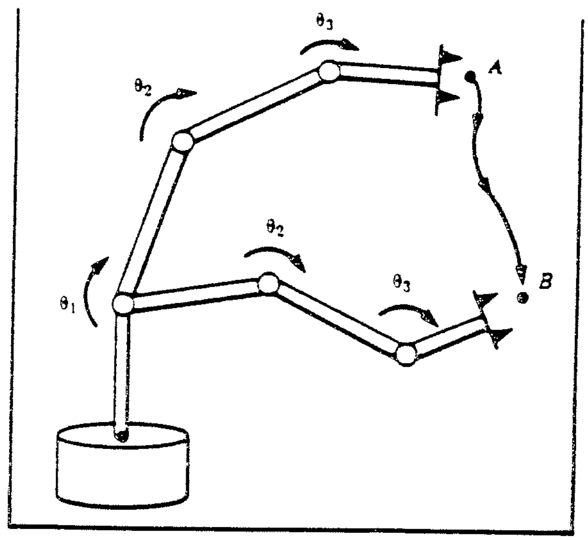


FIGURE 1.10 In order to move the end-effector through space from point A to point B we must compute a trajectory for each joint to follow.

For the end effector to move along certain path with certain velocity etc. requires, at every instant of time, a set of $\{\theta\}$ vectors i.e., a synchronized joint angle variations or coordinated joint angle variations. If this does not take place then the end effector will not traverse along its trajectory.

The path is defined by a series of points along the trajectory which are very close to each other. For each point, there exists <u>at least</u> one $\{\theta\}$ vector. These points are called VIA POINTS and the SMOOTH FUNCTION passing through these points is called a SPLINE FUNCTION.

MANIPULATOR DESIGN AND SENSORS

The parameters or the variables involved in the design are:

1) Size of links.
2) Number of joints.
3) Load capability.
4) Maximum tip speed.
5) Workspace size.
6) Deflection of links.
7) Choice and location of actuators.
8) Transmission system.
9) Position and Force sensors.

LINEAR POSITION CONTROL

Vast majority of manipulators are driven by actuators which supply a force or a torque to cause motion of links.

In the POSITION CONTROL SYSTEM, the errors in the trajectory chosen are minimized. The system computes the torques based on the LINEARIZATION of the Non-Linear Dynamic Equations. Such control methods are used in industries these days.

NON-LINEAR CONTROL

Some attempts have been made to use the Non-Linear Dynamic Control Algorithms in the CONTROLLERS. These algorithms are superior to the linear ones.

FORCE CONTROL

In certain applications, the forces of application must be very closely controlled. For example, if a manipulator is moving in free space then the POSITION CONTROL is important but when it touches a rigid object then a FORCE CONTROL may become important.

In HYBRID CONTROLS, a position control is applied in certain directions and force control

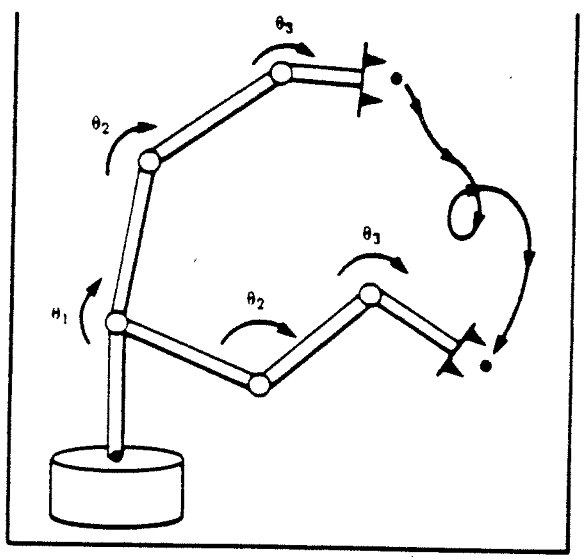


FIGURE 1.12 In order to cause the manipulator to follow the desired trajectory, a position control system must be implemented. Such a system uses feedback from joint sensors to keep the manipulator on course.

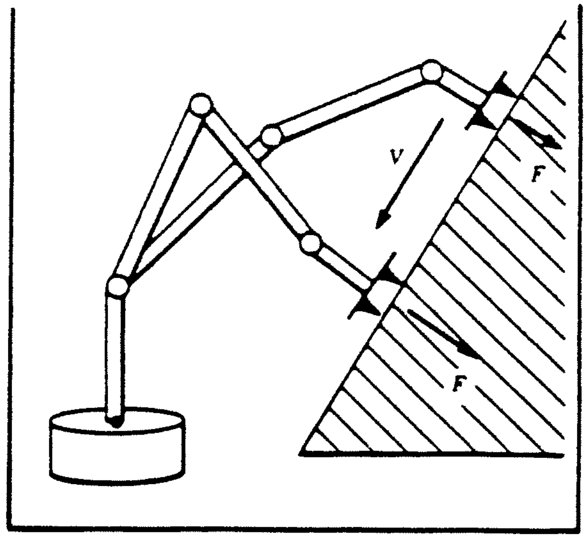


FIGURE 1.13 In order for a manipulator to slide across a surface while applying a constant force, a hybrid position-force control system must be used.

in some others.

NOTATIONS

1) $\underline{A}$, {A} VECTOR, Also (2) Co-ordinate System {A}

2) [B] matrix

3) $[B]^T$ Transpose of matrix [B]

4) $[B]^{-1}$ Inverse of matrix [B]

CHAPTER 2

In the manipulations of Robotic manipulators, control is applied on (a) forces and torques, etc. or (b) kinematic parameters such as displacement, velocity, acceleration, etc. of the end effector or the tool. Naturally, all of these have to be defined with respect to some inertial coordinate system - UNIVERSE COORDINATE SYSTEM which is a Cartesian frame.

2.2 DESCRIPTIONS : POSITIONS, ORIENTATIONS AND FRAMES

Once a coordinate system is established, we can locate any point in the space with a 3x1 POSITION VECTOR. In this course, the vector will be written with leading superscript which identifies the frame. For example a vector $\underline{P}$ or {P} will be expressed as

$$^{A}\underline{P} \text{ or } ^{A}\{P\}$$

In terms of the components we can write

$$^{A}\underline{P} = {}^{A}\{P\} = {}^{A}\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

fig

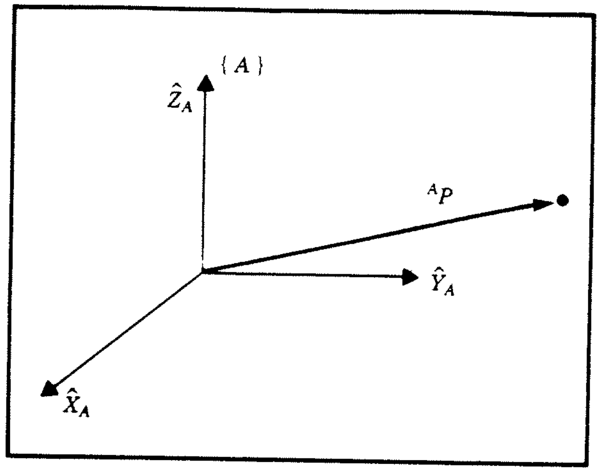


FIGURE 2.1 Vector relative to frame example.

ORIENTATION

The orientations of any of the axes of the system B are shown in the [R] matrices.

$$[R] = \begin{bmatrix} \hat{x}_B.\hat{x}_A & \hat{y}_B.\hat{x}_A & \hat{z}_B.\hat{x}_A \\ \hat{x}_B.\hat{y}_A & \hat{y}_B.\hat{y}_A & \hat{z}_B.\hat{y}_A \\ \hat{x}_B.\hat{z}_A & \hat{y}_B.\hat{z}_A & \hat{z}_B.\hat{z}_A \end{bmatrix} \qquad (2.3)$$

$$^A_B[R] = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \\ & & \end{bmatrix} \qquad (2.4)$$

Please note that all the vectors are unit vectors. Therefore, their magnitudes are equal to 1. The [R] has its elements, the direction cosines. It is an ORTHONORMAL MATRIX and the following relationships apply

$$ {}^{A}_{B}[R] = {}^{B}_{A}[R]^{-1} = {}^{B}_{A}[R]^{T} \tag{2.7}$$

fig

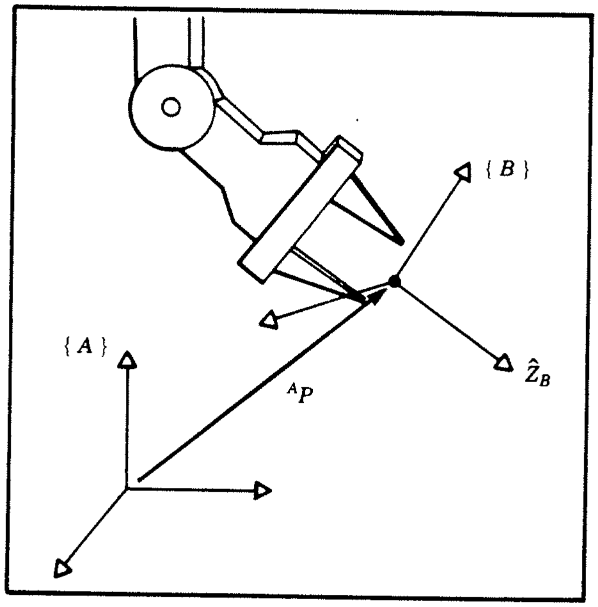


FIGURE 2.2 Locating an object in position and orientation.

DESCRIPTION OF A FRAME

A frame is described with respect to a REFERENCE FRAME by its [R] matrix and the position vector of its origin.

$$ {}^{U}_{A}[T] = [\ {}^{U}_{A}[R] \,\vdots\, {}^{U}P_{A\ ORIGIN}] $$

fig

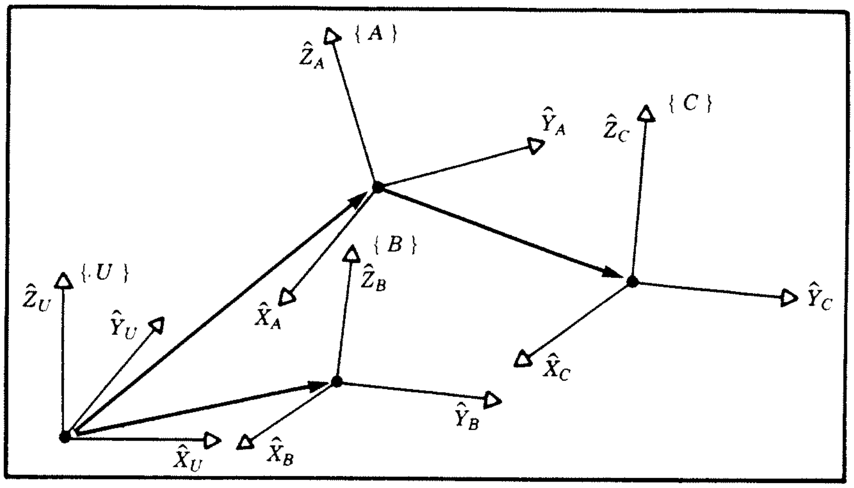


FIGURE 2.3 Example of several frames.

Naturally, ${}^{U}\underline{P}_{A\ ORIGIN}$ will have 3 components along the each of the axes of the reference frame U.

2.3 MAPPINGS: CHANGING DESCRIPTIONS FROM FRAME TO FRAME

A given point will have different position vector in different frames. In Fig 2.4, frames {A} and {B} have same orientation i.e., the corresponding axes are parallel. In this case {B} is displaced from {A} by a vector ${}^{A}\underline{P}_{B\ ORIGIN}$.

fig 2.4

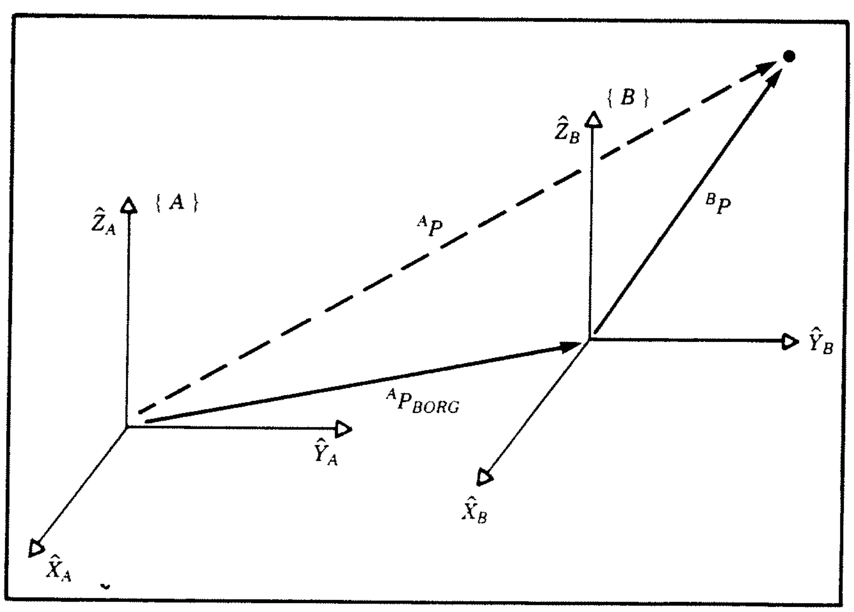


FIGURE 2.4 Translational mapping.

The unit vectors $\hat{X}_B$, $\hat{Y}_B$, and $\hat{Z}_B$ will be equal in magnitude. Therefore one can add vectors in system {A} and {B}.

$$
\begin{aligned}
{}^{A}\underline{P} &= {}^{B}\underline{P} + {}^{A}\underline{P}_{B\ ORIG} \\
{}^{A}\{P\} &= {}^{B}\{P\} + {}^{A}\{P_{B\ ORIG}\}
\end{aligned} \qquad (2.9)
$$

Here we have mapped ${}^{B}\{P\}$ into ${}^{A}\{P\}$.

fig

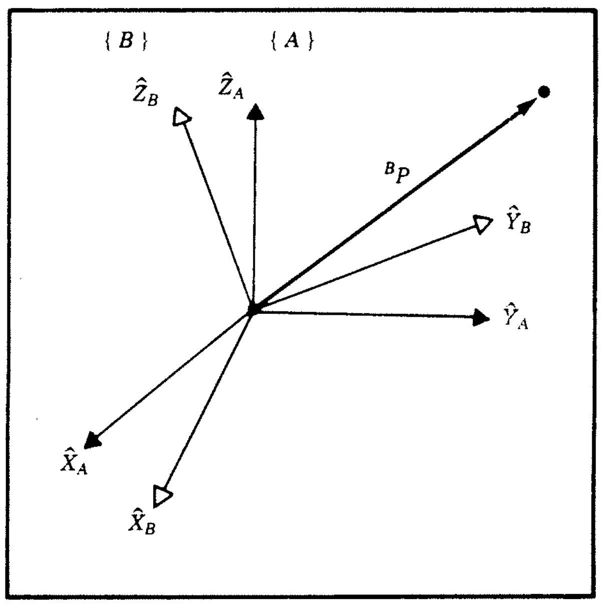


FIGURE 2.5 Rotating the description of a vector.

Here we say that ${}^{A}\{P_{B\ ORIGIN}\}$ defines this mapping. All the information necessary to perform the change in description is contained in this vector. A TRANSLATION VECTOR had sufficient information. On the other hand, as we will see next, when the ORIENTATION of two frames are different then we would need an <u>ADDITIONAL ROTATIONAL MATRIX</u> to completely define the mapping.

MAPPINGS INVOLVING ROTATED FRAMES

Fig 2.5 shows two frames {A} and {B} where the origins of the two systems coincide and there <u>does</u> exist an axis about which the frames {A} can be rotated to make it coincident with the frame {B}. It is also possible to rotate the frame {A} in three successive rotations about $\hat{x}_A$, $\hat{y}_A$, and $\hat{z}_A$ respectively to make it coincident with {B}. We will study these details later on. As shown in fig 2.5, we can write as columns, the direction cosines of $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$ to form a rotation matrix [R] as:

$$ {}^{A}_{B}[R] = \begin{bmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \\ & & \end{bmatrix} \qquad (2.11) $$

$$ = \begin{bmatrix} {}^{B}\hat{x}_A^T \\ {}^{B}\hat{x}_A^T \\ {}^{B}\hat{z}_A^T \end{bmatrix} = {}^{B}_{A}[R]^{-1} = [R]^T $$

Suppose we are given ${}^{B}\{P\}$ and we want to know ${}^{A}\{P\}$. In compact notation, the solution is

$$^{A}\{P\} = {}^{A}_{B}[R]\ {}^{B}\{P\} \qquad (2.13)$$

where

$$^{A}P_{x} = {}^{B}\hat{x}_{A} \cdot {}^{B}P \qquad (2.12)$$

$$^{A}P_{y} = {}^{B}\hat{y}_{A} \cdot {}^{B}P$$

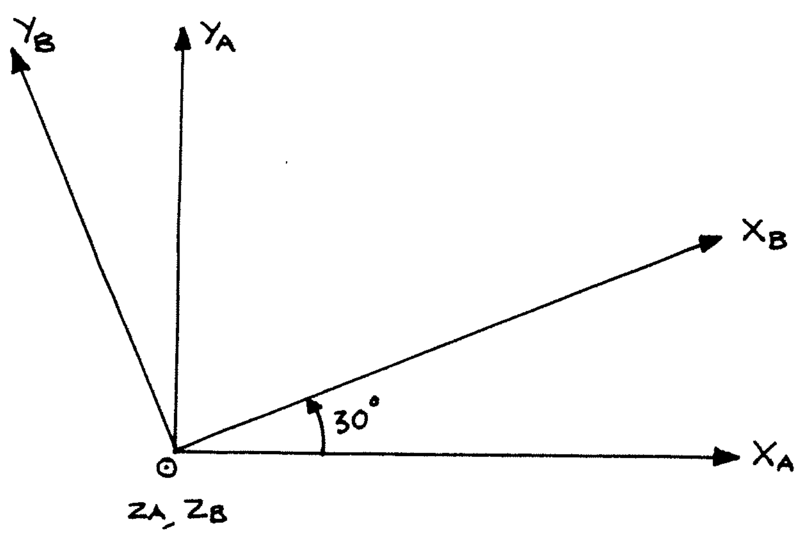
$$^{A}P_{z} = {}^{B}\hat{z}_{A} \cdot {}^{B}P$$

Please note that for this mapping, <u>the origins</u> of the two systems were coincident. We should always remember the inverse relationship

$${}^{A}_{B}[R] = {}^{B}_{A}[R]^{-1} = {}^{B}_{A}[R]^{T} \qquad (2.10)$$

<u>EXAMPLE</u>

fig



$$
{}^{A}_{B}[R] = \begin{bmatrix} \cos 30 & \cos 120 & \cos 90 \\ \cos 300 & \cos 30 & \cos 90 \\ \cos 270 & \cos 270 & \cos 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{A}[R] = \begin{bmatrix} \cos 330 & \cos 60 & \cos 90 \\ \cos 240 & \cos 330 & \cos 90 \\ \cos 270 & \cos 270 & \cos 0 \end{bmatrix}
$$

$$
{}^{B}_{A}[R]^{T} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

MAPPING INVOLVING GENERAL FRAMES

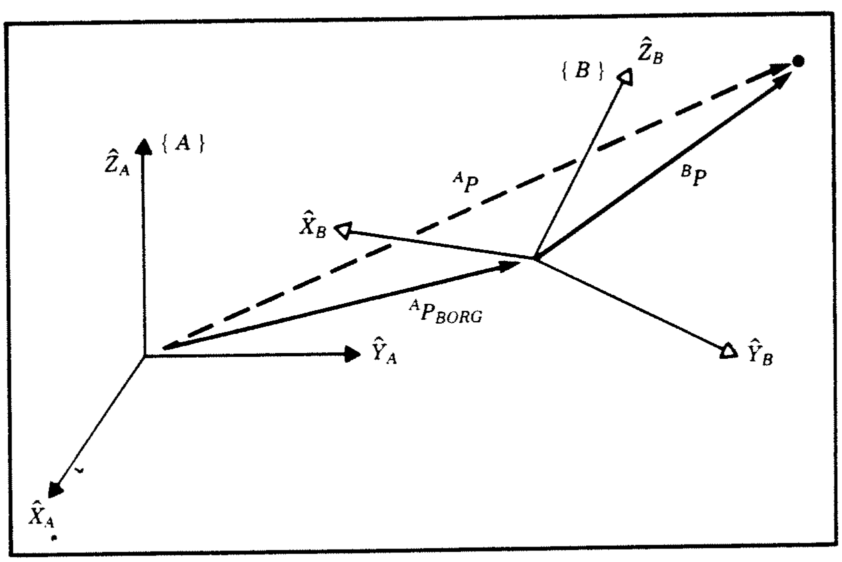


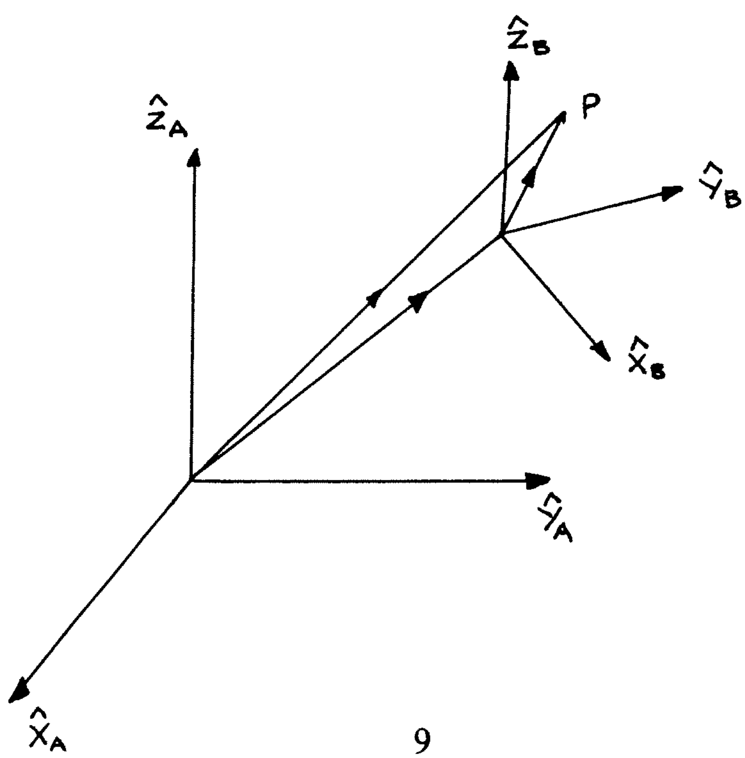
FIGURE 2.7 General transform of a vector.

In fig 2.7, the frame {B} has different orientation as well as the location of its origin is different from that of {A}. We are given the vector ${}^{B}\{P\}$ and we would like to know ${}^{A}\{P\}$. The formula is

$$\underset{4\times 1}{{}^{A}\{P\}} = \underset{4\times 4}{{}^{A}_{B}[T]}\ \underset{4\times 1}{{}^{B}\{P\}} \tag{2.18}$$

$$= \begin{bmatrix} [R] & | & {}^{A}P_{B\ ORIGIN} \\ 3\times 3 & | & 3\times 1 \\ -- & -- & -- \\ [0] & | & 1 \\ 1\times 3 & | & 1\times 1 \end{bmatrix} \begin{Bmatrix} P \\ -- \\ 1 \end{Bmatrix} \tag{2.18a}$$

fig



Suppose for the figure shown we have the following values

$$[R] = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}, \quad {}^{A}\{P_{B\ ORIGIN}\} = \begin{Bmatrix} 1 \\ 3 \\ 4 \end{Bmatrix}_{3\times 1}$$

$${}^{B}\{P\} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \hline 1 \end{bmatrix}$$

$$\begin{array}{cc} 3\times 3 & 3\times 1 \end{array}$$

$${}^{A}_{B}[T] = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 1 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cc} 1\times 3 & 1\times 1 \end{array}$$

${}^{A}_{B}[T]$ is made out of 4 sub matrices:

1) The [R] rotation matrix.

2) $\{P_{B\ ORIGIN}\}$ vector.

3) [0] 1×3 null matrix.

4) [1] 1×1 unit matrix.

Now we can multiply the submatrix in Eq.(2.18 a) and obtain

$$^{A}\{P\} = {}^{A}_{B}[R]\ {}^{B}\{P\} + {}^{A}\{P_{B\ ORIGIN}\} = \begin{Bmatrix} 0.866 \\ 5.23 \\ 7.00 \end{Bmatrix}$$

The first term(vector $\{^{A}X_1\}$) on the right hand side is nothing but projections of $^{B}\{P\}$ along ($\hat{x}_A$ - $\hat{y}_A$ - $\hat{z}_A$) system. Therefore the matrix ${}^{A}_{B}[R]$ projects a vector in {B} parallel to the coordinate axes of {A}.

The second term $^{A}\{P_{B\ ORIGIN}\}$ is already expressed in frame {A}. Now, these two vectors can be added because they are expressed in the same frame.

CONCLUSION

If the vector is expressed in a frame which is (a) oriented differently, and (b) its origin also does not coincide with the reference frame then one has to do two things:

1)   Project the vector parallel to the reference axes by pre-multiplying it with [R] matrix.

2)   Add the vector joining the two origins but these one also expressed in the reference frame.

$$\begin{bmatrix} {}^{A}\{P\} \\ ----- \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}[R] & | & {}^{A}\{P_{BORIG}\} \\ -- & -- & -- \\ [0] & | & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\{P\} \\ ----- \\ 1 \end{bmatrix} \tag{2.19}$$

If we expand the lower submatrices, we would get

$$[1] = \underset{1\times3}{[0]}\ \underset{3\times1}{{}^{B}\{P\}} + \underset{1\times1}{[1]}\underset{1\times1}{[1]}$$

$$= \underset{1\times1}{[0]} + \underset{1\times1}{[1]}$$

$$1 = 1$$

2.4 OPERATORS: TRANSLATION, ROTATIONS, TRANSFORMATIONS

The same mathematical forms which were used for mapping can also be used for translation of points, or rotation of vectors or both.

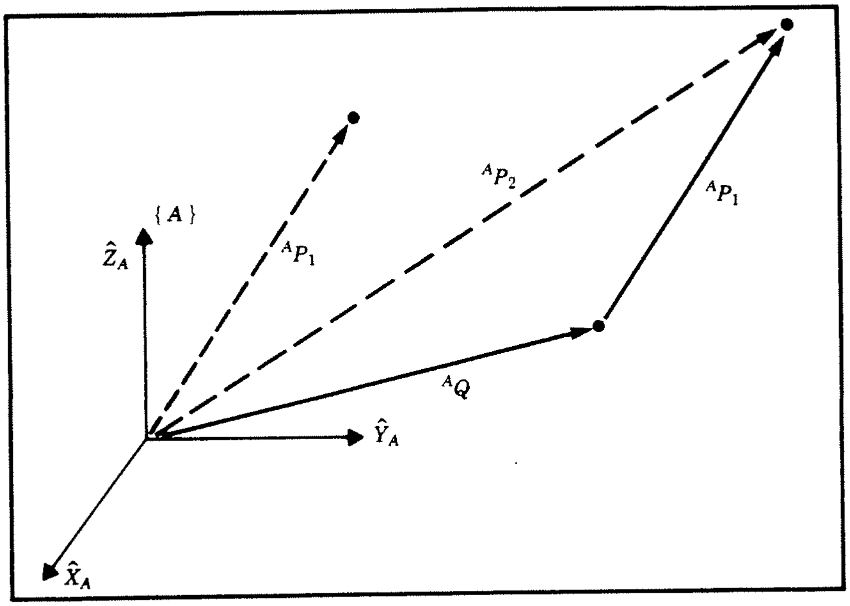


FIGURE 2.9 Translation operator.

A translation moves a point in space a finite distance along a given vector direction. In the fig 2.9, we would like to move a point P_1 along the direction of the vector $^A\{Q\}$. Since there is going to be only translation and no rotation i.e., $\theta = 0°$, we can write the transformation matrix D as

$$D_Q = \left[\begin{array}{ccc|c} 1 & 0 & 0 & Q_x \\ 0 & 1 & 0 & Q_y \\ 0 & 0 & 1 & Q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where the position vector of Q is written in the fourth column. The rotation matrix [R] is a unit matrix. The final position vector $\{P_2\}$ is obtained as

$$\underset{4\times 1}{{}^{A}\{P_2\}} = \underset{4\times 4}{[D_Q]}\ \underset{4\times 1}{{}^{A}\{P_1\}} \tag{2.25}$$

ROTATIONAL OPERATORS

A rotation matrix [R] will rotate a vector by certain angle θ about certain axis in the three dimensional space. While operating on a vector, it pre-multiplies it.

$$^{A}\{P_2\} = [R(\theta)]\ ^{A}\{P_1\} \tag{2.27}$$

when $\theta = 0°$, [R] becomes a unit matrix with 1 along its diagonal and 0 elsewhere. [R] rotated about z axis is written as

$$[R_z(\theta)] = \begin{bmatrix} \mathrm{Cos}\theta & -\mathrm{Sin}\theta & 0 \\ \mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.29}$$

The z axis is represented by a direction

$$[0 \quad 0 \quad 1]^T$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can have a general axis in space called 'k' axis and rotation about this axis whose directions are given by

$$^A\begin{Bmatrix} k_x \\ k_y \\ k_z \end{Bmatrix}$$

The corresponding rotation matrix is written as equation (2.80)

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}. \qquad \textbf{(2.80)}$$

Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and $^A\hat{K} = [k_x \; k_y \; k_z]^T$.

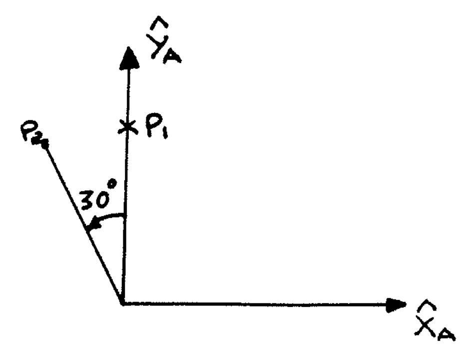
The expressions for $[R_y(\theta)]$ and $[R_x(\theta)]$ are

$$[R_x(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}\theta & -\mathrm{Sin}\theta \\ 0 & \mathrm{Sin}\theta & \mathrm{Cos}\theta \end{bmatrix}$$

$$[R_y(\theta)] = \begin{bmatrix} \mathrm{Cos}\theta & 0 & \mathrm{Sin}\theta \\ 0 & 1 & 0 \\ -\mathrm{Sin}\theta & 0 & \mathrm{Cos}\theta \end{bmatrix}$$

EXAMPLE

GIVEN: $^{A}\{P_1\} = \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$



Find $^{A}\{P_2\}$ which is obtained by rotating $^{A}\{P_1\}$ about $\hat{z}_A$ axis by 30°.

SOLUTION:

$${}^{A}\{P_2\} = [R_z(30°)]\ {}^{A}\{P_1\}$$

$$= \begin{bmatrix} \mathrm{Cos}30 & -\mathrm{Sin}30 & 0 \\ \mathrm{Sin}30 & \mathrm{Cos}30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$${}^{A}\{P_2\} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}$$

TRANSFORMATION OPERATORS

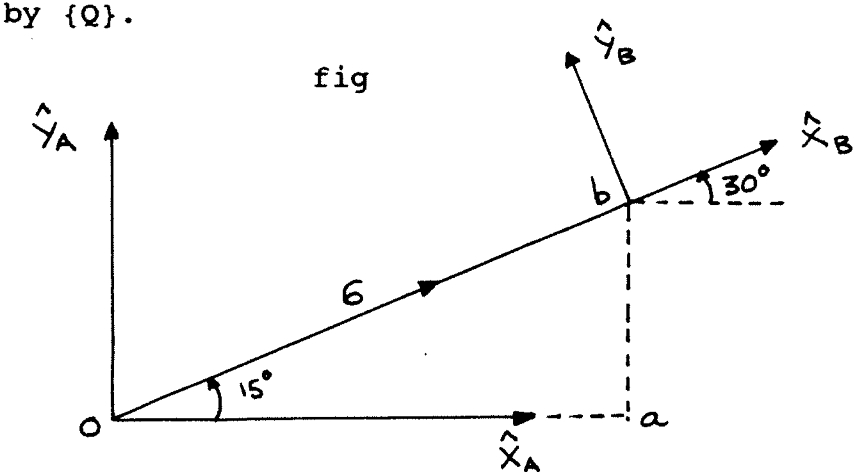
The combined operations of rotation and a translation are done using transformation operator [T] which has [R] and {Q} which are rotational and translational components as its sub-matrices.

$$[T] = \begin{bmatrix} [R] & | & \{Q\} \\ -- & -- & -- \\ [0] & | & 1 \end{bmatrix}$$

AN IMPORTANT THEOREM

The transform [T] which rotates by [R] and translates by {Q} is the same as the transform which describes a rotated frame by [R] and translated by {Q}.

fig



$$[T] = \begin{bmatrix} [R_z(30^\circ) & | & o\ a \\ & | & a\ b \\ & | & 0 \\ -- & -- & -- \\ 0\ \ 0\ \ 0 & | & 1 \end{bmatrix}$$

2.8 MORE ON REPRESENTATION OF ORIENTATION

The rotation matrices are special in that all columns are mutually orthogonal which means their dot products with other are equal to zero. Furthermore, the determinant is always equal to +1. They are called <u>proper orthonormal matrices</u>. Proper orthonormal matrices have determinant = +1, and the non proper have equal to

-1.

Next question is, what or how many independent parameters are there in 3x3 rotation matrices which has 9 elements. The answer comes from Cayley's formula for orthonormal matrices which states that for every rotation matrix [R] there exists a skew-symmetric matrix, [S], such that

$$[R] = [[I_3] - [S]]^{-1} [[I_3] + [S]] \tag{2.56}$$

Where $[I_3]$ is an identity matrix and [S] is given by

$$[S] = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix} \tag{2.57}$$

(Skew Symmetric)

One can see that [S] in Equation (2.57) above has only three independent parameters. If we see the Equation (2.56), we see that $[I_3]$ being an identity matrix is completely known; therefore, the right hand side contains only three unknowns or three independent parameters. It shows that the left hand side of this equation must also contain only three independent parameters.

The other way would be to express [R] as three columns as

$$[R] = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

where each of $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors. Then we should also have the following equations of constraints:

$$
\begin{aligned}
|\hat{x}| &= 1 \\
|\hat{y}| &= 1 \\
|\hat{z}| &= 1 \\
\hat{x} \cdot \hat{y} &= 0 \\
\hat{x} \cdot \hat{z} &= 0 \\
\hat{y} \cdot \hat{z} &= 0
\end{aligned}
\tag{2.59}
$$

To obtain the 9 elements of [R], we should have 9 equations which are subject to 6 equations of constraints. Therefore, there are only 3 independent parameters.

One should also remember that the products of rotation matrices are not commutative i.e.,

$$
{}^{A}_{B}[R]\ {}^{B}_{C}[R] \neq {}^{B}_{C}[R]\ {}^{A}_{B}[R]
$$

EXAMPLE:

$$\text{GIVEN} \quad {}^{A}_{B}[R] = \begin{bmatrix} 0.866 & -0.5 & 0.0 \\ 0.5 & 0.866 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$${}^{B}_{C}[R] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.866 & -0.5 \\ 0.0 & 0.5 & 0.866 \end{bmatrix}$$

$${}^{A}_{B}[R]\ {}^{B}_{C}[R] = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.5 & 0.75 & -0.43 \\ 0.0 & 0.5 & 0.87 \end{bmatrix}$$

$${}^{B}_{C}[R]\ {}^{A}_{B}[R] = \begin{bmatrix} 0.87 & -0.5 & 0.0 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix} \qquad (2.62)$$

In view of the fact that one can represent [R] by three independent parameters, there are representations which require only three independent parameters and are discussed below:

1 X-Y-Z FIXED ANGLES

Here, we are given the Reference Frame {A} and we have to specify the {B}.

fig

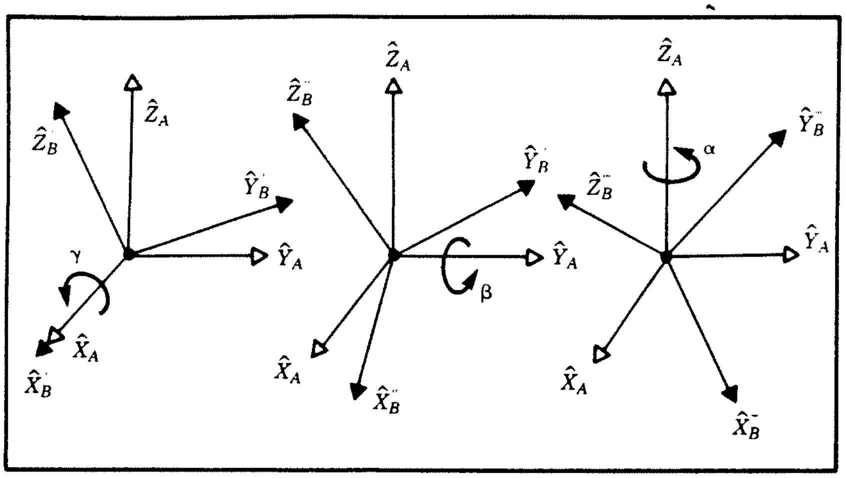


FIGURE 2.17 X-Y-Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

We start with a frame coincident with {A} and rotate this coincident but separate frame about various axes of {A}.

1) Rotate {B} about $\hat{X}_A$ by an angle γ.

2) Then rotate it about $\hat{Y}_A$ by an angle β.

3) Finally, rotate it about $\hat{Z}_A$ by an angle α.

It should be noted here that all the rotations were performed about the fixed or the Reference Axis. Representing the final matrix as

$$^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)]$$

the relationship between the various individual and the final matrix is written as

$$ {}^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)] = [R_z(\alpha)][R_y(\beta)][R_x(\gamma)] $$

In the equation above on the right hand side, the matrices have been pre-multiplied i.e., the rotation about the Y axis was performed after the X axis; so the rotation matrix corresponding to the Y axis rotations are pre-multiplied. It is an IMPORTANT RULE. Now we are in a position to write the complete matrices which are

$$ {}^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)] = \underset{3}{\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{2}{\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}} \underset{1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}} $$

$$
{}^{A}_{B}R_{xyz}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2.64}
$$

$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.65}
$$

If we want to determine α, β, and γ from the matrices given in equation (2.65) then we can use the following formulas in the GIVEN SEQUENCE:

$$
\begin{aligned}
\beta &= \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \\
\alpha &= \mathrm{Atan2}(r_{21}/c\beta,\ r_{11}/c\beta) \\
\gamma &= \mathrm{Atan2}(r_{32}/c\beta,\ r_{33}/c\beta)
\end{aligned} \tag{2.66}
$$

where $\mathrm{Atan2}(y,x) = \tan^{-1}(y/x)$. Here signs of both x and y are used. It is 4 quadrant arc tangent function.

Z - Y - X EULER ANGLES

This involves rotations about {B} of the system B as follows:

1) Start with a frame {B} coincident with {A}, and rotate about $\hat{z}_B$ by an angle as shown in fig 2.18.

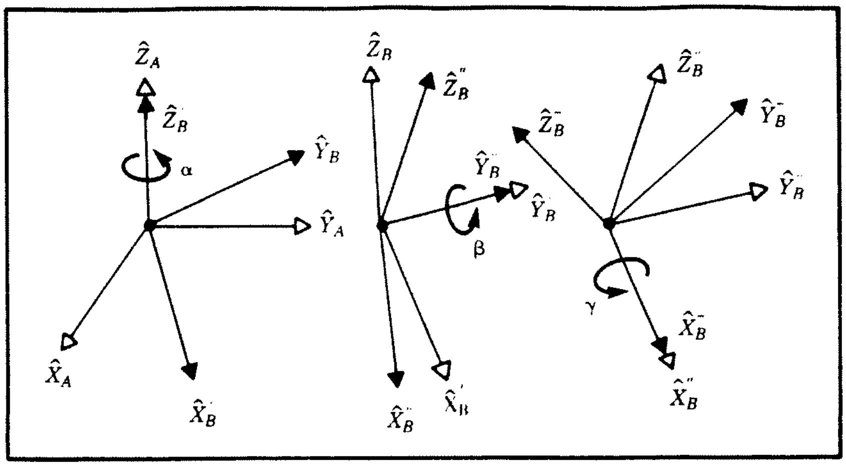


FIGURE 2.18 Z-Y-X Euler angles.

2) Then rotate about $\hat{Y}_B$ by an angle β.

3) Then rotate about $\hat{X}_B$ by an angle γ.

The final orientation matrix in this case will be

$$
{}^A_B[R_{\acute{z}\acute{y}\acute{x}}] = \underset{1}{[R_z(\alpha)]}\ \underset{2}{[R_y(\beta)]}\ \underset{3}{[R_x(\gamma)]}
$$

$$
{}^A_B[R_{\acute{z}\acute{y}\acute{x}}] = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \tag{2.70}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2.71}
$$

Z - Y - Z EULER ANGLES

In this case, the final expression is

$$
{}^{A}_{B}[R_{z'y'z'}] = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} \tag{2.72}
$$

The formulas for extracting α, β, and γ from the matrix on the right hand side of equation (2.72) are

$$
\begin{aligned}
\beta &= \mathrm{Atan2}((r_{31}{}^{2} + r_{32}{}^{2})^{1/2}, r_{33}) \\
\alpha &= \mathrm{Atan2}(r_{23}/s\beta\ ,\ r_{13}/s\beta) \\
\gamma &= \mathrm{Atan2}(r_{32}/s\beta\ ,\ -r_{31}/s\beta)
\end{aligned} \tag{2.74}
$$

EQUIVALENT ANGLE AXIS

Instead of three successive rotations in these three cases, it is also possible to rotate about an axis in space, only once to reach to the final orientation.

Eq(2.80)

$$
R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}. \tag{2.80}
$$

Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^{A}\hat{K} = [k_x\ k_y\ k_z]^T$.

where $c\theta$ = Cos θ, $s\theta$ = Sin θ, $v\theta$ = 1 - Cos θ and ${}^{A}\hat{K} = [K_x\ K_y\ K_z]^T$. If the matrix [R] is given and one wants to find out θ and $\hat{K}$, then one has to use the formulas

$$\theta = \mathrm{ACos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right] \qquad (2.81)$$

$$\hat{k} = \frac{1}{2\mathrm{Sin}\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (2.82)$$

In Equation (2.81), θ should lie between 0 and 180° which is obvious from the Fig 2.19

fig

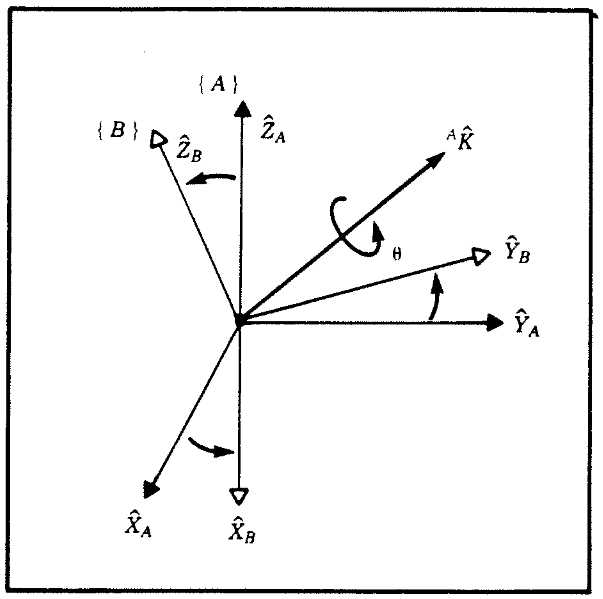


FIGURE 2.19 Equivalent angle-axis representation.

It would amount to a maximum of one complete rotation about the $\hat{K}$ axis.

TRANSFORMATION OF FREE VECTORS

So far we have discussed only the transformation of position vectors. However, there are other kinds of vectors such as velocity, force etc. These are transformed differently using <u>only the rotation matrices.</u>

1) Two vectors are said to be equal if they have (a) same dimensions, (b) same magnitude, and (c) same direction.

2) Two vectors are equivalent in certain capacity if each produces the very same effect in this capacity.

3) Vectors which are not equal may produce equivalent effects.

4) A line vector is one which has dependence on line of action besides having magnitude and direction. Force vector is an example.

5) A free vector is one which may be positioned anywhere in space without loss or change of meaning provided that magnitude and direction are preserved.

An example of this is a Moment Vector. Suppose we have a moment vector in frame B denoted by ${}^{B}\{N\}$. This vector in frame A will be

$$ {}^{A}\{N\} = {}^{A}_{B}[R]\ {}^{B}\{N\} \tag{2.93} $$

Similar relationships can be written about the velocity vector also

$$ {}^{A}\{V\} = {}^{A}_{B}[R]\ {}^{B}\{V\} \tag{2.94} $$

CHAPTER 2

WORKED OUT PROBLEMS

2.1 [15] A vector ${}^{A}P$ is rotated about Z_A by θ (30) degrees and is subsequently rotated about X_A by φ (20) degrees. Give the rotation matrix which accomplishes these rotations in the given order.

GIVEN $\theta = 30^{\circ}$; $\phi = 20^{\circ}$

ONE CAN USE PROGS 12 OR 2

SOLUTION

$$[R] = [ROT(\hat{x}, 20^{\circ})]\,[ROT(\hat{z}, 30^{\circ})]$$

USING PROG 2

THETA VECTOR

$$\{\theta\} = \begin{Bmatrix} 20 \\ 30 \\ 0 \end{Bmatrix}$$

IDENTITY MATRIX (ALL DIAGONALS = 1)

AXIS VECTOR

$$\{I\}_{AXIS} = \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix} \begin{matrix} \hat{x} \\ \hat{z} \\ \hat{x} \end{matrix}$$

ONE CAN USE ANY AXIS 1, 2, 3 AS LONG AS $\theta = 0$

POSITION VECTORS OF ALL AXES.

$$[\,0.0 \quad 0.0 \quad 0.0\,]^T \longrightarrow \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

RESULT

$$\begin{bmatrix} 0.8659 & -0.5 & 0.0 & 0.0 \\ 0.4699 & 0.8136 & -0.34 & 0.0 \\ 0.1711 & 0.2962 & 0.9396 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

4 X 4 MATRIX

2.2 [15] A vector ^{A}P is rotated about Y_A by 30 degrees and is subsequently rotated about X_A by 45 degrees. Give the rotation matrix which accomplishes these rotations in the given order.

USE PROG 2

$$[R] = [Rot(\hat{X}, 45^\circ)][Rot(\hat{Y}, 30^\circ)]$$

$$\{\theta\} = \begin{Bmatrix} 45 \\ 30 \\ 0 \end{Bmatrix} \rightarrow \text{THETA VECTOR}$$

$$\{I\} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} \rightarrow \text{AXIS VECTOR}$$

POSITION VECTORS FOR EACH CO-ORDINATE SYSTEM

$$\begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}, \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}.$$

RESULT

$$[T] = \begin{bmatrix} 0.8659 & 0.0 & 0.5 & 0.0 \\ 0.3537 & 0.7068 & -0.6124 & 0.0 \\ -0.3537 & 0.7073 & 0.6121 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

2.3 [16] A frame $\{B\}$ is located as follows: initially coincident with a frame $\{A\}$, we rotate $\{B\}$ about Z_B by θ (20) degrees and then we rotate the resulting frame about X_B by φ (25) degrees. Give the rotation matrix which will change the description of vectors from ${}^{B}P$ to ${}^{A}P$.

USE $\theta = 20^\circ$ AND $\phi = 25^\circ$.

PART A

$${}^{A}_{B}[R] = [ROT(\hat{z}, 20)]\,[ROT(\hat{x}, 25)]$$

USE PROG 2 OR 12

BY FOLLOWING A PROCEDURE SIMILAR TO EARLIER TWO PROBLEMS THE RESULT IS

$${}^{A}_{B}[T] = \begin{bmatrix} 0.9396 & -0.31 & 0.1446 & 0.0 \\ 0.3421 & 0.8515 & -0.3972 & 0.0 \\ 0.0 & 0.4227 & 0.9062 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

PART B

$${}^{B}_{A}[T] = {}^{A}_{B}[T]^{-1} = \begin{bmatrix} 0.9396 & 0.3421 & 0.0 & 0.0 \\ -0.31 & 0.8515 & 0.42 & 0.0 \\ 0.1446 & -0.3972 & 0.9 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$${}^{A}\{P\} = {}^{A}_{B}[T]\,{}^{B}\{P\}$$

2.4 [16] A frame $\{B\}$ is located as follows: initially coincident with a frame $\{A\}$, we rotate $\{B\}$ about Z_B by 30 degrees and then we rotate the resulting frame about X_B by 45 degrees. Give the rotation matrix which will change the description of vectors from ${}^{B}P$ to ${}^{A}P$.

$$ {}^{A}_{B}[R] = \left[ROT(\hat{z}, 30^{\circ}) \right] \left[ROT(\hat{x}, 45) \right] $$

USE PROG. 2

$$ \{\theta\} = \begin{Bmatrix} 30 \\ 45 \\ 0 \end{Bmatrix} ; \quad \{I\} = \begin{Bmatrix} 3 \\ 1 \\ 1 \end{Bmatrix} $$

RESULT

$$ {}^{A}_{B}[T] = \begin{bmatrix} 0.8659 & -0.35355 & 0.3537 & 0 \\ 0.500 & 0.6121 & -0.6124 & 0 \\ 0.0 & 0.7073 & 0.7068 & 0 \\ 0.0 & 0.0 & 0.0 & 1 \end{bmatrix} $$

2.12 [14] A velocity vector is given by

$$ {}^{B}V = \begin{bmatrix} 10.0 \\ 20.0 \\ 30.0 \end{bmatrix} $$

Given

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 11.0 \\ 0.500 & 0.866 & 0.000 & -3.0 \\ 0.000 & 0.000 & 1.000 & 9.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

compute ${}^{A}V$.

METHOD 1

$$ {}^{A}\left\{ V \right\} = \underset{4\times 4}{{}^{A}_{B}\left[\; T \;\right]}\; {}^{B}\left\{ \begin{matrix} 10.0 \\ 20.0 \\ 30.0 \\ 0.0 \end{matrix} \right\} $$

4 x 1 VECTOR

USE PROG 12

$$ {}^{A}\left\{ V \right\} = \left\{ \begin{matrix} -1.34 \\ 22.32 \\ 30.0 \\ 0.0 \end{matrix} \right\} $$

METHOD 2

USE

FIND θ USING PROG 11

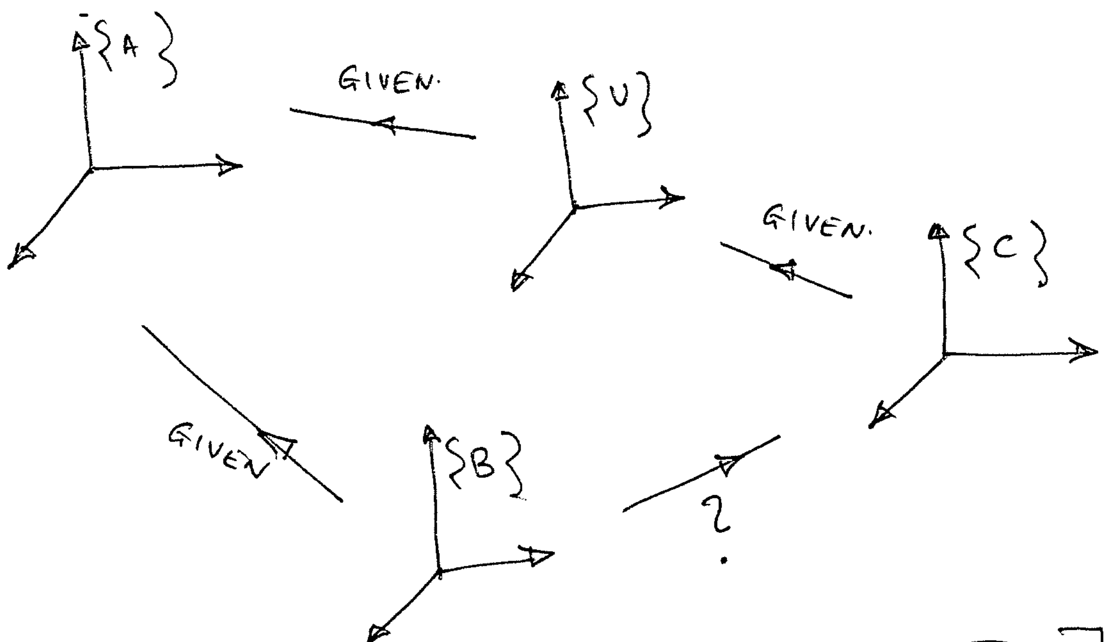
$$ {}^{A}\left\{ V \right\} = \underbrace{\underset{3\times 3}{{}^{A}_{B}\left[\; R \;\right]} \; \underset{3\times 1}{\left\{ \begin{matrix} 10.0 \\ 20.0 \\ 30.0 \end{matrix} \right\}}}_{\text{PROG 12}} $$

2.13 [21] The following frame definitions are given as known. Draw a frame diagram (like that of Fig. 2.15) which qualitatively shows their arrangement. Solve for ${}^B_C T$.

$$
{}^U_A T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 11.0 \\ 0.500 & 0.866 & 0.000 & -1.0 \\ 0.000 & 0.000 & 1.000 & 8.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^B_A T = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.0 \\ 0.000 & 0.866 & -0.500 & 10.0 \\ 0.000 & 0.500 & 0.866 & -20.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^C_U T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & -3.0 \\ 0.433 & 0.750 & -0.500 & -3.0 \\ 0.250 & 0.433 & 0.866 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
{}^B_C[T] = {}^B_A[T]\ {}^A_U[T]\ {}^U_C[\cancel{T}]
$$

$$
= {}^B_A[A]\ {}^U_A[T]^{-1}\ {}^C_U[T]^{-1}
$$

STEP 1

CALCULATE

$${}^{U}_{A}[T]^{-1} \quad \text{USING PROG 3}$$

$${}^{C}_{U}[T]^{-1} \quad \text{USING PROG 3}$$

STEP 2

MULTIPLY TWO MATRICES AT A TIME USING PROG 12

STEP 1

RESULT

$${}^{U}_{A}[T]^{-1} = \begin{bmatrix} 0.866 & 0.5 & 0.0 & -9.026 \\ -0.5 & 0.866 & 0.0 & 6.366 \\ 0.0 & 0.0 & 1.0 & -8.00 \\ 0.0 & 0.0 & 0.0 & 1.00 \end{bmatrix}$$

$${}^{C}_{U}[T] = \begin{bmatrix} 0.866 & 0.433 & 0.250 & 3.147 \\ -0.5 & 0.75 & 0.433 & -0.549 \\ 0.0 & -0.5 & 0.866 & -4.098 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

STEP 2

MULTIPLY TWO MATRICES AT A TIME USING PROG 12

FINAL RESULT

$${}^{B}_{C}[T] = \begin{bmatrix} 0.499 & 0.749 & 0.433 & -6.575 \\ -0.749 & 0.625 & -0.217 & 14.783 \\ -0.433 & -0.216 & 0.874 & -28.311 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

CHAPTER 3

MANIPULATOR KINEMATICS

3.1 INTRODUCTION

In this chapter, we will study the location and orientation of the end effector relative to the base. There are no motions involved in this chapter. This will be achieved by first defining a stationary frame at the base and a moving frame attached to each of the links. By knowing the displacement of the origins of each of the moving frames with respect to each other and also the orientations of the axes, it will be possible to calculate the position and the orientation of the axes at the end effector with respect to the base coordinate system.

3.2 LINK DESCRIPTION

fig

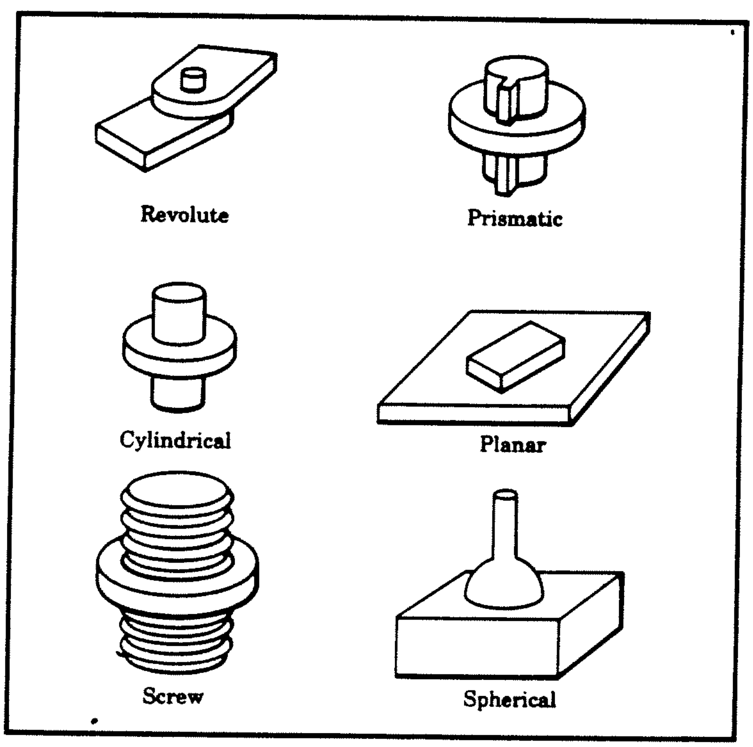


FIGURE 3.1 The six possible lower pair joints.

The manipulator may be thought of as a series of links connected to each other through the joints which are either revolute or sliding, in this course. In other words, these joints have only single degree of freedom.

A typical manipulator has six degrees of freedom. In the field of Robotics, the transformation matrices between two coordinate systems are written in accordance with the work of Denevit and Hartenberg. These matrices are also 4 x 4 but they require only 4 independent parameters, instead of 6 as seen in the last chapter. In this convention, the coordinate system i-1 is fixed on the link i-1 such that z_{i-1} axis points in the direction of the rotation vector, if it is a rotating or pinned connection, or along the sliding direction, in the case of the prismatic joint. The x_{i-1} axis is located or aligned along the link length. By fixing these two axes, one automatically fixes the y_{i-1} axis.

In this convention, there are two screw axes involved. At first one rotates about x_{i-1} axis such that z_{i-1} axis coincides with the z_i axis and this rotational angle is called α_i. Next, the second rotation is given about this z_{i-1} axis which is now coincident with the z_i axis until x_{i-1} axis coincides with the x_i axis. In the first rotation i.e., when screw rotates by α degrees, it also advances or translates by a_i meters. Similarly, in the second rotation, the translation involved is d_i meters and the rotation of the z_{i-1} axis is by θ_i degrees. This is clearly shown in fig 3.5. In actual robotic manipulators the design is such that only θ_i is a variable and the other parameters are fixed for a given manipulator. It

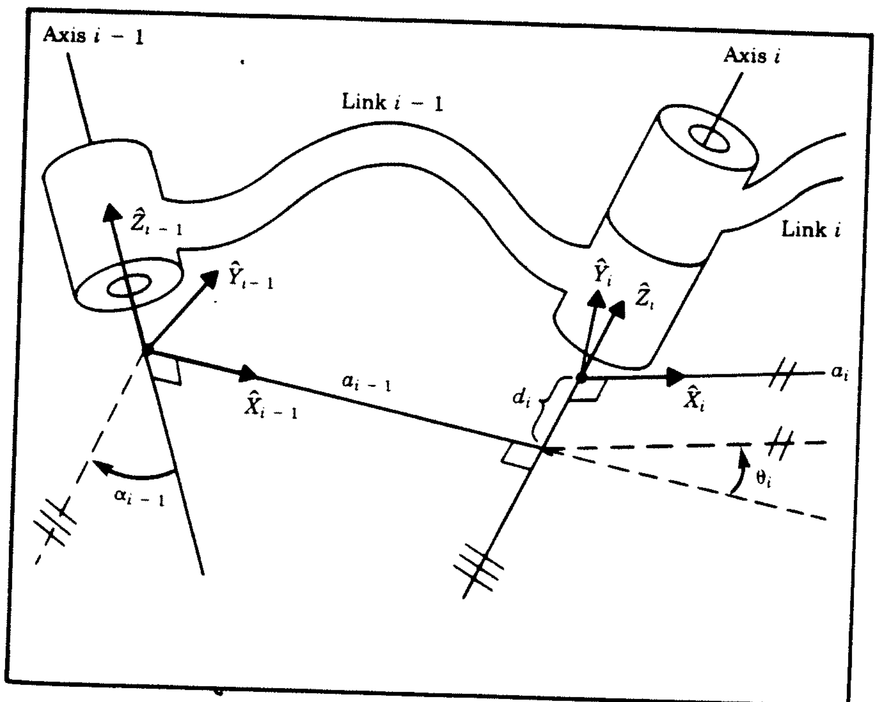


FIGURE 3.5 Link frames are attached so that frame $\{i\}$ is attached rigidly to link i.

will be the convention in this course to use

1) $a_o = a_n = 0.0$

2) $\alpha_o = \alpha_n = 0.0$

3) The origin or the coordinate system will be located at the beginning of the link.

4) If the joint 1 is a revolute then the zero position is selected at θ_1 degrees from the base coordinate system.

The Denevit-Hartenberg Transformation matrices can be written as a product of four individual matrices and the expression will be

$$ {}^{i-1}_{i}[\,T\,] = [R_x(\alpha_{i-1})]\ [D_x(a_{i-1})]\ [R_z(\theta_i)]\ [D_z(d_i)] $$

$$ = [Screw_x(a_{i-1},\alpha_{i-1})][Screw_z(d_i,\theta_i)] $$

$$ {}^{i-1}_{i}[T] = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

THREE LINK MANIPULATOR (PLANAR)

The D-H parameters for this manipulator are given in fig 3.8. It is a good example where one can verify our concepts from planar kinematics. Suppose we use a numerical value for $\theta_1 = 30°$, $\theta_2 = 40°$ $\theta_3 = 30°$ and $L_1 = L_2 = 10$ cm. If $L_3 = 5$ cm then

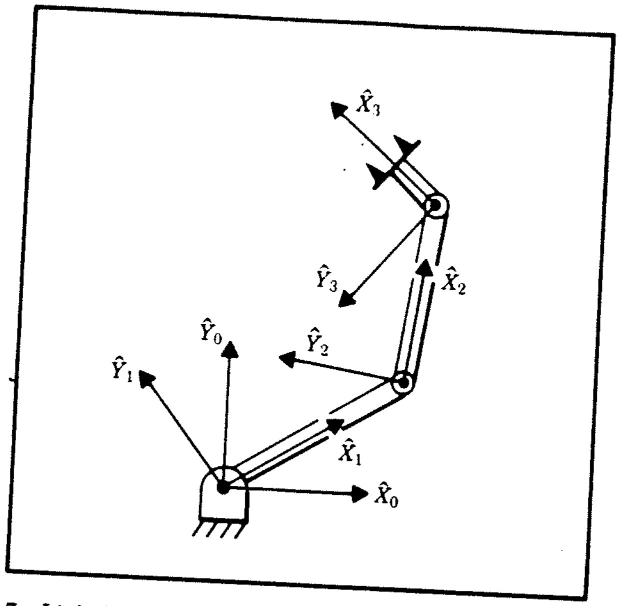


FIGURE 3.7 Link frame assignments.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

FIGURE 3.8 Link parameters of the three-link planar manipulator.

$$ {}^{0}_{3}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T] $$

$$ = {}^{0}_{3}\begin{bmatrix} -0.174 & -.0984 & 0 & 1.207 \times 10^{-1} \\ 0.984 & -0.174 & 0 & 0.144 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[R_z\ (\theta - 108)\right] $$

From the fig 3.7, we can see that

$$ {}^{0}\{E.E.\} = {}^{0}_{3}[T]\begin{Bmatrix} 0.05 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.11 \\ 0.193 \\ 0 \\ 1 \end{Bmatrix} $$

In the polar coordinates the location will be

E E = 10 $\angle 30$ + 10 $\angle 70$ + 5 $\angle 100$

= 22.339 $\angle 59.873$ = 11.212i + 19.321j + 0k cm

= 0.112i + 0.193j + 0k m

PUMA 560

The D-H Parameters for PUMA 560 manipulator are given in the Table (P 91). The Position vector of the origin at the end effector can be calculated using the formula

$$ {}^{0}_{6}[T] = {}^{0}_{1}[A] \ldots {}^{5}_{6}[A] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} $$

where the symbol ${}^{i-1}{}_{i}[A]$ has been used instead of ${}^{i-1}{}_{i}[T]$ to indicate the transformation between the coordinate systems attached to the moving links.

fig

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0	a_2	d_3	θ_3
4	−90°	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	−90°	0	0	θ_6

FIGURE 3.21 Link parameters of the PUMA 560.

If we do not substitute the numerical values for α_i, a_i, etc. and keep it in the symbolic form then we can obtain the expression

$$ {}^{0}_{6}[T] = \begin{bmatrix} r_{11} & . & . & . & r_{13} & p_x \\ . & . & . & . & . & . \\ r_{31} & . & . & . & r_{33} & p_z \\ 0 & . & . & . & 0 & 1 \end{bmatrix} $$

where

Eq(3.14)

$$
\begin{aligned}
r_{11} &= c_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] + s_1(s_4c_5c_6 + c_4s_6), \\
r_{21} &= s_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] - c_1(s_4c_5c_6 + c_4s_6), \\
r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6, \\
\\
r_{12} &= c_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] + s_1(c_4c_6 - s_4c_5s_6), \\
r_{22} &= s_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] - c_1(c_4c_6 - s_4c_5s_6), \\
r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, \\
\\
r_{13} &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, \\
r_{23} &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, \\
r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
\\
p_x &= c_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] - d_3s_1, \\
p_y &= s_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] + d_3c_1, \\
p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
\tag{3.14}
$$

This equation constitutes the kinematics of the PUMA 560. It specifies how to compute the position and orientation of the end effector.

3.8 FRAMES WITH STANDARD NAMES

In the task performance or planning, several "Standard" frames are necessary. All robot motions are described in terms of these frames.

(a) The Base Frame {B}

It is located at the base of the manipulator. It is merely another name for frame ${}^{0}\{\ \}$.

(b) The Station Frame {S}

The {S} is located at a task relevant location. It serves as the standard frame or the reference frame for the task. As far as the robot system is concerned, {S} serves as the universal frame and all actions of the robot are made relative to it. It is also called the WORLD FRAME or the TASK FRAME.

(c) The Wrist Frame {W}

{W} is affixed to the last link of the manipulator. It is another name for frame {N}, the link frame attached to the last link of the robot.

(d) The Tool Frame {T}

It is affixed to the end of the tool, the robot happens to be holding. The tool frame is specified with respect to the wrist frame.

(e) The Goal Frame {G}

{G} is a description of the location to which the robot has to move the tool to. In other words, at the end of the motion the tool frame {T} should be identical to the Goal Frame {G}. It requires

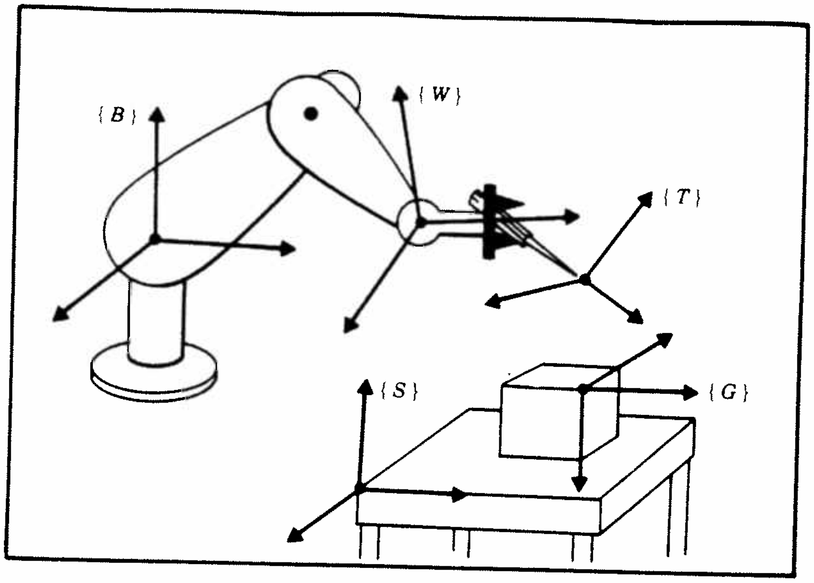


FIGURE 3.27 The standard frames.

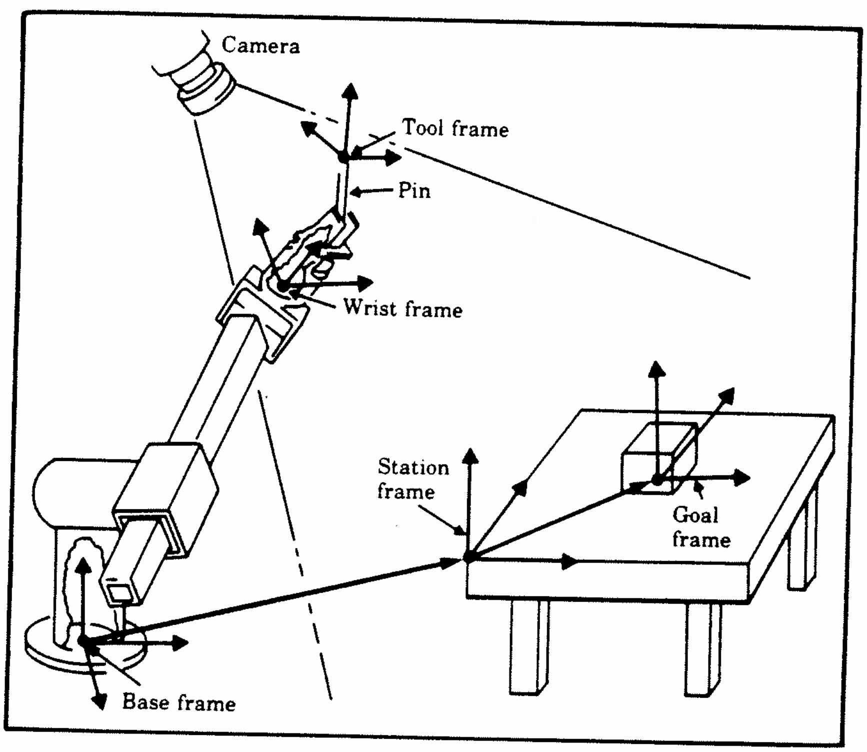


FIGURE 3.28 Example of the assignment of standard frames.

identity in both the location as well as the orientation.

3.9 WHERE IS THE TOOL ?

One of the first capabilities a robot must have is to be able to calculate the position and orientation of the tool it is holding with respect to {S}. This can be done using the relationship

$$ {}^{S}_{T}[T] = {}^{B}_{S}[T]^{-1} \, {}^{B}_{W}[T] \, {}^{W}_{T}[T] \tag{3.18} $$

CHAPTER 4

INVERSE MANIPULATOR KINEMATICS

In the last chapter we considered the problem of calculating the position and orientation of the tool relative to the user's workstation given the joint angles of the manipulator. Of course, the other design parameters such as α_i, a_i, and d_i etc. are also known. However, in this chapter we will attempt and solve a more DIFFICULT problem of solving for the joint angles, given the position and orientation of the end effector. The reason it is more difficult is that the equations involved are many and also, they are nonlinear in nature which is quite clear by inspecting Eq. below. Here the matrix is

Eq(3.14)

$$
{}^0_6T = {}^0_1T \, {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

where

$$
\begin{aligned}
r_{11} &= c_1 \left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] + s_1(s_4c_5c_6 + c_4s_6), \\
r_{21} &= s_1 \left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] - c_1(s_4c_5c_6 + c_4s_6), \\
r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,
\end{aligned}
$$

$$
\begin{aligned}
r_{12} &= c_1 \left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] + s_1(c_4c_6 - s_4c_5s_6), \\
r_{22} &= s_1 \left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] - c_1(c_4c_6 - s_4c_5s_6), \\
r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,
\end{aligned}
$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$$
$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$$
$$r_{33} = s_{23}c_4s_5 - c_{23}c_5,$$

$$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1,$$
$$p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1,$$
$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}. \tag{3.14}$$

$$C_{23}=C_2C_3-S_2S_3=Cos(\theta_2+\theta_3)$$
$$S_{23}=C_2S_3+S_2C_3=Sin(\theta_2+\theta_3)$$
$$S_2=Sin\theta_2$$
$$C_2=Cos\theta_2$$

$$^{S}_{T}[T] = \begin{bmatrix} r_{11} & . & . & . & p_x \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is given. Naturally, the left hand side of Equation (3.14) is known but all θ_i, i = 1...6 are unknown. Since this set of simultaneous transcendental equations has more than one solution~~s~~, an obvious conclusion is that a given end effector orientation and position can be attained in more than one ways.

On the other hand, in certain situations when it is beyond the reach of the manipulator, then no solution can be found. It simply means that the location and the orientation are not in the WORKSPACE of the manipulator.

A DEXTROUS WORKSPACE is that volume of space where the manipulator can reach with all orientations.

A REACHABLE WORKSPACE is one where the manipulator can reach with at least one orientation.

Clearly, a dextrous workspace is a subset of the reachable workspace.

fig

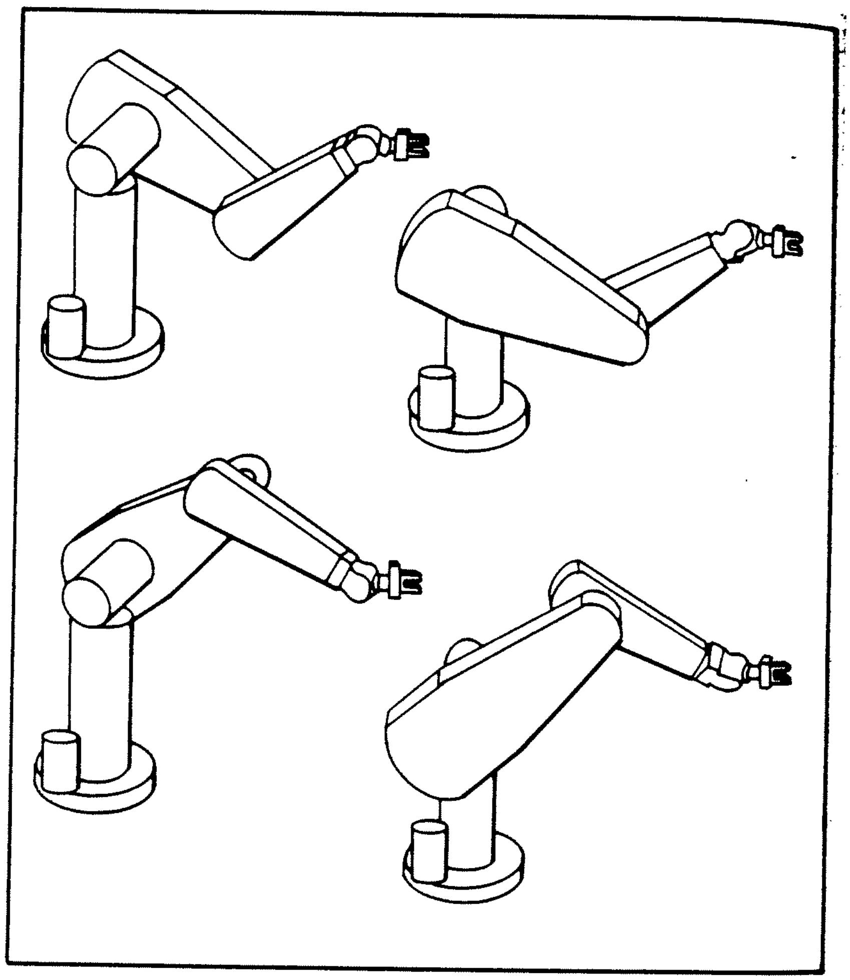


FIGURE 4.4 Four solutions of the PUMA 560.

METHOD OF INVERSE SOLUTION

There are no general methods available to solve these nonlinear set of equations. The solution methods are divided into two categories: (a) closed form, and (b) numerical.

Closed form solutions are preferred because the computational times are negligible as compared to the numerical solution (a) Newton Raphson Technique, or (b) Optimization Techniques.

Within the class of closed form solution are: (a) algebraic method, and (b) geometric method.

A major recent result is that all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series of chain are now solvable(Numerical Solution). In the practical design of manipulators a closed form solution is a must.

A sufficient condition that a manipulator with six revolute joints will have a closed form solution is that three neighbouring joint axes intersect at a point. Almost every manipulator with six degrees of freedom built today has three axes intersecting. For example, axes 4,5, and 6 of the PUMA 560 intersect.

4.3 The Motion of Manipulator Subspace When $n < 6$

The set of reachable goal frames for a given manipulator constitutes its reachable workspace. For example a description of the subspace for a three link manipulator (Planar) is given by

$$
{}^{B}_{W}[T] = \begin{bmatrix} c\phi & -s\phi & 0.0 & x \\ s\phi & c\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

where x, and y give the position of the wrist and Φ, the orientation. As x,y, and Φ are assigned arbitrary values, the subspace is generated.

4.4 Algebraic Verses Geometric Solutions

For the three link manipulator the inverse solution (only a few steps) are:

fig

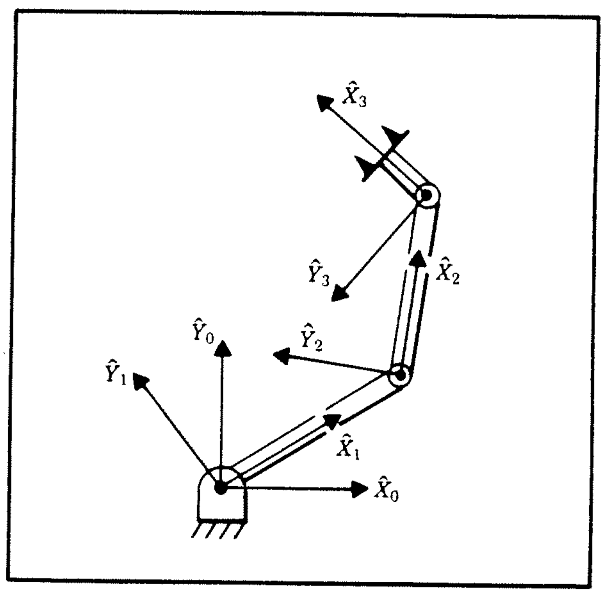


FIGURE 3.7 Link frame assignments.

PROBLEM

GIVEN THE MATRIX FIND θ_1, θ_2, θ_3?

$$
{}^{B}_{W}[T] = \begin{bmatrix} c\phi & -s\phi & 0.0 & x \\ s\phi & c\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{4.7}
$$

$$
= \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1+l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1+l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{4.6}
$$

where $c_{12} = \text{Cos}(\theta_1+\theta_2)$ etc.

$$
c_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2} \tag{4.14}
$$

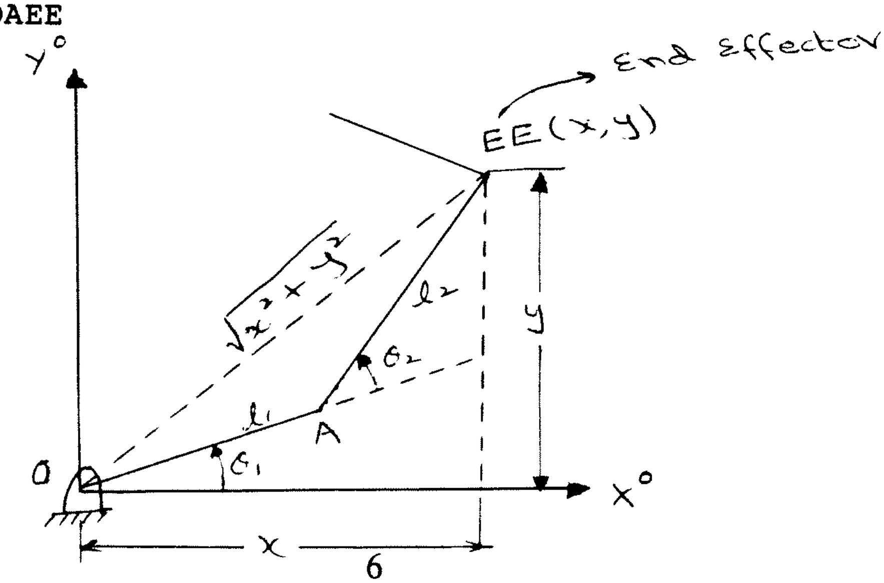
$$
s_2 = \pm\sqrt{1-c_2^2} \tag{4.15}
$$

$$
\theta_2 = \text{Atan2}(s_2,c_2) \tag{4.16}
$$

$$
K_1 = l_1+l_2c_2 \tag{4.19}
$$

$$
K_2 = l_2s_2 \tag{4.19}
$$

In triangle OAEE



$$\mathrm{Cos}(180-\theta_2) = \frac{l_1^2+l_2^2-(x^2+y^2)}{2l_1l_2}$$

$$-\mathrm{Cos}\theta_2 = \frac{l_1^2+l_2^2-(x^2+y^2)}{2l_1l_2}$$

$$\mathrm{Cos}(\theta_2) = \frac{(x^2+y^2)-l_1^2-l_2^2}{2l_1l_2}$$

$$\theta_1 = \mathrm{Atan2}(y,x) - \mathrm{Atan2}(k_2,k_1) \tag{4.27}$$

$$\theta_1 + \theta_2 + \theta_3 = \mathrm{Atan2}(s\phi,c\phi) \tag{4.28}$$

from which we can solve for θ_3 because θ_2 and θ_1 are already known.In the Geometric Solution, we try to decompose the spatial Geometry into several plane geometry.

4.7 EXAMPLES OF INVERSE MANIPULATOR KINEMATICS - PUMA 560

Here we wish to solve

$$
{}^{0}_{6}[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$= {}^{0}_{1}[T(\theta_1)]\ {}^{1}_{2}[T(\theta_2)]\ {}^{2}_{3}[T(\theta_3)] \ldots\ {}^{5}_{6}[T(\theta_6)]$$

Without going into the details,the sequence in which the solution can be obtained is as follows:

We are given

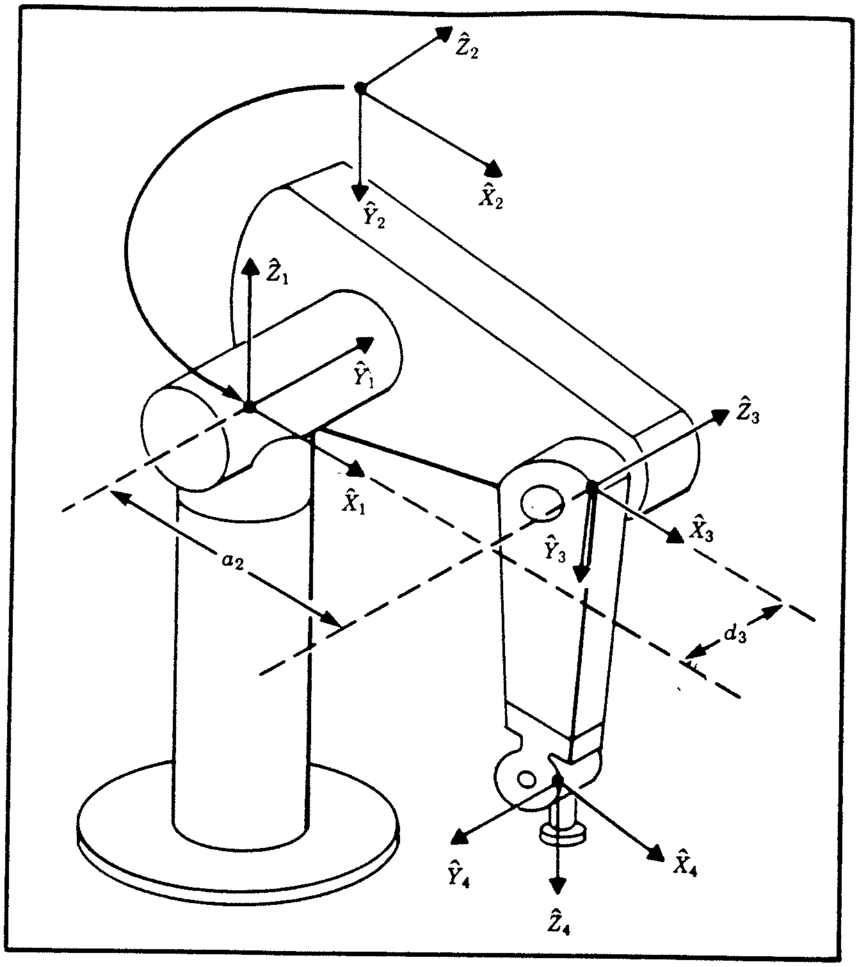


FIGURE 3.18 Some kinematic parameters and frame assignments for the PUMA 560 manipulator.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

FIGURE 3.21 Link parameters of the PUMA 560.

1)

$$ {}^0_6[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2)

All α_i, d_i, a_i

$$\theta_1 = \text{Atan2}(p_y,p_x) - \text{Atan2}(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}) \qquad (4.64)$$

$$K = \frac{p_x^2+p_y^2+p_z^2-a_2^2-a_3^2-d_3^2-d_4^2}{2a_2} \qquad (4.67)$$

$$\theta_3 = \text{Atan2}(a_3,d_4) - \text{Atan2}(k, \pm\sqrt{a_3^2+d_4^2-k^2}) \qquad (4.68)$$

$$\theta_{23} = \text{Atan2}[(-a_3-a_2c_3)p_z-(c_1p_x+s_1p_y)(d_4-a_2s_3),(a_2s_3-d_4)p_z-(a_3+a_2c_3)(c_1p_x+s_1p_y)] \qquad (4.73)$$

$$\theta_2 = \theta_{23} - \theta_3 \qquad (4.74)$$

$$\theta_4 = \text{Atan2}(-r_{13}s_1+r_{23}c_1, -r_{13}c_1c_{23}-r_{23}s_1c_{23}+r_{33}s_{23}) \qquad (4.76)$$

$$-s_5 = r_{13}(c_1c_{23}c_4+s_1s_4)+r_{23}(s_1c_{23}c_4-c_1s_4)-r_{33}(s_{23}c_4)$$

$$c_5 = r_{13}(-c_1s_{23})+r_{23}(-s_1s_{23})+r_{33}(-c_{23}) \qquad (4.79)$$

$$\theta_5 = \text{Atan2}(s_5, c_5) \qquad (4.80)$$

$$s_6 = -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4)$$

$$c_6 = r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] - r_{31}(s_{23}c_4c_5 + c_{23}s_5)]$$

$$\theta_6 = \text{Atan2}(s_6, c_6) \qquad (4.82)$$

NOTE:

1) Because of the + or - sign appearing in Equations(4.64) and Equations(4.68), these equations compute 4 solutions.

2) Additionally, ~~from~~ four more solutions are obtained by "flipping" the wrist which is expressed mathematically as

$$\left.\begin{aligned} \theta_4' &= \theta_4 + 180^\circ \\ \theta_5' &= -\theta_5 \\ \theta_6' &= \theta_6 + 180^\circ \end{aligned}\right] \qquad (4.83)$$

After 8 solutions are computed, some or all of them may need to be discarded because joint limit limitations.

CHAPTER 5

JACOBIANS : VELOCITIES AND STATIC FORCES

5.1 INTRODUCTION

In this chapter, we move beyond static considerations and discuss the transformations of static forces, velocities and angular velocities. We have to remember that the motion or forces involved are in three dimensions and that the body is rigid.

The transformations are done using a matrix called a Jacobian. It is worth to discuss the notation of the vectors and superscripts here.

$$^{A}(^{B}\{V\}_{Q}) = {}^{A}_{B}[R]\,^{B}\{V\}_{Q} \tag{5.4}$$

The subscript Q indicates the velocity of a point Q in the frame {B}. The Pre superscript indicates that we wish to know this vector in the frame {A}.

The right hand shows that we can obtain this vector in the frame {A} by pre-multiplying $\{V\}_Q$ by $^{A}_{B}[R]$ matrix. By pre multiplication with [R] we project the vector $\{V\}_Q$ parallel to the axes of the frame {A}, which is geometricaly shown in fig. 5.1.a.

In short, the procedure of calculating the velocity of any point S on a link would be to calculate (a) the velocity of the

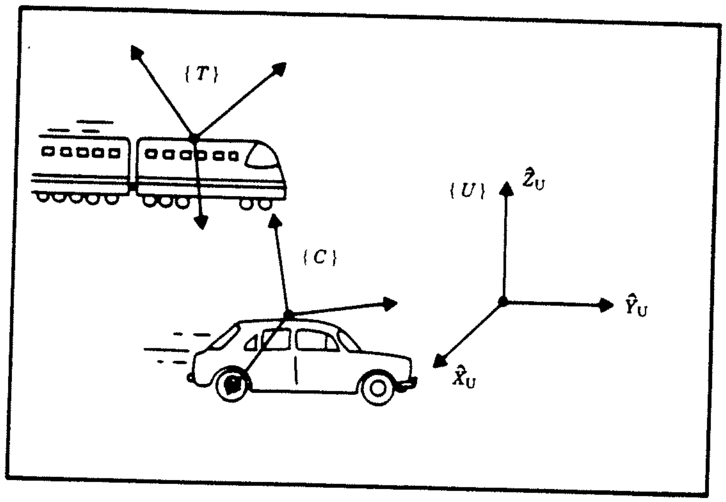


FIGURE 5.1 Example of some frames in linear motion.

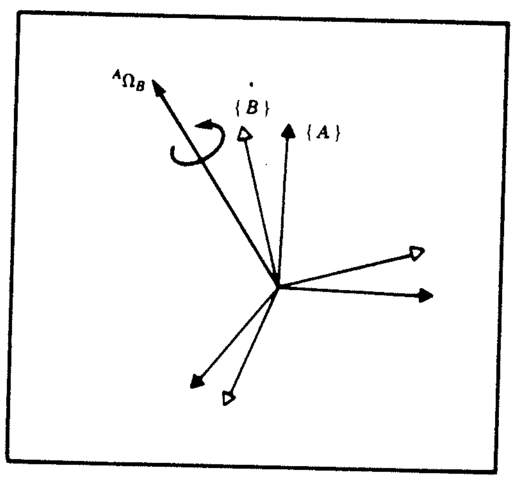


FIGURE 5.2 Frame $\{B\}$ is rotating with angular velocity $^{A}\Omega_{B}$ relative to frame $\{A\}$.

origin of the coordinate system attached to the link, and (b) to this velocity, one has to vectorially add the velocity difference of the point S and that of this origin.

THE ANGULAR VELOCITY VECTOR

We will use the symbol Ω to indicate the angular velocity vector. Though the linear velocity describes the attribute of a point, the angular velocity describes the atribute of a body. Since we always attach a frame to the body, we can also consider the angular velocity as the attribute of the frame. The symbol ${}^{A}\Omega_{B}$ describes the rotation of the frame {B} relative to {A}. It is a vector; therefore it can be expressed in any frame using an appropriate [R] matrix.

5.3.LINEAR AND ROTATIONAL VELOCITIES OF RIGID BODIES

We can express the velocity of the points either in their own frames or in some other frames. For example, the velocity of the point Q in the Fig 5.3 in its own frame will be

$$\underset{3\times1}{{}^{B}\{V\}_{Q}} = \underset{3\times1}{{}^{B}\{V_{BOrigin}\}} + \underset{3\times1}{{}^{B}\Omega_{B} \times {}^{B}\{Q\}} + \underset{3\times1}{{}^{B}\{V\}_{Q}}$$

and in the frame {A} will be

$${}^{A}\{V\}_{Q} = {}^{A}\{V_{B\ ORIG}\} + {}^{A}_{B}[R]\ {}^{B}\{V\}_{Q} + {}^{A}\Omega_{B} \times ({}^{A}_{B}[R]\ {}^{B}\{Q\}) \quad (5.13)$$

It should be noted that in the third term, the cross product is

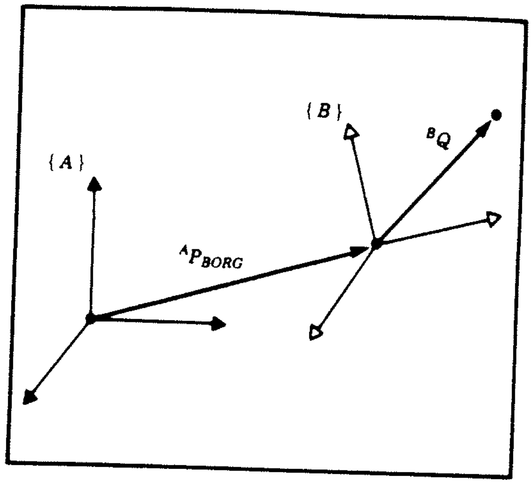


FIGURE 5.3 Frame $\{B\}$ is translating with velocity $^AV_{BORG}$ relative to frame $\{A\}$.

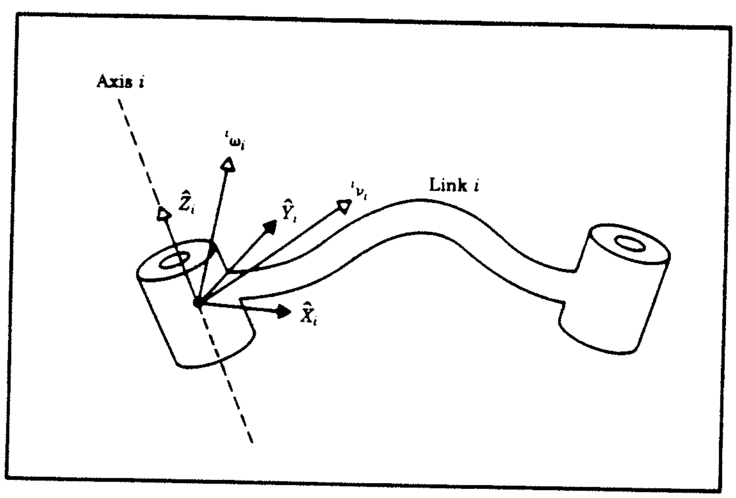


FIGURE 5.6 The velocity of link i is given by vectors v_i and ω_i which may be written in any frame, even frame $\{i\}$.

defined because the vector ${}^{B}\{Q\}$ has already been transformed into the frame {A} using ${}^{A}_{B}[R]$ matrix; otherwise the cross product in two different systems would not have been valid.

5.5.MOTION OF LINKS OF A ROBOT

Each link frame will have some linear motion (velocity) and angular velocity. We will represent these motions of the origin and the frame as V_i and ω_i respectively, where the subscript i stands for the frame {i}.

5.6.VELOCITY PROPAGATION FROM LINK TO LINK

In kinematics, the velocities, etc., are calculated starting from the base to the increasing link numbers upto the end effector. This is a very important fact to remember. Another point to note is that $\dot{\theta}$ are the relative angular velocities; therefore we can write the relationship.

$$ {}^{i}_{i+1}\{\omega\} = {}^{i}_{i}\{\omega\} + {}^{i}_{i+1}[R]\ {}^{i}_{i+1}(\dot{\theta})\ {}^{i+1}\{\hat{Z}\} \qquad (5.43)$$

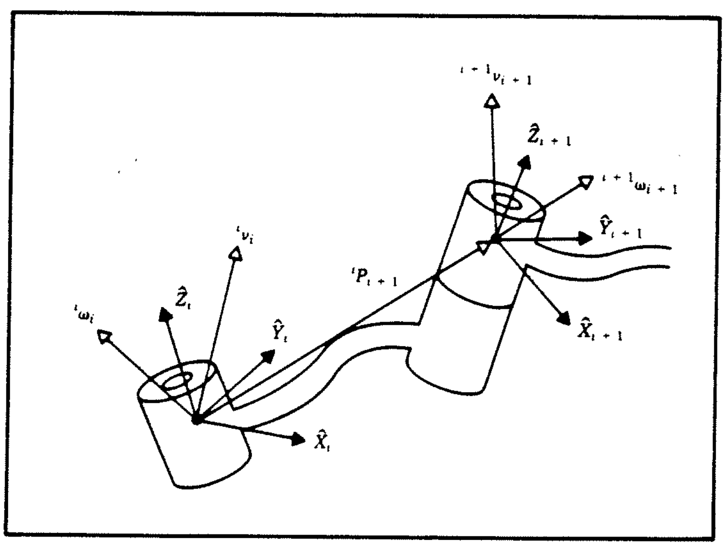


FIGURE 5.7 Velocity vectors of neighboring links.

In Eq. (5.43), the fist term on the right hand side (RHS) is the angular velocity of the link i. The second term contains ${}^{i}_{i+1}(\dot{\theta})$, a scalar which is the relative angular velocity of the link i+1 with respect to link i and the vector ${}^{i+1}\{\hat{Z}\}$ is mathematically expressed as

$$^{i+1}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Therefore the product can also be expressed as

$$^{i+1}\begin{Bmatrix} 0 \\ 0 \\ {}^{i+1}\dot{\theta} \end{Bmatrix}$$

This product is expressed in the $(i+1)^{th}$ frame. To add both terms on RHS, they must be expressed in the same frame i, which requires the premultiplication with the matrix ${}^{i}_{i+1}[R]$.

However, if we wish to know the angular velocity of the link i+1 in the $(i+1)^{th}$ frame then the equation will be

$${}^{i+1}_{i+1}\{\omega\} = {}^{i+1}_{i}[R]\ {}^{i}_{i}\{\omega\} + (\dot{\theta}_{i+1})\ {}^{i+1}_{i+1}\{\hat{Z}\} \qquad (5.45)$$

The velocity of the $(i+1)^{th}$ frame (its origin) will be equal to the velocity of the i^{th} frame plus the velocity difference between these two. The corresponding equation will be

$$ {}^{i+1}_{i+1}\{v\} = {}^{i+1}_{i}[R] \ ({}^{i}_{i}\{v\} + {}^{i}_{i}\{\omega\} \times {}^{i}_{i+1}\{P\}) \qquad (5.47) $$

where the vector ${}^{i}_{i+1}\{P\}$ is the Position Difference Vector between the origins. The equivalent relationships for the case when the joint (i+1) is prismatic are

$$ {}^{i+1}_{i+1}\{\omega\} = {}^{i+1}_{i}[R] \ {}^{i}_{i}\{\omega\} \qquad (5.48) $$

$$ {}^{i+1}_{i+1}\{v\} = {}^{i+1}_{i}[R] \ ({}^{i}_{i}\{v\} + {}^{i}_{i}\{\omega\} \times {}^{i}_{i+1}\{P\}) + \dot{d}_{i+1} \ {}^{i+1}_{i+1}\{\hat{Z}\} $$

Applying these equations successively from link to link, we can compute ${}^{N}_{N}\{\omega\}$ and ${}^{N}_{N}\{v\}$. If we want these velocities in the base coordinate system then we can write

$$ {}^{0}_{N}\{\omega\} = {}^{0}_{N}[R] \ {}^{N}_{N}\{\omega\} $$

and

$$ {}^{0}_{N}\{v\} = {}^{0}_{N}[R] \ {}^{N}_{N}\{v\} $$

5.7. JACOBIANS

The Jacobian is a multidimensional form of the derivative. Suppose we have six functions

$$ y_1 = f_1 \ (x_1, x_2, \ldots\ldots, x_6) $$
$$ y_2 = f_2 \ (x_1, x_2, \ldots\ldots, x_6) $$
$$ \cdot $$
$$ \cdot $$
$$ y_6 = f_6 \ (x_1, x_2, \ldots\ldots, x_6) $$

which can be expressed in compact notation as

$Y = F(X)$

Then we can write the relationship for the partial derivate as

$$\{\delta Y\} = [\partial F/\partial x]\ \{\delta x\} \tag{5.61}$$

Where $$\{\delta Y\} = \begin{Bmatrix} \delta y_1 \\ \delta y_2 \\ : \\ : \\ \delta y_6 \end{Bmatrix}$$

$$[\partial F/\partial x] = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \ldots\ldots \partial f_1/\partial x_6 \\ : & \\ \partial f_6/\partial x_1 & \ldots\ldots\ldots\ldots\ldots\ \partial f_6/\partial x_6 \end{bmatrix}$$

If the functions f_1, f_2, etc. are nonlinear then clearly $[\partial F/\partial x]$ will be a function of x_1, x_2,.....etc. Differentiating Eq.(5.61) we can write

$$\begin{aligned} \{\delta \dot{Y}\} &= [\partial F/\partial x]\ \{\delta \dot{x}\} \\ &= [J]\ \{\delta \dot{x}\} \end{aligned}$$

Therefore [J] at a given set of values of x_1, x_2,... etc. is a linear transformation between $\{\delta\dot{x}\}$ and $\{\delta\dot{Y}\}$. In the field of Robotics, [J(x)] contains θ_i , l_i, etc. In others words, the joint angles and other design parameters. For example, for a two link manipulator shown in Fig. 5.8, the corresponding Jacobian is

$$^{0}[J(\theta)] = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ & \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{bmatrix} \tag{5.67}$$

and

$$^{3}[J(\theta)] = \begin{bmatrix} l_1S_2 & 0 \\ l_1c_2 + l_2 & l_2 \end{bmatrix} \tag{5.66}$$

The Jacobians transform the joint velocities to the Cartesian Velocities, i.e.

$$^{0}\{v\} = {}^{0}[J]\ {}^{0}\{\dot{\theta}\} \tag{5.64}$$

or

$$^{A}\begin{Bmatrix} v \\ \cdots \\ \omega \end{Bmatrix} = \left[\begin{array}{c:c} {}^{A}_{B}[R] & [0] \\ \hdashline [0] & {}^{A}_{B}[R] \end{array}\right] {}^{B}\begin{Bmatrix} v \\ \cdots \\ \omega \end{Bmatrix} \tag{5.69}$$

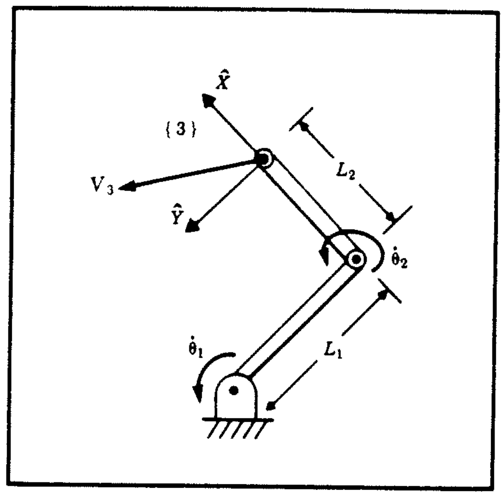
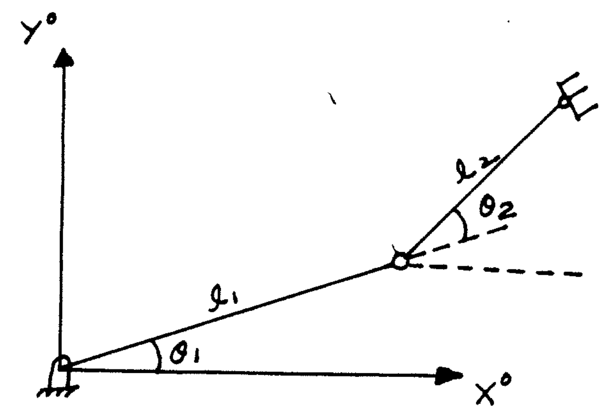


FIGURE 5.8 A two-link manipulator.

Then the transformation equation for a Jacobian will be

$$
{}^{A}[J(\theta)] = \left[\begin{array}{c:c} {}^{A}_{B}[R] & [0] \\ \hdashline [0] & {}^{A}_{B}[R] \end{array}\right] {}^{B}[J(\theta)] \tag{5.71}
$$

JACOBIAN OF A TWO LINK MANIPULATOR



DISPLACEMENT EQN.

$$X^0_{EE} = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)$$

$$Y^0_{EE} = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)$$

$$
\begin{Bmatrix} \dot{X}^0_{EE} \\ \dot{Y}^0_{EE} \end{Bmatrix} = \left[\begin{array}{c:c} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ \hdashline l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{array}\right] \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}
$$

$\{\underline{V}\} = [J]\{\underline{\dot{\theta}}\} \rightarrow$ form of eqn.

$$\begin{Bmatrix} \ddot{X}^0_{EE} \\ \ddot{Y}^0_{EE} \end{Bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -l_1 \cos\theta_1\dot{\theta}_1 - l_2\{\cos(\theta_1+\theta_2)\}(\dot{\theta}_1+\dot{\theta}_2) & -l_2 \cos_{12}(\dot{\theta}_1+\dot{\theta}_2) \\ \\ -l_1 \sin\theta_1\dot{\theta}_1 - l_2\{\sin(\theta_1+\theta_2)\}(\dot{\theta}_1+\dot{\theta}_2) & -l_2 \sin_{12}(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \\ \dot{\theta}_2 \end{bmatrix}$$

5.8 SINGULARITIES

We can rewrite Eq. (5.64) in the form

$$^0\{\dot{\theta}\} = {}^0[J]^{-1}\ {}^0\{v\} \qquad (5.72)$$

provided matrix [J] in non singular

SINGULARITIES

1) Workspace boundary singularities are those which occur when the manipulator is fully stretched out or folded back on itself such that the end effector is near or at the boundary of the workspace.

2) Workspace interior singularities are those which occur away from the workspace boundary and are caused by two or more joint axes lining up.

When a manipulator is in singular configuration, it has lost one or more degrees of freedom as viewed from the Cartesian space. It means that in some direction (or subspace) in Cartesian space along which it is impossible to move the hand no matter which joint rates are selected.

To find the condition of singularities one has to equate the determinant of $[J(\theta)] = 0$.

5.9.STATIC FORCES IN MANIPULATORS

The analysis of static and dynamic forces are of great importance in the field of robotics. In moving a load, the motors must apply sufficient torques to cause the motion. In the static situation, there has to be locking at the joints and therefore, there must be resisting torques there. In these static situations, the forces or torques are calculated by considering the robot as a structure which means there are no degrees of freedom at the joints.

Defining the symbols

${}_i\{f\} = f_i$ = Force exerted by link i-1 on link i

on ${}_i\{n\} = n_i$ = Torque exerted by link i-1 on link i

Summing the forces and setting them equal to zero we have

$${}^if_i - {}^if_{i+1} = 0 \qquad (5.76)$$

and summing torques about the origin of the frame {i} we get

$${}^in_i - {}^in_{i+1} - {}^iP_{i+1} \times {}^if_{i+1} = 0 \qquad (5.77)$$

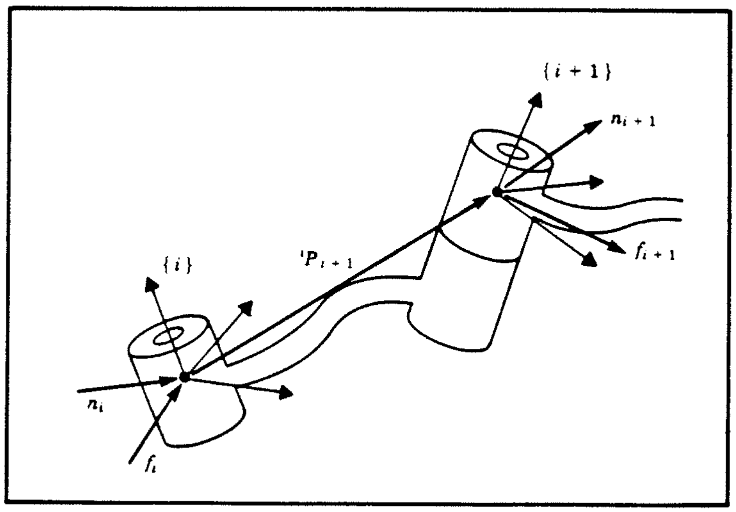


FIGURE 5.11 Static force-moment balance for a single link.

These two equations can be rewritten as

$$ {}^{i}_{i}\{f\} = {}^{i}_{i+1}\{f\} \tag{5.78}$$

$$ {}^{i}_{i}\{n\} = {}^{i}_{i+1}\{n\} + {}^{i}\{P_{i+1}\} \times {}^{i}_{i+1}\{f\} \tag{5.79}$$

Normally ${}^{i}_{i+1}\{f\}$ is expressed in the frame $\{i+1\}$. So we have to resolve these forces parallel to the i^{th} frame. If we do so, we can write

$$ {}^{i}_{i}\{f\} = {}^{i}_{i+1}[R] \; {}^{i+1}_{i+1}\{f\} \tag{5.80}$$

$$ {}^{i}_{i}\{n\} = {}^{i}_{i+1}[R] \; {}^{i+1}_{i+1}\{n\} + {}^{i}_{i+1}\{P\} \times {}^{i}_{i}\{f\} \tag{5.81}$$

The torque required to keep the system in static equilibrium will be

$$ \tau_i = {}^{i}_{i}\{n\}^{T} \cdot {}^{i}_{i}\{\hat{Z}\} = \{n_1 \;\; n_2 \;\; n_3\} \left\{ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\} \tag{5.82}$$

For a prismatic joint, it will be

$$ \tau_i = {}^{i}_{i}\{f\}^{T} \; {}^{i}_{i}\{\hat{Z}\} \tag{5.83}$$

5.10 JACOBIANS IN THE FORCE DOMAIN

It can be shown that the relationship

$$\{\tau\} = [J]^T \{F\} \qquad (5.96)$$

can be useful in tranforming the Cartesian forces acting at the hand into the equivalent joint forces. The corresponding equation in the frame {0} is

$$\{\tau\} = {}^0[J]^T\{F\} \qquad (5.97)$$

When the Jacobian loses rank at certain configurations, a small joint torque can sustain high forces at the end effector. For example, in the case of a two link manipulator near the outstretched position, a small torque can sustain large forces.

CHAPTER 5
VEL. ANALYSIS

PROBLEM 1

1) A two-link manipulator with rotational joint is shown in fig. 1 Calculate the velocity of the tip of the arm using

$L_1 = 5$ cm, $L_2 = 10$ cm; $L_3 = 5$ cm;
$\theta_1 = 30^\circ$, $\theta_2 = 80^\circ$, $\theta_3 = 0^\circ$
$\dot{\theta}_1 = 5$ rad/s $\dot{\theta}_2 = 10$ rad/s $\dot{\theta}_3 = 7$ rad/s

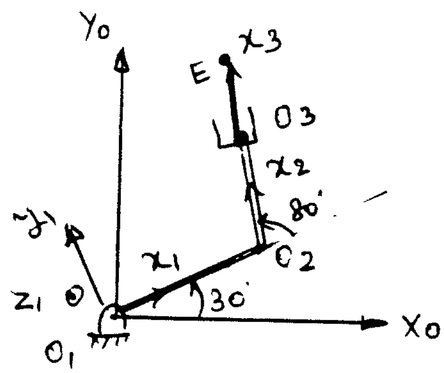


fig. 1

	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	30
2	0	0.05	0	80
3	0	0.10	0	0

Table 1 D-H Parameters

SOLUTION

$$ {}^{0}_{1}[T] = \begin{bmatrix} 0.8659 & -0.5001 & 0.0 & 0.0 \\ 0.5001 & 0.8659 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} $$

→ Use 2:in Robotics Software

$$
{}^{1}_{2}[T] = \begin{bmatrix} 0.1730 & -0.9849 & 0.0 & 0.05 \\ 0.9849 & 0.1730 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$
→ From 2: in Robotics Software

$$
{}^{2}_{3}[T] = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.10 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

$$
{}^{0}_{3}[T] = {}^{0}_{1}[T] * {}^{1}_{2}[T] * {}^{2}_{3}[T]
$$

$$
= \begin{bmatrix} -0.3427 & -0.9394 & 0.0 & 0.009 \\ 0.9394 & -0.3427 & 0.0 & 0.118 \\ 0.0 & 0.0 & 1.0 & 0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

All the above four D-H matrices are got from Case no: 2 (D-H parameters) in ROBOTICS SOFTWARE

For calculations below, use 12 for matrix multiplication 3 for inverse & 8 for cross product of vectors;

Note: (i) [R] is got from [T] by removing the last column and row

(ii) ${}^{1}_{0}[R] = {}^{0}_{1}[R]^{-1}$

in general ${}^{i+1}_{i}[R] = {}^{i}_{i+1}[R]^{-1}$

} use 3 to find the inverse

$${}^{1}\{\omega_1\} = {}^{1}_{0}[R]\,{}^{0}\{\underbrace{\cancel{\omega_0}}_{}\}^{=0} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{Bmatrix}$$

$${}^{1}\{\omega_1\} = \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix}$$

$${}^{1}\{v_{0_1}\} = {}^{1}_{0}[R]\left({}^{0}\{\cancel{v_0}\}^{=0} + {}^{0}\{\cancel{\omega_0}\}^{=0} \times \{\cancel{R_{01}}\}^{=0}\right)$$

$${}^{1}\{v_{0_1}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$${}^{2}\{\omega_2\} = \underbrace{{}^{2}_{1}[R]^{*}\,{}^{1}\{\omega_1\}}_{\text{use } \underline{\underline{12}}} + {}^{2}\begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{Bmatrix} \qquad {*}\ {}^{2}_{1}[R] = {}^{1}_{2}[R]^{-1}$$

$$= \begin{bmatrix} 0.173 & 0.9849 & 0.0 \\ -0.9849 & 0.173 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 15 \end{Bmatrix}$$

$${}^{2}\{v_{0_2}\} = {}^{2}_{1}[R]\left({}^{1}\{\cancel{v_{0_1}}\}^{=0} + \underbrace{{}^{1}\{\omega_1\} \times {}^{1}\{R_{0_2 0_1}\}}_{\text{use } 8}\right)$$

$$= \begin{bmatrix} 0.173 & 0.9849 & 0.0 \\ -0.9849 & 0.173 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \left(\begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix}^{*} \times \begin{Bmatrix} 0.05 \\ 0 \\ 0 \end{Bmatrix} \right)$$

$$= \begin{bmatrix} 0.173 & 0.9849 & 0.0 \\ -0.9849 & 0.173 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \underbrace{\begin{Bmatrix} 0 \\ 0.25 \\ 0 \end{Bmatrix}}_{\text{use } \underline{12}}$$

$$= \begin{Bmatrix} 0.246 \\ 0.04325 \\ 0.0 \end{Bmatrix}$$

$${}^3\{\omega_3\} = {}^3_2[R]\,{}^2\{\omega_2\} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 15.0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 22.0 \end{Bmatrix}$$

$${}^3\{v_{O3}\} = {}^3_2[R]\left({}^2\{v_{O2}\} + {}^2\{\omega_2\} \times \begin{Bmatrix} 0.10 \\ 0 \\ 0 \end{Bmatrix} \right)$$

$$= \underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \left(\begin{Bmatrix} 0.246 \\ 0.04325 \\ 0.0 \end{Bmatrix} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 15 \end{Bmatrix} \times \overbrace{\begin{Bmatrix} 0.10 \\ 0 \\ 0 \end{Bmatrix}}^{\text{use } 8}}_{\text{use } 4} \right)}_{\text{use } 12}$$

$$= \begin{Bmatrix} 0.246 \\ 1.54325 \\ 0.0 \end{Bmatrix}$$

$${}^3\{v_{E3}\} = {}^3\{v_{O3}\} + {}^3\{\omega_3\} \times \begin{Bmatrix} {}^3 O_3E \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.246 \\ 1.54325 \\ 0.0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 22.0 \end{Bmatrix} \times \begin{Bmatrix} 0.05 \\ 0.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.246 \\ 1.543 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1.10 \\ 0 \end{Bmatrix}$$

$${}^{3}\{V_{E_3}\} = \begin{Bmatrix} 0.246 \\ 2.643 \\ 0 \end{Bmatrix}$$

$${}^{0}\{V_{E_3}\} = {}^{0}_{3}[R] \begin{Bmatrix} 0.246 \\ 2.643 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2.567 \\ -0.674 \\ 0 \end{Bmatrix}$$

PLANAR ANALYSIS

$$\underline{V}_{O_1} = \underline{0}$$

$$\underline{V}_{O_2} = \cancel{\underline{V}_{O_1}}^{0} + \underline{V}_{O_2 O_1}$$

$$= \omega_1 \times \underline{R}_{O_2 O_1}$$

$$= 5 \times 0.05 \angle(30+90)$$

$$= 0.25 \angle 120^{\circ}$$

PLANAR ANALYSIS

$$\underline{V}_{\theta_3} = \underline{V}_{\theta_2} + \underline{\omega}_3 \times \underline{R}_{\theta_3 O_2}$$

$$= 0.25 \angle 120 + \underbrace{15}_{\omega_2} \times \underbrace{0.1}_{R_{O_3 O_2}} \angle (110+90)$$

$$= 0.25 \angle 120 + 1.5 \angle 200^0$$

$$= 1.563 \angle 190.94 = \begin{Bmatrix} -1.534 \\ -0.296 \\ 0 \end{Bmatrix}$$

$$\underline{V}_{E_3} = \underline{V}_{O_3} + \underline{\omega}_3 \times R_{E_3 O_3}$$

$$= 1.563 \angle 190.94 + 22 \times .05 \angle 200$$

$$= 1.563 \angle 190.94 + 1.1 \angle 200$$

$$= 2.654 \angle 194.68 = \begin{Bmatrix} -2.567 \\ -0.672 \\ 0 \end{Bmatrix}$$

PROBLEM 2

II A three-link manipulator is shown in fig. 2:

$L_1 = 5$ cm, $L_2 = 10$ cm, $L_3 = 5$ cm
$\theta_1 = 30^\circ$; $\theta_2 = ?$ $\theta_3 = ?$
$\dot{\theta}_1 = 5$ rad/s $\dot{\theta}_2 = 10$ rad/s $\dot{d}_3 = 4$ cm/s

Calculate ?

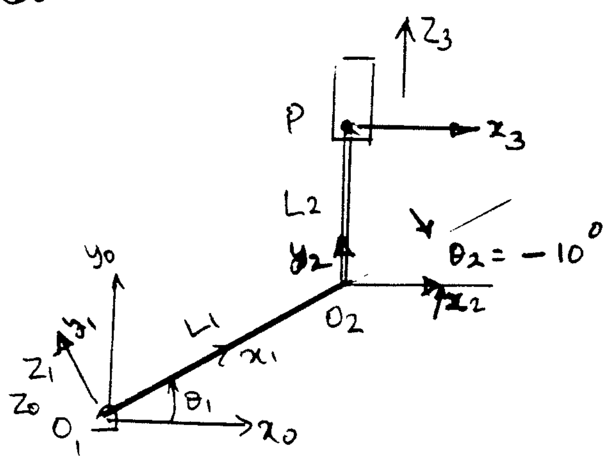


fig. 2

	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	30°
2	0	0.05	0	−10°
3	−90°	0	0.10	0

Table 2. D-H Parameters

SOLUTION

$$ {}^0_1[T] = \begin{bmatrix} 0.8659 & -0.5001 & 0.0 & 0.0 \\ 0.5001 & 0.8659 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \rightarrow \text{Use 2: in Robotics Software} $$

$$
{}^{1}_{2}[T] = \begin{bmatrix} 0.98479 & 0.17371 & 0.0 & 0.05 \\ -0.17371 & 0.98479 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \rightarrow \text{from 2 in Robotics software}
$$

$$
{}^{2}_{3}[T] = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -6.32e-4 & 0.9999 & 0.099 \\ 0 & -0.999 & -6.32e-4 & -6.32e^{-5} \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \rightarrow \text{"}
$$

$$
{}^{0}_{3}[T] = {}^{0}_{1}[T] * {}^{1}_{2}[T] * {}^{2}_{3}[T] \qquad \rightarrow \text{USING PROG 2}
$$

$$
= \begin{bmatrix} 0.93964 & 2.1635E-4 & -3.4215E-1 & 9.0807E-3 \\ 3.4212E-1 & -5.94165E-4 & 9.3964E-1 & 1.18973E-1 \\ 0 & -9.999E-1 & -6.3232E-4 & -6.3232E-5 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

All the above four D-H matrices are got from case no 2 (D-H parameters) in ROBOTICS SOFTWARE

For calculations given below: use <u>12</u> for matrix multiplication, <u>3</u> for inverse & <u>8</u> for cross product of vectors.

<u>Note</u>: (i) [R] is got from [T] by removing the last column and row

(ii) ${}^{i+1}_{i}[R] = {}^{i}_{i+1}[R]^{-1} = {}^{i}_{i+1}[R]^{T}$

Use <u>3</u> to find inverse.

$${}^{1}\{\omega_1\} = {}^{1}_{0}[R]\,{}^{0}\{\cancel{\omega_0}\}^{=0} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{Bmatrix} \qquad \text{EQ (5-45)}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix}$$

$${}^{1}\{v_{01}\} = {}^{1}_{0}[R]\left({}^{0}\{\cancel{v_0}\}^{=0} + {}^{0}\{\cancel{\omega_0}\}^{=0} \times {}^{0}\{\cancel{R_{01}}\}^{=0} \right) \qquad \text{EQ (5-47)}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$${}^{2}\{\omega_2\} = {}^{2}_{1}[R]\,{}^{1}\{\omega_1\} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 0.98479 & -0.17371 & 0.0 \\ 0.17371 & 0.98479 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 15 \end{Bmatrix}$$

$${}^{2}\{v_{02}\} = \underbrace{{}^{2}_{1}[R]}_{\text{use 12 — OPER 3}}\Big(\underbrace{{}^{1}\{\cancel{v_{01}}\}^{=0} + \underbrace{\{\omega_1\} \times {}^{1}\{R_{O_2O_1}\}}_{\text{use 8 — OPER 1}}}_{\text{use 4 — OPER 2}} \Big)$$

$$= \underbrace{\begin{bmatrix} 0.98479 & -0.17371 & 0 \\ 0.17371 & 0.98479 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^{2}_{1}[R]} \left(\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix}}_{{}^{1}_{1}\{\omega\}} \times \underbrace{\begin{Bmatrix} 0.05 \\ 0 \\ 0 \end{Bmatrix}}_{{}^{1}\{R_{O_2O_1}\}} \right)$$

$$= \begin{bmatrix} 0.98479 & -0.17371 & 0 \\ 0.17371 & 0.98479 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.25 \\ 0 \end{Bmatrix}$$

$${}^{2}\{v_{O_2}\} = \begin{Bmatrix} -0.043427 \\ 0.24619 \\ 0.0 \end{Bmatrix}$$

$${}^{3}\{\omega_3\} = {}^{3}_{2}[R]\,{}^{2}\{\omega_2\} + \begin{Bmatrix} 0 \\ 0 \\ \cancel{\dot{\theta}_3} \end{Bmatrix} = 0$$

$\rightarrow$ SLIDING CONNECTION

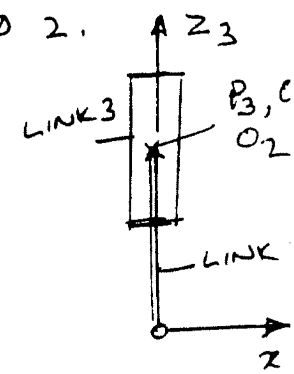
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 15 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -15 \\ 0 \end{Bmatrix}$$ $\rightarrow$ ROTATING CW FROM LOCAL Y_3 DIRECTION

NOTE:

P$_3$ AND O$_3$ ARE SAME POIN

P$_3$ AND P$_2$ ARE COINCIDEN POINTS BUT ON LINKS 3 AND 2.



$${}^{3}\{v_{P_2}\} = {}^{3}_{2}[R]\left({}^{2}\{v_{O_2}\} + {}^{2}\{\omega_2\} \times \begin{Bmatrix} 0 \\ 0.10 \\ 0 \end{Bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{Bmatrix} -0.043427 \\ 0.24619 \\ 0.0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 15 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0.10 \\ 0 \end{Bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -1.543427 \\ 0.24619 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} -1.543427 \\ 0 \\ 0.24619 \end{Bmatrix}$$

$${}^{3}\{v_{P_3}\} = {}^{3}\{v_{O_3}\} = {}^{3}\{v_{P_2}\} + {}^{3}\begin{Bmatrix} 0 \\ 0 \\ \dot{d}_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.543427 \\ 0.0 \\ 0.24619 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \underbrace{0.04}_{\dot{d}_3 = 4\,cm/s} \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.543427 \\ 0 \\ 0.28619 \end{Bmatrix}$$

$${}^{0}\{v_{03}\} = {}^{0}_{3}[R]\,{}^{3}\{v_{03}\}$$

$$= \begin{bmatrix} 0.9396 & 0. & -0.34215 \\ 0.34215 & 0. & 0.9396 \\ 0 & -1.0 & 0.0 \end{bmatrix} \begin{Bmatrix} -1.543427 \\ 0 \\ 0.28619 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.548124 \\ -0.26 \\ 0 \end{Bmatrix}$$

CHAPTER 6

MANIPULATOR DYNAMICS

In this chapter we study the forces required to cause the motion. There are two types of problems.

1) Given the vectors $\{\theta\}$, $\{\dot{\theta}\}$, and $\{\ddot{\theta}\}$ find $\{\tau\}$

2) Given $\{\tau\}$ find $\{\theta\}$, $\{\dot{\theta}\}$ and $\{\ddot{\theta}\}$

The first type of problem is useful in controlling the manipulator. The second type is useful in simulation.

6.3 MASS DISTRIBUTION

We identify an inertia tensor ${}^{A}[I]$ as

$$ {}^{A}[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} $$

where

$$ I_{xx} = \iiint_V (y^2+z^2)\rho \, dv $$

$$ I_{yy} = \iiint_V (x^2+z^2)\rho \, dv $$

$$ I_{xy} = \iiint_V \rho xy \, dv $$

The elements I_{xx}, I_{yy}, I_{zz} are mass moment of inertia and the other terms are mass products of inertia. $^{A}[I]$ is a symmetric matrix, so it has six independent elements. If the reference frame is the principal axes then the product of inertia terms are zero and the matrix is diagonal.

6.4 NEWTON'S EQUATION, EULER'S EQUATION

We will consider each link of a manipulator as a rigid body. If the location of the centre of mass and inertia tensor are given then the mass distribution is completely characterized. Newton's equations and their rotational analogies, Euler's equation, describe how forces, inertias and accelerations relate.

$$\{F\} = m\,\{\dot{v}_c\} \tag{6.29}$$

Where $\{\dot{v}_c\}$ is the acceleration of the mass centre. Equation (6.29) is the Newton's equation of motion and

$$\{N\} = {}^{C}[I]\{\dot{\omega}\} + \{\omega\} \times {}^{C}[I]\{\omega\} \tag{6.30}$$

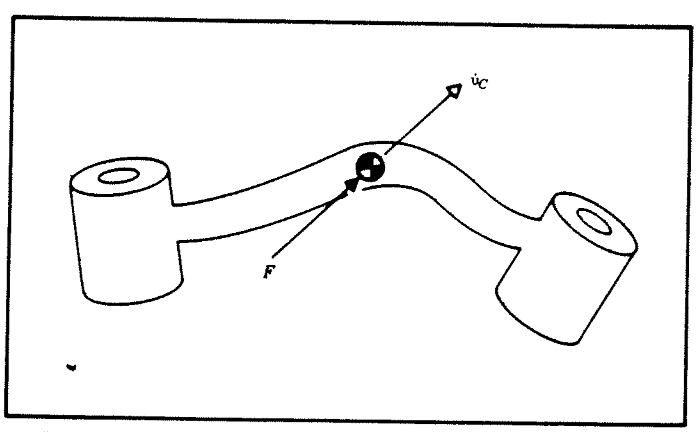


FIGURE 6.3 A force F acting at the center of mass of a body causes the body to accelerate at $\dot{v}_C$.

is the Euler's Equation for this link in frame {C} where the origin is located at the centre of the mass.

6.5 ITERATIVE NEWTON-EULER DYNAMIC FORMULATION

We now consider the problem of computing the torques that correspond to a given trajectory of a manipulator. We assume that we know the position, velocity and acceleration of the joints i.e.,

$$\{\theta\},\ \{\dot{\theta}\},\ \text{and}\ \{\ddot{\theta}\}$$

are known. With this knowledge, and the knowledge of the kinematics and inertia tensors and mass of the links, we can calculate the joint torques required to cause the motion. To use the following algorithm it is also know that ${}^{0}_{0}\{\omega\} = {}^{0}_{0}\{\dot{\omega}\} = \{0\}$, i.e. the base is fixed.

The angular velocity and angular acceleration for a <u>rotational</u> joint are given by

$$^{i+1}\{\omega\}_{i+1} = {}^{i+1}_{i}[R]\,{}^{i}\{\omega\}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\{\hat{Z}_{i+1}\} \tag{6.31}$$

$$^{i+1}\{\dot{\omega}\}_{i+1} = {}^{i+1}_{i}[R]\,{}^{i}\{\dot{\omega}\}_{i} + {}^{i+1}_{i}[R]\,{}^{i}\{\omega\}_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\{\hat{Z}_{i+1}\} + \ddot{\theta}_{i+1}\,{}^{i+1}\{\hat{Z}_{i+1}\} \tag{6.32}$$

The linear equation for each link frame origin in <u>such cases</u> is

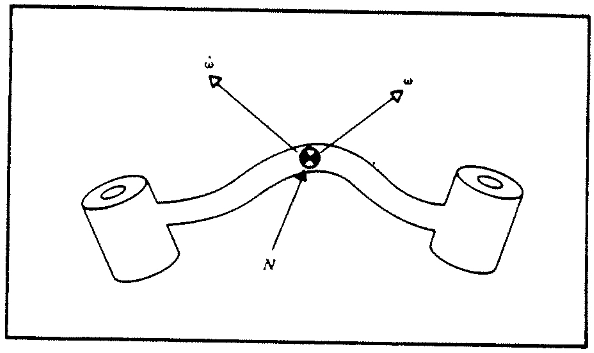


FIGURE 6.4 A moment N is acting on a body, and the body is rotating with velocity ω and accelerating at $\dot{\omega}$.

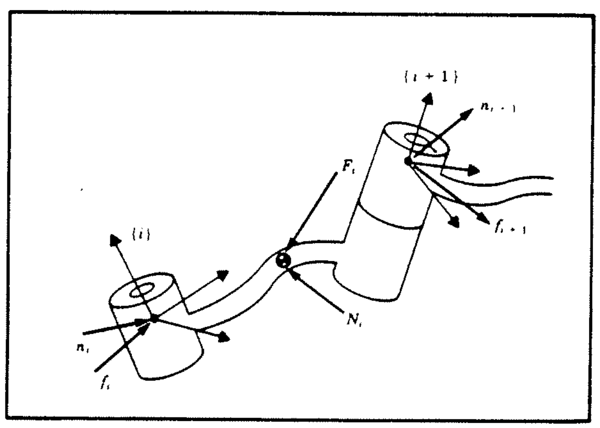


FIGURE 6.5 The force balance. including inertial forces, for a single manipulator link.

given by

$$ {}^{i+1}\{\dot{v}\}_{i+1} = {}^{i+1}{}_{i}[R]({}^{i}\{\dot{\omega}\}_{i} \times {}^{i}\{P_{i+1}\} + {}^{i}\{\omega\}_{i} \times ({}^{i}\{\omega\}_{i} \times {}^{i}\{P_{i+1}\}) + {}^{i}\{\dot{v}\}_{i}) \tag{6.34} $$

on the other hand, if the joint is prismatic, then one has to use

$$ {}^{i+1}\{\dot{\omega}\}_{i+1} = {}^{i+1}{}_{i}[R]\,{}^{i}\{\dot{\omega}\}_{i} \tag{6.33} $$

$$ {}^{i+1}\{\dot{v}\}_{i+1} = {}^{i+1}{}_{i}[R]({}^{i}\{\dot{\omega}\}_{i} \times {}^{i}\{P_{i+1}\}+{}^{i}\{\omega\}_{i} \times ({}^{i}\{\omega\}_{i} \times {}^{i}\{P_{i+1}\})+{}^{i}\{\dot{v}\}_{i}) + 2\,{}^{i+1}\{\omega\}_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\{\hat{Z}_{i+1}\} + \ddot{d}_{i+1}\,{}^{i+1}\{\hat{Z}_{i+1}\} \tag{6.35} $$

The acceleration of the centre of the link in either of the two cases will be

$$ {}^{i}\{\dot{v}\}_{Ci} = {}^{i}\{\dot{\omega}\}_{i} \times {}^{i}\{P_{Ci}\} + {}^{i}\{\omega\}_{i} \times ({}^{i}\{\omega\}_{i} \times {}^{i}\{P_{Ci}\}) + {}^{i}\{\dot{v}\}_{i} \tag{6.36} $$

The inertial force $\{\underline{F}_i\}$ and torque $\{N_i\}$ will be given by

$$ {}^{i}{}_{i}\{F\} = m\,{}^{i}{}_{i}\{\dot{v}_{Ci}\} \tag{6.37} $$

$$ {}^{i}\{N\} = {}^{Ci}[I]\,{}^{i}{}_{i}\{\dot{\omega}\} + ({}^{i}{}_{i}\{\omega\} \times {}^{Ci}[I]\,{}^{i}{}_{i}\{\omega\}) $$

INCLUSION OF GRAVITY FORCES IN THE DYNAMICS ALGORITHM

The effect of gravity on the links can be included quite easily by setting ${}^{0}{}_{0}\{\dot{v}\} = {}^{0}{}_{0}\{g\}$ where g = 9.81. This fictitious upward acceleration causes exactly the same effect on the links as gravity would. This does not introduce additional computational expense.

6.8. THE STRUCTURE OF THE MANIPULATOR DYNAMIC EQUATIONS

THE STATE SPACE EQUATION

It is quite often convenient to express the dynamic equations in the general form as

$$\underset{n\times 1}{\{\tau\}} = \underset{n\times n}{[M(\underline{\theta})]}\{\ddot{\theta}\} + \underset{n\times 1}{\{V(\underline{\theta},\underline{\dot{\theta}})\}} + \underset{n\times 1}{\{G(\underline{\theta})\}} \tag{6.59}$$

Where

$[M(\underline{\theta})]$ is the n×n mass matrix

$\{V(\underline{\theta},\underline{\dot{\theta}})\}$ is an n×1 vector having coriolis and centrifugal terms

and $\{G(\theta)\}$ is an n×1 vector of gravity terms.

For a two link manipulator shown in fig.6.6, these vectors and matrices are

$$[M(\underline{\theta})] = \begin{bmatrix} l_2^2m_2+2l_1l_2m_2c_2+l_1^2(m_1+m_2) & l_2^2m_2+l_1l_2m_2c_2 \\ l_2^2m_2+l_1l_2m_2c_2 & l_2^2m_2 \end{bmatrix} \tag{6.60}$$

$$\{V(\underline{\theta},\underline{\dot{\theta}})\} = \underset{2\times 1}{\begin{Bmatrix} -m_2l_1l_2s_2\dot{\theta}_2^2- 2m_2l_1l_2s_2\dot{\theta}_1\dot{\theta}_2 \\ m_2l_1l_2s_2\dot{\theta}_1^2 \end{Bmatrix}} \tag{6.61}$$

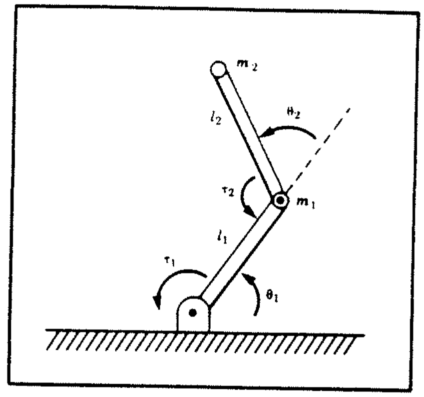


FIGURE 6.6 Two-link with point masses at distal end of links.

$$\{G(\underline{\theta})\} = \underset{2\times 1}{\begin{Bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ \\ m_2 l_2 g c_{12} \end{Bmatrix}} \tag{6.62}$$

THE CONFIGURATION SPACE EQUATION

One can rewrite Eq. (6.59) in the form

$$\{\tau\} = \underset{n\times n}{[M(\underline{\theta})]}\underset{n\times 1}{\{\ddot{\theta}\}} + \overset{\text{coriolis coefficients}}{\underset{n\times(n-1)/2}{[B(\underline{\theta})]}}\{\dot{\underline{\theta}}\;\dot{\underline{\theta}}\} + [C(\underline{\theta})]\{\dot{\underline{\theta}}^2\} + \{G(\underline{\theta})\} \tag{6.63}$$

where

$$\{\dot{\theta}\;\dot{\theta}\} = [\dot{\theta}_1\dot{\theta}_2 \ldots \dot{\theta}_{n-1}\dot{\theta}_n]^T \tag{6.64}$$

$[C(\underline{\theta})]$- centrifugal coefficients

$$\{\dot{\underline{\theta}}^2\} = [\dot{\theta}_1^2\dot{\theta}_2^2 \ldots \dot{\theta}_n^2]^T \tag{6.65}$$

Eq.(6.63) gives a form in which parameters are only a function of joint position. It shows that the dynamics is computation intensive.

6.12. DYNAMIC SIMULATION

Problem $\longrightarrow$ Given $\{\tau\}$, find $\{\theta\}$, $\{\dot{\theta}\}$, $\{\ddot{\theta}\}$.

Given the dynamics equations written in the closed form as in Eq. (6.59), simulation requires solving the dynamic equation for acceleration

$$\{\ddot{\theta}\} = [M]^{-1}\{\tau - V(\underline{\theta},\underline{\dot{\theta}}) - G(\underline{\theta}) - F(\theta,\dot{\theta})\} \tag{6.115}$$

We may then apply any of the several known Numerical Integration Schemes to integrate acceleration to compute future position and velocities.

Given initial conditions of the manipulator, usually in the form

$$\{\theta(0)\} = {}_0\{\theta\}$$

and

$$\{\dot{\theta}(0)\} = \{0.0\}.$$

We numerically integrate Eq. (6.115) forward in time steps of size $\triangle t$.

EULER INTEGRATION SCHEME

At first calculate

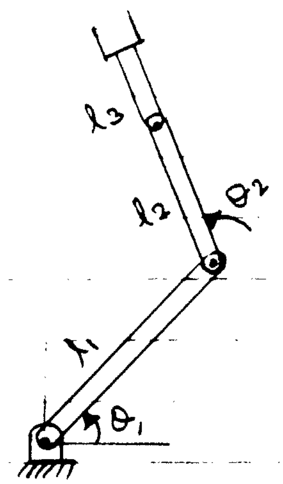
$$\{\dot{\theta}(t+\triangle t)\} = \{\dot{\theta}(t)\} + \{\ddot{\theta}(t)\}\triangle t$$

$$\{\theta(t+\triangle t)\} = \{\theta(t)\} + \{\dot{\theta}(t)\}\triangle t + \tfrac{1}{2}\triangle t^2\{\ddot{\theta}(t)\} \tag{6.117}$$

and then substitute these two $\{\dot{\theta}(t+\Delta t)\}$ and $\{\theta(t+\Delta t)\}$ in Eq.(6.115) to calculate $\{\ddot{\theta}(t+\Delta t)\}$. In this way, the position, velocity, and acceleration caused by a certain torque function can be calculated.

DYNAMIC ANALYSIS

PLANAR - 3 LINK MANIPULATOR :



LINK	l_i (m)	θ_i	$\dot{\theta}_i$ (r/s)	$\ddot{\theta}_i$ (r/s²)	m_i (kg)	TYPE OF CROSS-SECTION	MATERIAL
1	0.3	30°	3.0	4.0	2.965	CIRCULAR	1% Cr-Steel
2	0.25	80°	2.0	5.0	2.4708	CIRCULAR	1% Cr-Steel
3	0.2	0°	4.0	7.0	1.976	CIRCULAR	1% Cr-Steel

Each of the link is assumed to be cylinder of length l_i & radius $R = 0.02$ m

To calculate Mass moments of inertia :

$\rho = 7865$ kg/m³ (1% Cr. Steel)

LINK 1 :

$$m_1 = \pi R^2 L_1 \times \rho$$

$$= \pi \times (0.02)^2 \times 0.3 \times 7865 = 2.965 \text{ kg}$$

$$I_{xx} = \frac{\pi R^4}{2} \times \rho \times L_1$$

$$= \frac{\pi \times (0.02)^4 \times 7865 \times 0.3}{2} = 5.93007 \times 10^{-4} \,\text{kg-m}^2$$

$$I_{yy} = I_{zz} = \frac{1}{12} m_1 (3R^2 + L_1^2)$$

$$= \frac{1}{12} m_2 (3 \times (0.02)^2 + (0.3)^2)$$

$$= 2.2534 \times 10^{-2} \,\text{kg-m}^2$$

Similarly

LINK 2:

$$m_2 = \pi R^2 L_2 \rho = 2.4708 \text{ kg}$$

$$I_{xx} = \frac{\pi R^4}{2} \times \rho \times L_2 = 4.94172 \times 10^{-4} \,\text{kg-m}^2$$

$$I_{yy} = I_{zz} = \frac{1}{12} \times m_2 \times (3R^2 + L_2^2) = 1.13115 \times 10^{-2} \,\text{kg-m}^2$$

LINK 3:

$$m_3 = \pi R^2 L_3 \rho = 1.9767 \text{ kg}$$

$$I_{xx} = \frac{\pi R^4}{2} \times \rho \times L_3 = 3.9534 \times 10^{-4} \,\text{kgm}^2$$

$$I_{yy} = I_{zz} = \frac{1}{12} m_3 (3R^2 + L_3^2) = 6.7886 \times 10^{-3} \,\text{kg-m}^2$$

PLANAR - 3 LINK MANIPULATOR

i LINK	m_{i+1}	$\dot{\theta}_{i+1}$	$\ddot{\theta}_{i+1}$	$^{i}\{\omega\}_i$	$^{i}\{\dot{\omega}\}_i$	$^{i}_{i+1}[R]$	$^{C_{i+1}}_{i+1}[I]$	$^{i}\{P\}_{i+1}$	$^{i}\{P_c\}_{i+1}$	$^{i}\{\dot{v}\}_i$	$^{i}\{\dot{v}_c\}_i$	$^{i}\{F\}_i$	$^{i}\{N\}_i$	$^{i}\{f\}_i$	$^{i}\{n\}_i$
UNITS	kg	r/s	r/s²	r/s	r/s²		kg-m²	m	m	m/s²	m/s²	N	N-m	N	N-m
0	2.965	3	4	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{matrix}0.8659 & -0.5001 & 0\\0.5001 & 0.8659 & 0\\0 & 0 & 1\end{matrix}$	$\begin{matrix}5.93e-4 & 0 & 0\\0 & 2.253e-2 & 0\\0 & 0 & 2.253e^{-2}\end{matrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.15\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\9.81\\0\end{Bmatrix}$	–	–	–	–	–
1	2.471	2	5	$\begin{Bmatrix}0\\0\\3\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\4\end{Bmatrix}$	$\begin{matrix}0.1730 & -0.9849 & 0\\0.9849 & 0.1730 & 0\\0 & 0 & 1\end{matrix}$	$\begin{matrix}4.94e-4 & 0 & 0\\0 & 1.31e-2 & 0\\0 & 0 & 1.31e-2\end{matrix}$	$\begin{Bmatrix}0.3\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.125\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}4.905\\8.494\\0\end{Bmatrix}$	$\begin{Bmatrix}3.556\\9.094\\0\end{Bmatrix}$	$\begin{Bmatrix}10.543\\26.965\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\9.012E-2\end{Bmatrix}$	$\begin{Bmatrix}0.4468\\45.726\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\4.7552\end{Bmatrix}$
2	1.977	4	7	$\begin{Bmatrix}0\\0\\5\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\9\end{Bmatrix}$	$\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}$	$\begin{matrix}3.95e-4 & 0 & 0\\0 & 6.78e-3 & 0\\0 & 0 & 6.78e-3\end{matrix}$	$\begin{Bmatrix}0.25\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.1\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}9.929\\-0.495\\0\end{Bmatrix}$	$\begin{Bmatrix}6.804\\0.630\\0\end{Bmatrix}$	$\begin{Bmatrix}16.812\\1.557\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0.118\end{Bmatrix}$	$\begin{Bmatrix}16.732\\13.190\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\4.992\end{Bmatrix}$
3	–	–	–	$\begin{Bmatrix}0\\0\\9\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\16\end{Bmatrix}$	–	–	$\begin{Bmatrix}0.2\\0\\0\end{Bmatrix}$	–	$\begin{Bmatrix}3.679\\1.755\\0\end{Bmatrix}$	$\begin{Bmatrix}-4.421\\3.355\\0\end{Bmatrix}$	$\begin{Bmatrix}-8.740\\6.633\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0.1085\end{Bmatrix}$	$\begin{Bmatrix}-7.97e-2\\11.633\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\1.7718\end{Bmatrix}$

```
THE JOINT IS ROTATIONAL
STEP 1:
TYPE ROTATIONAL MATRIX [R] NOT ITS TRANSPOSE
 ENTER THE MATRIX (3X3) ROW BY ROW
0.8659 -0.5001 0.
0.5001  0.8659 0.
0.      0.     1.
ENTER OMEGA[I-1] VECTOR ?
0. 0. 0.
ENTER THETADOTDOT (SCALAR) ?
4.
ENTER THETADOT (SCALAR) ?
3.
ANGULAR VELOCITY OMEGA[I] IS
0.0  0.0   3.0

STEP 2:
ENTER OMEGADOT[I-1] VECTOR ?
0. 0. 0.
ANGULAR ACCELERATION OMEGADOT[I] IS
0.0  0.0   4.0

STEP 3:
ENTER  VDOT[I-1] VECTOR ?
0. 9.81 0.
ENTER P[I-1] VECTOR ?
0. 0. 0.
THE VDOT[I] VECTOR IS :
4.905 8.494 0.

STEP 4:
ENTER PC[I] VECTOR ?
0.15 0.0 0.0
THE VDOT[CI] VECTOR IS :
3.556 9.094 0.

STEP 5:
ENTER MASS M ?
2.965
THE FORCE VECTOR F(I) IS :
 10.543 26.965  0.0

STEP 6
ENTER I MATRIX [3X3] ROW BY ROW ?
5.93e-4 0. 0.
0. 2.253e-2 0.
0. 0. 2.253e-2
THE TORQUE VECTOR N(I) IS :
0.0000000E+00  0.0000000E+00  9.012e-2

ENTER
 1: CONTINUE
 2: OVER
1


1
```

(e) (5

```
THE JOINT IS ROTATIONAL
STEP 1:
TYPE ROTATIONAL MATRIX [R] NOT ITS TRANSPOSE
 ENTER THE MATRIX (3X3) ROW BY ROW
0.1730  -0.9849  0.0
0.9849   0.1730  0.0
0.0      0.0     1.0
ENTER OMEGA[I-1] VECTOR ?
0. 0. 3.
ENTER THETADOTDOT (SCALAR) ?
5.
ENTER THETADOT (SCALAR) ?
2.
ANGULAR VELOCITY OMEGA[I] IS
0.0  0.0   5.000000

STEP 2:
ENTER OMEGADOT[I-1] VECTOR ?
0. 0. 4.
ANGULAR ACCELERATION OMEGADOT[I] IS
0.00  0.00   9.000000

STEP 3:
ENTER  VDOT[I-1] VECTOR ?
4.905 8.494 0.
ENTER P[I-1] VECTOR ?
0.3  0. 0.
THE VDOT[I] VECTOR IS :
9.929 -0.495 0.

STEP 4:
ENTER PC[I] VECTOR ?
0.125 0.0 0.0
THE VDOT[CI] VECTOR IS :
6.804 0.630 0.

STEP 5:
ENTER MASS M ?
2.471
THE FORCE VECTOR F(I) IS :
16.812 1.557 0.0

STEP 6
ENTER I MATRIX [3X3] ROW BY ROW ?
4.94E-4  0.           0.
0.         1.31e-2      0.
0.          0.        1.31e-2
THE TORQUE VECTOR N(I) IS :
0.0000000E+00  0.0000000E+00  0.118

ENTER
 1: CONTINUE
 2: OVER
1
```

```
THE JOINT IS ROTATIONAL
STEP 1:
TYPE ROTATIONAL MATRIX [R] NOT ITS TRANSPOSE
 ENTER THE MATRIX (3X3) ROW BY ROW
1.0       0.0      0.0
0.0       1.0      0.0
0.0       0.0      1.0
ENTER OMEGA[I-1] VECTOR ?
0. 0. 5.
ENTER THETADOTDOT (SCALAR) ?
7.
ENTER THETADOT (SCALAR) ?
4.
ANGULAR VELOCITY OMEGA[I] IS
0.0  0.0   9.000000

STEP 2:
ENTER OMEGADOT[I-1] VECTOR ?
0. 0.  9.
ANGULAR ACCELERATION OMEGADOT[I] IS
0.0  0.0   16.0

STEP 3:
ENTER  VDOT[I-1] VECTOR ?
9.929 -0.495 0.00
ENTER P[I-1] VECTOR ?
0.25 0.0 0.0
THE VDOT[I] VECTOR IS :
3.679 1.755  0.0

STEP 4:
ENTER PC[I] VECTOR ?
0.1 0.0 0.0
THE VDOT[CI] VECTOR IS :
-4.421 3.355 0.0

STEP 5:
ENTER MASS M ?
1.977
THE FORCE VECTOR F(I) IS :
-8.740 6.633 0.00

STEP 6
ENTER I MATRIX [3X3] ROW BY ROW ?
3.95E-4  0.          0.
0.         6.78E-3   0.
0.          0.         6.78E-3
THE TORQUE VECTOR N(I) IS :
0.0000000E+00  0.0000000E+00  0.1085

ENTER
 1: CONTINUE
 2: OVER
2
```

(a) 7

```
THE FORWARD RECURSION IS OVER
BACKWARD ITERATION STARTS HERE

STEP 7:
ENTER [R] MATRIX FOR I TO I+1 ?
1. 0. 0.
0. 1. 0.
0. 0. 1.
ENTER VECTOR fo(I+1)?
8.6602 5.0 0.0
ENTER VECTOR F OF LINK I?
-8.740 6.633 0.0
f VECTOR OF LINK I:
-7.97E-2 11.633 0.0

CROSS CHECKING OF FORCE RESULTS:
eqn(6.   )
L.H.S - R.H.S = { 0 0 0}
CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW:
0.0 -4.7E-7 0.

STEP 8

ENTER N(I) VECTOR?
0.0   0.0   0.1085
ENTER n(I+1) VECTOR?
0. 0. 0.
ENTER PC(I) VECTOR?
0.1 0.0 0.0
ENTER P(I) VECTOR?
0.2 0.0 0.0
n(I) vector is ?
 0.0   0.0  1.7718

CROSS CHECKING OF TORQUE RESULTS
eqn(6.   )
L.H.S - R.H.S = {0 0 0}
CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW
0.0 0. 6.705E-8

ENTER
1:CONTINUE
2:OVER
1
```

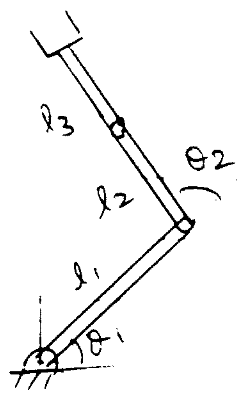
```
THE FORWARD RECURSION IS OVER
BACKWARD ITERATION STARTS HERE

STEP 7:
ENTER [R] MATRIX FOR I TO I+1 ?
1. 0. 0.
0. 1. 0.
0. 0. 1.
ENTER VECTOR fo(I+1)?
-7.97E-2 11.633 0.
ENTER VECTOR F OF LINK I?
16.812 1.557 0.
f VECTOR OF LINK I:
16.732 13.190 0.

CROSS CHECKING OF FORCE RESULTS:
eqn(6.   )
L.H.S - R.H.S = {0 0 0}
CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW:
-4.69E-7 1.19E-7 0.0

STEP 8

ENTER N(I) VECTOR?
0.0   0.0   0.118
ENTER n(I+1) VECTOR?
0. 0. 1.7718
ENTER PC(I) VECTOR?
0.125 0.0 0.0
ENTER P(I) VECTOR?
0.25 0.0 0.0
n(I) vector is ?
 0.0  0.0  4.992

 CROSS CHECKING OF TORQUE RESULTS
 eqn (6.   )
 L.H.S - R.H.S = {0 0 0}
 CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW
0. 0. -3.129E-7

ENTER
1:CONTINUE
2:OVER
1
```

6 9

```
THE FORWARD RECURSION IS OVER
BACKWARD ITERATION STARTS HERE

STEP 7:
ENTER [R] MATRIX FOR I TO I+1 ?
0.1730  -0.9849 0.0
0.9849   0.1730 0.0
0.0      0.0    1.0
ENTER VECTOR fo(I+1)?
16.732 13.190 0.
ENTER VECTOR F OF LINK I?
10.543 26.965 0.0
f VECTOR OF LINK I:
0.4468 45.726 0.

CROSS CHECKING OF FORCE RESULTS:
eqn (6.   )
L.H.S - R.H.S = {0 0 0}
CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW
0. 1.907E-6 0.

STEP 8

ENTER N(I) VECTOR?
0.0   0.0   0. 0. 9.012E-2
ENTER n(I+1) VECTOR?
0. 0. 0. 0. 4.992
ENTER PC(I) VECTOR?
0.15 0.0 0.0
ENTER P(I) VECTOR?
0.3 0.0 0.0
n(I) vector is ?
 0.0   0.0  14.7552

CROSS CHECKING OF TORQUE RESULTS
eqn(6.   )
L.H.S - R.H.S = {0 0 0}
CORRECT IF RESIDUE VECTOR IS ZERO

PRINTING RESIDUE VECTOR BELOW
0. 0. -1.639E-7

ENTER
1:CONTINUE
2:OVER
2
```

Static force Analysis

PLANAR - 3 LINK MANIPULATOR



PARAMETERS	LINK		
	1	2	3
ANGLE (Degrees)	30°	80°	0°
LENGTH (m)	0·3	0·25	0·2.
MASS (kg)	2·965	2·4708	1·976
CROSS-SECTION TYPE	CIRCULAR	CIRCULAR	CIRCULAR
MATERIAL	1% Cr·Steel	1% Cr·Steel	1% Cr·steel
RADIUS (m)	0·02	0·02	0·02.

MASS MOMENTS OF INERTIA calculations are same as that in the Dynamic analysis

STATIC FORCE ANALYSIS

3 LINK PLANAR MANIPULATOR - REVOLUTE JOINT

i LINK	m_{i+1}	$\dot{\theta}_{i+1}$	$\ddot{\theta}_{i+1}$	${}^{i}\{\omega\}_{i}$	${}^{i}\{\dot{\omega}\}_{i}$	${}^{i}_{i+1}[R]$	${}^{ci+1}_{i+1}[I]$	${}^{i}\{P\}_{i+1}$	${}^{i}\{P_c\}_{i+1}$	${}^{i}\{\dot{v}\}_{i}$	${}^{i}\{\dot{v}_c\}_{i}$	${}^{i}\{F\}_{i}$	${}^{i}\{N\}_{i}$	${}^{i}\{f\}_{i}$	${}^{i}\{n\}_{i}$
UNITS	kg	r/s	r/s²	r/s	r/s²		kg m²	m	m	m/s²	m/s²	N	N m	N	N-m.
0	2.965	0	0	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{matrix}0.8659 & -0.5001 & 0\\0.5001 & 0.8659 & 0\\0.0 & 0.0 & 1.0\end{matrix}$	$\begin{matrix}5.93e-4 & 0 & 0\\0 & 2.253e-2 & 0\\0 & 0 & 2.253e-2\end{matrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.15\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\9.81\\0\end{Bmatrix}$	-	-	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	-	-
1	2.471	0	0	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{matrix}0.1730 & -0.9849 & 0\\0.9849 & 0.1730 & 0\\0 & 0 & 1.0\end{matrix}$	$\begin{matrix}4.94e-4 & 0 & 0\\0 & 1.131e-2 & 0\\0 & 0 & 1.131e-2\end{matrix}$	$\begin{Bmatrix}0.3\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.125\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}4.9059\\8.4944\\0\end{Bmatrix}$	$\begin{Bmatrix}4.9059\\8.4944\\0\end{Bmatrix}$	$\begin{Bmatrix}14.5462\\25.1861\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}32.9397\\72.3594\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\17.929\end{Bmatrix}$
2	1.977	0	0	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{matrix}1.0 & 0.0 & 0.0\\0.0 & 1.0 & 0.0\\0.0 & 0.0 & 1.0\end{matrix}$	$\begin{matrix}3.95e-4 & 0 & 0\\0 & 6.78e-3 & 0\\0 & 0 & 6.78e-3\end{matrix}$	$\begin{Bmatrix}0.25\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0.1\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}9.214\\-3.362\\0\end{Bmatrix}$	$\begin{Bmatrix}9.214\\-3.362\\0\end{Bmatrix}$	$\begin{Bmatrix}22.769\\-8.3082\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}49.6452\\-9.9552\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\-1.4502\end{Bmatrix}$
3	-	-	-	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	-	-	$\begin{Bmatrix}0.2\\0\\0\end{Bmatrix}$	-	$\begin{Bmatrix}9.214\\-3.362\\0\end{Bmatrix}$	$\begin{Bmatrix}9.214\\-3.362\\0\end{Bmatrix}$	$\begin{Bmatrix}18.216\\-6.647\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}26.8762\\-1.647\\0\end{Bmatrix}$	$\begin{Bmatrix}0\\0\\0.3353\end{Bmatrix}$

Chapter 9

LINEAR CONTROL OF MANIPULATORS

9.1 INTRODUCTION

We would want to control the wrist motions to take the tool through a specified space curve. To achieve this, we would represent the dynamics of the system by a linear differential equation whose solutions are simpler and known. A linear differential equation has constant coefficients. Since these coefficients are a function of joint variables, the solution will be valid in the neighborhood of a point in space. Since the space curve is made up of large number of points, one has to recompute each of the coefficients and the forces involved, point by point.

$$\tau = M(\theta_d)\ddot{\theta}_d + V(\theta_d \dot{\theta}_d) + G(\theta_d) \qquad \text{Equation (9.1)}$$

STEP 1

In robotic control problems, one carries out the inverse kinematic analysis to calculate

$\theta_d \, \dot{\theta}_d \, \ddot{\theta}_d$ from the specification of the task.

STEP 2

Compute the (τ) by substituting kinematic parameters θ etc. on the right hand side of Equation (9.1).

If our dynamic model was accurate then the end effector would move along the specified path. However, this would not happen. Actually, there would be errors. In an open loop control, one applies the (τ)) vector and no effort is made to correct for the errors.

On the other hand, to achieve greater accuracy, one modifies the applied torque based on the error at the previous instant of time. These errors are used to calculate additional torques, besides the theoretical torque vector. This process of correction is called the feed-back which is dependent upon the actual position, actual velocity of the end effector which is sensed by the sensors, and the corresponding desired position, or desired velocity etc, one can define the errors as:

$$
\begin{aligned}
E &= \theta_d - \theta_d \\
\dot{E} &= \dot{\theta}_d - \dot{\theta}_a
\end{aligned}
\qquad (9.2)
$$

Such a system of torque based control is called closed - loop system. A stable system is one where the errors build up with time.

9.4 CONTROL OF SECOND-ORDER SYSTEMS

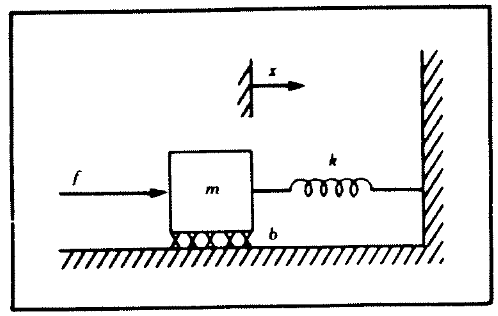


FIGURE 9.6 A damped spring-mass system with an actuator.

A second order system with a forcing function is written as:

$$m\ddot{x} + b\dot{x} + kx = f \qquad (9.34)$$

Suppose we have sensors which can measure and report $x\ ,\ \dot{x}$ etc.

We propose a control law which computes the force which should be applied by the actuator (motor for rotary motion or solenoid for linear motion). This force is proportional to the displacement and velocity errors (by applying in opposite direction). Here the proportionality constaints are k_p and k_v. The minus sign is to compensate for the positive or negative values of x *and* $\dot{x}$

In this case, the resulting differential equation will be

$$m\ddot{x}+bx+kx=-k_p x-k_v x \qquad \text{Equation 9.36}$$

$$m\ddot{x}+(b+k_v)x+(k+k_p)x=0 \qquad \text{Equation 9.37}$$

$$m\ddot{x}+b^1x+k^1x=0 \qquad \text{Equation 9.38}$$

Here we have

$$b^I = b + k_v ;\ k^I = k + k_p$$

Here k_p, k_v are control gains. Often these k_p and k_v are chosen such that we have a critically damped system. These values should not make b^I or k^I negative; in which case if negative the system will become unstable.

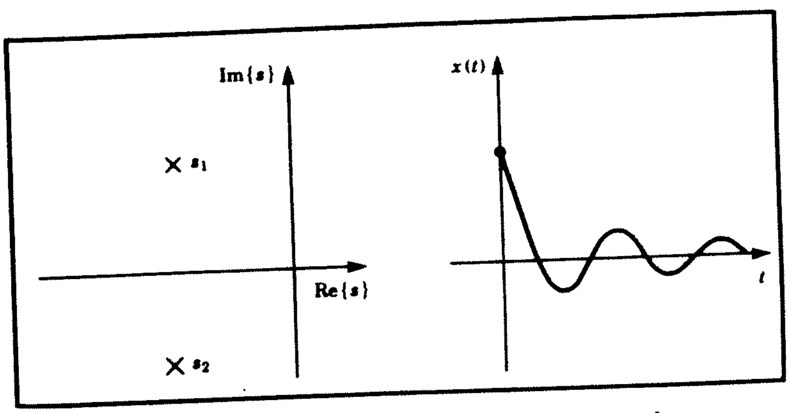


FIGURE 9.4 Root location and response to initial conditions for an underdamped system.

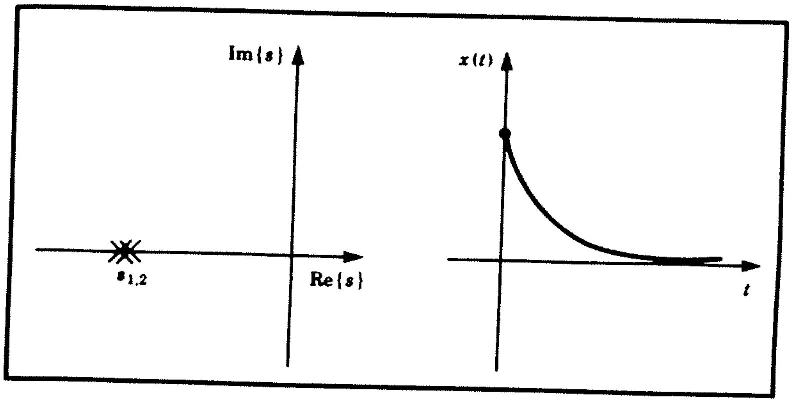


FIGURE 9.5 Root location and response to initial conditions for a critically damped system.

9.6 TRAJECTORY FOLLOWING CONTROL

A trajectory is specified where the subscript d represents the desired value. Defining error as $e = x_d - x$, a servo control law which will cause trajectory following is

$$f^1 = x_d + k_v \dot{e} + k_p e \qquad (9.50)$$

If we combine this equation with the equation of motion of a unit mass equation (9.44), which is

$$\ddot{x} = f^1$$

we get

$$\ddot{x} = \ddot{x}_d + k_v \dot{e} + k_p e \qquad (9.52)$$

This is a second order differential equation where, we can choose the coefficients k_v, and k_p to make the error e, stable.

9.7 DISTURBANCE REJECTION

We have the equation

$$\ddot{e} + k_v \dot{e} + k_p e = 0$$

and , we want to maintain good performance (small errors) even in the presence of some external disturbance or noise.

The error equation in this case is

$$\ddot{e} + k_v \dot{e} + k_p e = f_{dist} \qquad (9.53)$$

STEADY STATE ERROR

Let us consider the simplest case when f_{dist} = constant (like a static force).

The steady state response will also be a constant value

$$k_p e = f_{dist} \qquad (9.55)$$

$$e = f_{dist} / k_p \qquad (9.56)$$

The higher the k_p value, the smaller will be the error.

ADDITION OF AN INTEGRAL TERM

Suppose we have an additional integral term

$$f^1 = \ddot{x}_c + k_v \dot{e} + k_p e + k_i \int e\, dt \tag{9.57}$$

This results in the error equation as

$$e + k_v e + k_p e + k_i \int e\, dt = f_{dist} \tag{9.58}$$

This term has been added to make the steady state error = 0

If e(t) < 0 for t < 0. We can write for t > 0 by differentiating Equation (9.58) which in the case of constant distanbance becomes {here $f_{dist} = 0$}.

$$k_i e = 0$$

$$or \qquad e = 0$$

Here one can solve Equation (9.59) to calculate the error e as a function of time.

The form of Equation (9.57) is called PID (Proportional Integral Derivative Control Law).

IMAGE PROCESSING AND ANALYSIS WITH VISION SYSTEMS

INTRODUCTION

There is a very large body of work associated with vision systems, image processing, and pattern recognition that addresses many different hardware- and software- related topics This information has been accumulated since the 1950s, and with the added interest in the subject from different sectors of the industry and economy, it is growing rapidly. The enormous number of papers published every year indicates that there must be many useful techniques constantly appearing in the literature. At the same time, it also means that a lot of these techniques may be unsuitable for other applications. In this chapter, we will study and discuss some fundamental techniques for image processing and image analysis, with a few examples of routines developed for certain purposes. The chapter does not profess to be a complete survey of all possible vision routines, but only an introduction.. It is recommended that the interested reader continue studying the subject through other references.

The next few sections present some fundamental definitions of terms and basic concepts that we will use throughout the chapter.

IMAGE PROCESSING VERSUS IMAGE ANALYSIS

Image processing relates to the preparation of an image for later analysis and use. Images captured by a camera or a similar technique (e.g., by a scanner) are not necessarily in a form that can be used by image analysis routines. Some may need improvement to reduce noise, others may need to be simplified, and still others may need to be enhanced, altered, segmented, filtered, etc. Image processing is the collection of routines and techniques that improve, simplify, enhance, or otherwise alter an image.

What Is an Image ?

Image analysis is the collection of processes in which a captured image that is prepared by image processing is analyzed in order to extract information about the image and to identify objects or facts about the object or its environment.

TWO- AND THREE-DIMENSIONAL IMAGES

Although all real scenes are three dimensional, images can either be two or three dimensional. Two-dimensional images are used when the depth of the scene or its features need not be determined. As an example, consider defining the surrounding contour or the silhouette of an object. In that case, it will not be necessary to determine the depth of any point on the object. Another example is the use of a vision system for inspection of an integrated circuit board. Here too, there is no need to know the depth relationship between different parts, and since all parts are fixed to a flat plane, no information about the surface is necessary. Thus, a two-dimensional image analysis and inspection will suffice.

Three-dimensional image processing deals with operations that require motion detection, depth measurement, remote sensing, relative positioning, and navigation. CAD/CAM-related operations also require three-dimensional image processing, as do many inspection and object recognition tasks. Other techniques, such as computed tomography (CT) scan, are also three dimensional. In computed tomography, either X-rays or ultrasonic pulses are used to get images of one slice (Section) of the object at a time, and later, all of the images are put together to create' a three-dimensional image of the internal characteristics of the object.

All three-dimensional vision systems share the problem of coping with many- to-one mappings of scenes to images. To extract information from these scenes, image-processing techniques are combined with artificial intelligence techniques. When the system is working in environments with known characteristics (e.g., controlled lighting), it functions with high accuracy and speed. On the contrary, when the environment is unknown or noisy and uncontrolled (e.g., in underwater operations), the systems are not very accurate and require additional processing of the information. Thus, they operate at low speeds. In addition, a three-dimensional coordinate system has to be dealt with.

WHAT IS AN IMAGE ?

An image is a representation of a real scene, either in black and white or in color, and either in print form or in a digital form. Printed images may have been reproduced either by multiple colors and gray scales (as in color print or half-tone print) or by a single ink source. For example, in order to reproduce a photograph with real half tones, one has to use multiple gray inks, which, when combined, produce an image that is somewhat realistic. However, in most print applications, only one color of ink is available (such as black ink on white paper in a newspaper or copier). In that case, all gray levels must be produced by changing the ratio of black versus white areas (the size of the black dot). Imagine that a picture to be printed is divided into small sections.

(a)

(b)

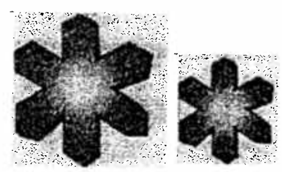


Figure 8.1 Examples of gray intensity creation in printed images. In print, only one color of ink is used, while the ratio of the black to the white area of the pixel is changed to create different gray levels.

In each section, if the ink portion of the section is smaller compared to the white-blank area, the section will look lighter gray. (See examples in Figure 8.1.) If the black ink area is larger compared to the white area, it will look darker gray. By changing the size of the printed dot, many gray levels may be produced, and collectively, a gray-scale picture may be printed.

Unlike printed images, television and digital images are divided into small sections called picture cells, or pixels (in three-dimensional images, they are called volume cells or voxels), where the size of all pixels are the same, while the **intensity of light in each pixel** is varied to create the gray images. Since we deal with digital images, we will always refer to pixels of the same size with varying intensities.

ACQUISITION OF IMAGES

There are two types of vision cameras: analog and digital. Analog cameras are not very common any more, but are still around; they used to be standard at television stations. Digital cameras are much more common and are mostly similar to each other. A video camera is a digital camera with an added videotape recording section. Otherwise, the mechanism of image acquisition is the same as in other cameras that do not record an image. Whether the captured image is analog or digital, in vision systems the image is eventually digitized. In a digital form, all data are binary and are stored in a computer file or memory chip.

The following short discussion is about **analog** and digital cameras and how their images are captured. Although analog cameras are not common anymore, **since the television sets available today are still mostly analog,** understanding the way the camera works will help in understanding how the television set works. Thus, both analog and digital cameras are examined here.

8.5.1 Vidicon Camera

A vidicon camera is an analog camera that transforms an image into an analog electrical signal. The signal, a variable voltage (or current) versus time, can be stored, digitized, broadcast, or reconstructed into an image. Figure 8.2 shows a simple schematic of a vidicon camera. With the use of a lens, the scene is projected onto a screen made up of two layers: a transparent metallic film and a **photoconductive mosaic** that is sensitive to light. The mosaic reacts to the varying intensity of light by varying its resistance. As a result, as the image is projected onto it, the magnitude of the **resistance at each location varies** with the intensity of the light. An electron gun generates and sends a continuous cathode beam (a stream of electrons with a negative

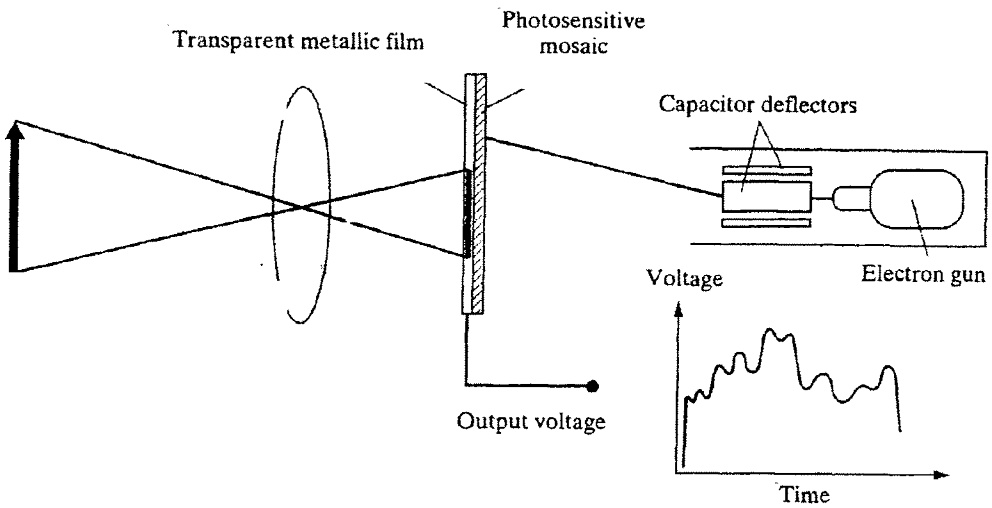


Figure 8.2 Schematic of a vidicon camera.

Voltage Electron gun Time

Acquisition Photosensitive Transparent metallic film mosaic Capacitor deflectors

Figure 8.2 Schematic of a vidicon camera.

---

charge) through two pairs of capacitors (deflectors) that are perpendicular to each other. Depending on the charge on each pair of capacitors, the electron beam is deflected up or down, and left or right, and is projected onto the photoconductive mosaic. At each instant, as the beam of electrons hits the mosaic, the charge is conducted to the metallic film and can be measured at the output port. The voltage measured at the output is V = IR, where I is the current (of the beam of electrons), and R is the resistance of the mosaic at the point of interest.

Now suppose that we routinely change the charges in the two capacitors and thus deflect the beam both sideways and up and down, so as to cause it to scan the mosaic (a process called a raster scan). As the beam scans the image, at each instant the **output** is proportional to the resistance of the mosaic or **proportional to the intensity of the light** on the mosaic. By reading the output voltage continuously, an analog representation of the image can be obtained.

To create moving images in televisions, the image is scanned and reconstructed **30 times a second**. Since human eyes possess a temporary hysteresis effect of about 1/10 second, images changing at 30 times a second are perceived as continuous and thus moving. The image is divided into two 240-line sub-images, interlaced onto each other. Thus, a television image is composed of **480 image lines**, **changing 30 times a second**. In order to return the beam to the top of the mosiac, another **45 lines are used,** creating a total of **525 lines**. In most other countries, 625 lines are the standard. Figure 8.3 depicts a raster scan in a vidicon camera.

If the signal is to be broadcast, it is usually frequency modulated (FM); that is, the frequency of the carrier signal is a function of the amplitude of the signal. The signal is broadcast and is received by a receiver, where it is de-modulated back to the original signal, creating a variable voltage with respect to time. To re-create the image — for example, in a television set — this voltage must be converted back to an image. To do this, the voltage is fed into a cathode-ray tube (CRT) with an electron gun and similar deflecting capacitors, as in a vidicon camera. The intensity of the electron beam in the television is now proportional to the voltage of the signal, and is scanned similar to the way a camera does. In the television set, however, the beam Output voltage

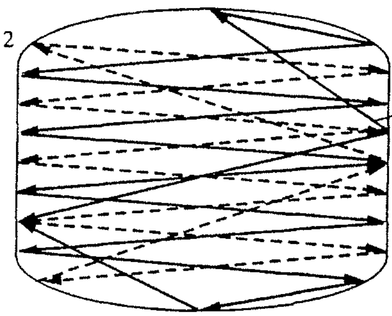


Figure 8.3 A raster scan depiction of a vidicon camera.

Image Processing and Analysis with Vision Systems

Scan 2

Scan 1

Figure 8.3 A raster scan depiction of a vidicon camera.

Return beam is projected onto a phosphorous-based material on the screen, which glows proportionally to the intensity of the beam, thus re-creating the image.

For color images, the projected image is decomposed into the **three colors of red, green, and blue (ROB).** The exact same process is repeated for the three images, and three simultaneous signals are produced and broadcast. In the television set, three electron guns regenerate three simultaneous images in RGB on the screen, except that the screen has three sets of small dots (pixels) that react by glowing in ROB colors and are repeated over the entire screen. All color images in any system are divided into ROB images and are dealt with as three separate images.

If the signal is not to be broadcast, it either is recorded for later use, is digitized (as discussed later), or is fed into a monitor for direct viewing.

8.5.2 Digital Camera

A digital camera is based on solid-state technology. As with other cameras, a set of lenses is used to project the area of interest onto the image area of the camera. The main part of the camera is a solid-state silicon wafer image area that has hundreds of thousands of extremely small photosensitive areas called **photo sites printed on it**. Each small area of the wafer is a pixel. As the image is projected onto the image area, at each pixel location of the wafer a charge is developed that is proportional to the intensity of light at that location. (Thus, a digital camera is also called a charge coupled device, or CCD camera, and a charge integrated device, or CID camera). The collection of charges, if read sequentially, would be a representation of the image pixels. (See Figure 8.4).

The wafer may have as many as 520,000 pixels in an area with dimensions of a fraction of an inch (/16 x ¼). Obviously, it is impossible to have direct wire connections to all of these pixels to measure the charge in each one. To read such an enormous number of pixels, 30 times a second the charges are moved to optically isolated shift registers next to each photo site, are moved down to an output line, and then are read [1,2]. The result is that every thirtieth of a second the charges in all pixel locations are read sequentially and stored or recorded. The output is a discrete representation of the image — a voltage sampled in time — as shown in Figure 8.5(a). Figure 8.5(b) is the CCD element of a VHS camera.

Similar to CCD cameras for visible lights, long-wavelength infrared cameras yield a television like image of the infrared emissions of a scene [3].

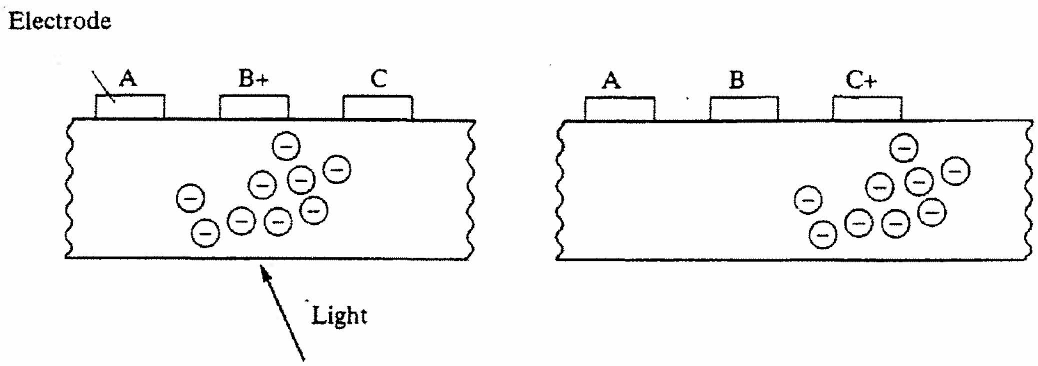
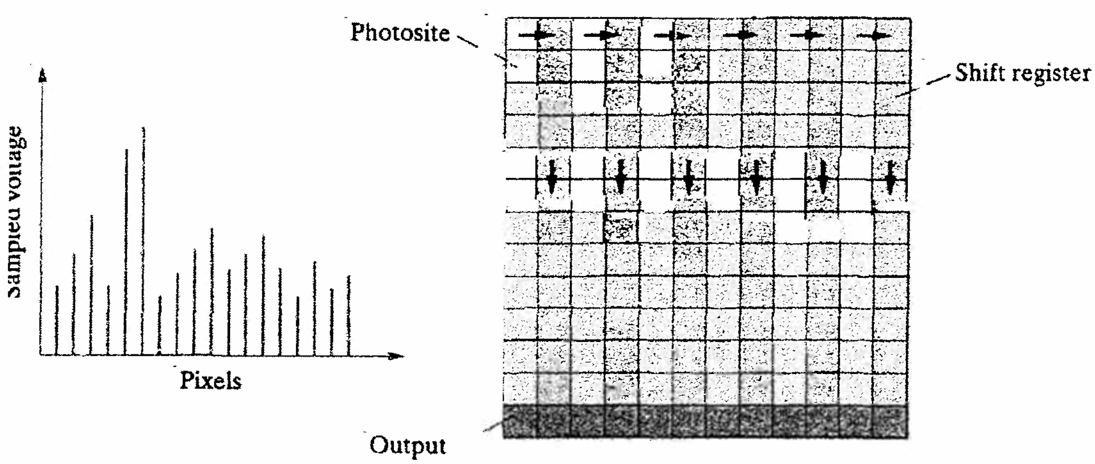
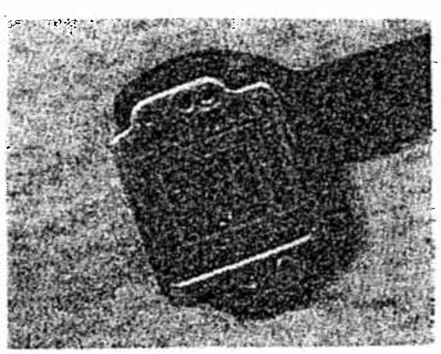


Figure 8.4 Image acquisition with a digital camera involves the development, at each pixel location, of a charge proportional to the light at the pixel. The image is then read by moving the charges to optically isolated shift registers and reading them at a known rate.



(*a*)



(*b*)

Figure 8.5 (a) Image data collection model. (b) The CCD element of a VHS camera

8.6 DIGITAL IMAGES

The sampled voltages from the aforementioned process are first digitized through an analog-to-digital converter (ADC) and then either stored in the computer storage unit in an image format such as TIFF, JPG, Bitmap, etc., or displayed on a monitor. Since it is digitized, the stored information is a collection of U's and l's that represent the intensity of light at each pixel; a digitized image is nothing more than a computer file that contains the collection of these 0's and l's, sequentially stored to **represent the intensity of light at each pixel.** The files can be accessed and read by a program. can be duplicated and manipulated, or can be rewritten in a different form. Vision routines generally access this information, perform some function on the data, and either display the result or store the manipulated result in a new file.

An image that has different gray levels at each pixel location is called a gray image. The gray values are digitized by a digitizer, yielding strings of 0's and l's that are subsequently displayed or stored. **A color image is obtained by superimposing three images of red, green, and blue hues, each with a varying intensity and each equivalent to a gray image (but in a colored state).** Thus, when the image is digitized, it will similarly have strings of U's and l's for each hue. **A binary image is an image such that each pixel is either fully light or fully dark — a 0 or a 1.** To achieve a binary image, in most cases a gray image is converted by using the histogram of the image and a cut-off value called a threshold. A histogram determines the distribution of the different gray levels. One can pick a value that best determines a cutoff level with least distortion and use that value as a threshold to assign 0's (or "off") to **all pixels whose gray levels are below the threshold value and to assign l's (or "on") to all pixels whose gray values are above the threshold**. Changing the threshold will change the binary image. The advantage of a binary image is that it requires far less memory and can be processed much faster than gray or colored images.

8.7 FREQUENCY DOMAIN VS. SPATIAL DOMAIN

Many processes that are used in image processing and analysis are based on the frequency domain or the spatial domain. In frequency-domain processing, the frequency spectrum of the image is used to alter, analyze, or process the image. In this case, the individual pixels and their contents are not used. Instead, **a frequency representation of the whole image** is used for the process. **In spatial-domain** processing, the process is applied to the **individual pixels** of the image. As a result, each pixel is affected directly by the process. However, the two techniques are equally important and powerful and are used for different purposes. Note that although spatial- and frequency-domain techniques are used differently, they are related. For example, suppose that a spatial filter is used to reduce noise in an image. As a result, noise level in the image will be reduced, but at the same time, the frequency spectrum of the image will also be affected, due to the reduction in noise.

The next several sections discuss some fundamental issues about frequency and spatial domains. The discussion, although general, will help us throughout the entire chapter.

8.8 FOURIER TRANSFORM AND FREQUENCY CONTENT OF A SIGNAL

As you may remember from your mathematics or other courses, any periodic signal may be decomposed into a number of sines and cosines of different amplitudes and frequencies as follows:

$$f(t) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t.$$

If you add these sines and cosines together again, you will have reconstructed the original signal. Equation (1) is called a Fourier series, and the collection of different frequencies present in the equation is called the frequency spectrum or frequency content of the signal. Of course, although the signal is in the amplitude—time domain, the frequency spectrum is in the amplitude—frequency domain. To understand this concept better, let’s look at an example.

Consider a signal in the form of a simple sine function like f(t) = sin (t). Since this signal consists of only one frequency with a constant amplitude, if we were to plot the signal in the frequency domain, it would be represented by a single line at the given frequency, as shown in Figure 8.6. Obviously, if we plot the function represented by the arrow in Figure 8.6(b) with the given frequency and amplitude, we will have reconstructed the same sine function. The plots in Figure 8.7 are similar and represent

$$f(t) = \Sigma_{n=1,3\ldots15}(1/\mathrm{n})\ \sin(nt)$$

The frequencies are also plotted in the frequency—amplitude domain. Clearly, when the number of frequencies contained in f(t) increases, the summation becomes closer to a square function.

Theoretically, to reconstruct a square wave from sine functions, an infinite number of sines must be added together. Since a square wave function represents a sharp change, this means that rapid changes (such as an impulse, a pulse, a square wave, or anything else similar to them or modeled by them) have a large number of frequencies. The sharper the change, the higher is the number of frequencies needed to reproduce it. Thus, any video (or other) signal that contains sharp changes (such as noise, high contrasts, or an impulse or step function) or that has

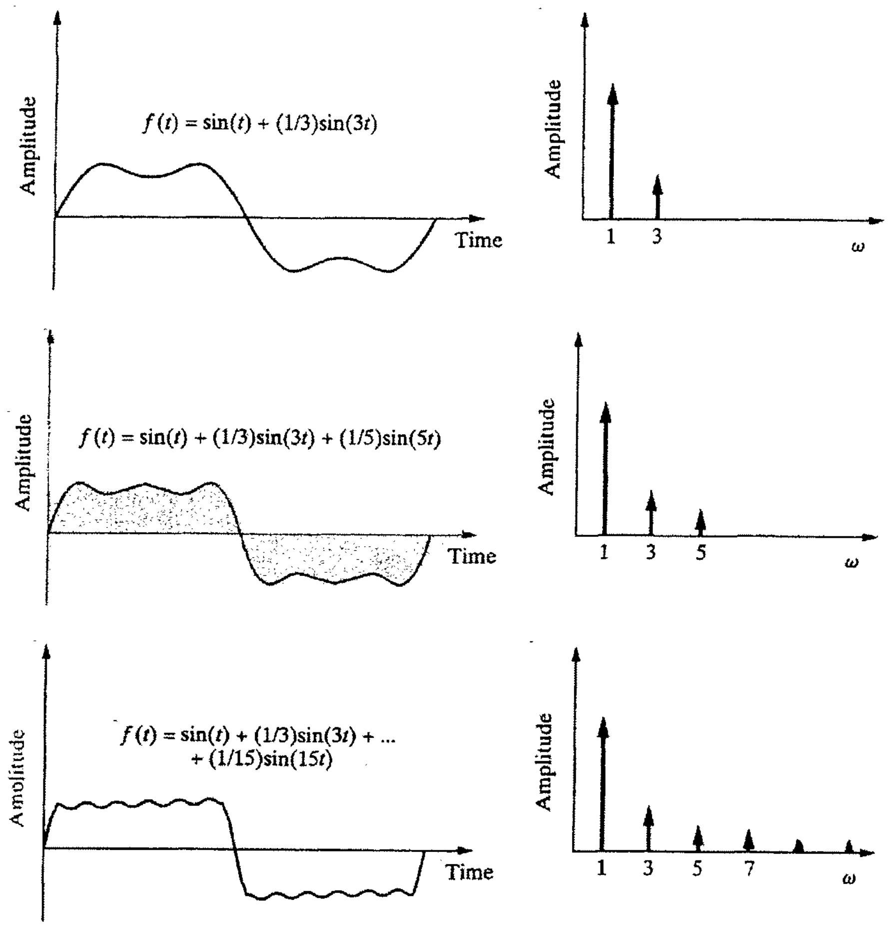


Figure 8.7 Sine functions in the time and frequency domain for a successive set of frequencies. As the number of frequencies increases, the resulting signal becomes closer to a square function.

detailed information (high-resolution signals with fast, varying changes) will have a larger number of frequencies in its frequency spectrum.

A similar analysis can be applied to **non-repeating signals** as well. (**The equation used is a Fourier transform or, sometimes, a fast Fourier Transform, or FFT**) Although we will not discuss the details of the Fourier transform in this book, suffice it to say that an **approximate frequency spectrum of any signal can be found.** Although, theoretically, there will be infinite frequencies in the spectrum, generally, some of the major frequencies within the spectrum will have **larger amplitudes. These major frequencies, or harmonics,** are used in identifying and labeling a signal, including recognizing voices, shapes, objects, etc.

8.9 FREQUENCY CONTENT OF AN IMAGE: NOISE, EDGES

Consider sequentially plotting the gray values of the pixels of an image (on the y axis) against (a) time or (b) pixel location (on the x-axis) as the image is scanned. (See Section 8.5.) The result will be a discrete time plot of varying amplitudes showing the intensity of light at each pixel, as indicated in Figure 8.8. Let's say that we are on the ninth row and are looking at pixels 129—144. The intensity of pixel **136** is very different from the intensities of the pixels around it and may be considered to be **noise**. (Generally, noise is information that does not belong to the surrounding environment.) The intensities of pixels **134 and 141** are also different from the neighboring pixels and may indicate a **transition** between the object and the background; thus, these pixels can be construed as representing the **edges** of the object.

Although we are discussing a discrete (digitized) signal, it may be transformed into a large number of sines and cosines with different amplitudes and frequencies that, if added together, will reconstruct the signal. As discussed earlier, slowly changing signals (such as small changes between succeeding pixel gray values) will require few sines and cosines in order to be reconstructed, and thus have low frequency content. On the other hand, quickly varying signals (such as large differences between pixel gray levels) will require many more frequencies to be reconstructed and thus have high frequency content. Both noises and edges are instances in which one pixel value is very different from the neighboring ones. Thus, **noises and edges create the larger frequencies of a typical frequency spectrum**, whereas slowly varying gray level sets of pixels, representing the object, contribute to the lower frequencies of the spectrum.
However, if a high-frequency signal is passed through a low-pass filter — a filter that allows lower frequencies to go through without much attenuation in amplitude, but that severely attenuates the amplitudes of the higher frequencies in the

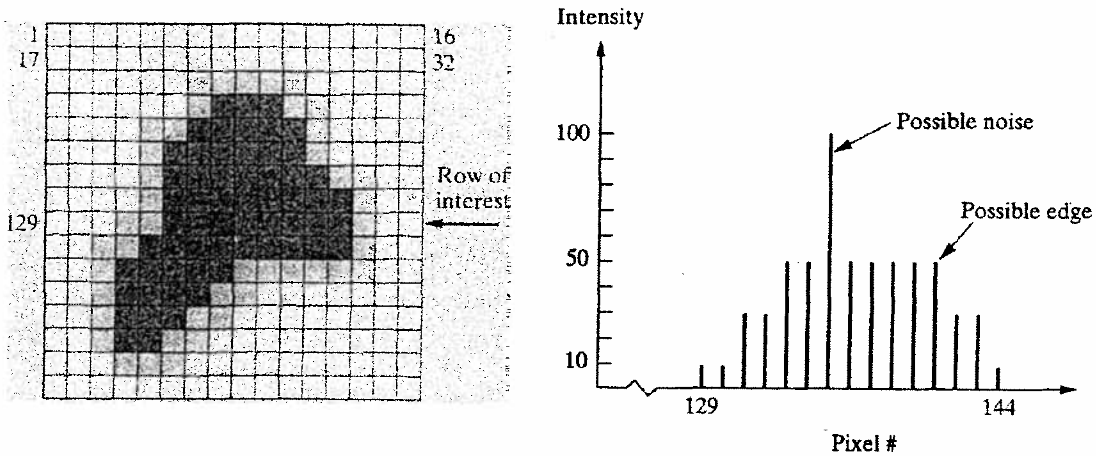


Figure 8.8 Noise and edge information in an intensity diagram of an image. The pixels with intensities that are much different from the intensities of neighboring pixels can be considered to be edges or noise.

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signal — the filter will reduce the influence of all high frequencies, including the noises and edges. This means that, although a **low-pass filter** will reduce noises, it will also reduce the clarity of an image by **attenuating the edges**, thus softening the image throughout. A **high-pass filter**, on the other hand, will increase the apparent effect of higher frequencies by **severely attenuating the low-frequency** amplitudes. In such cases, noises and edges will be left alone, but **slowly changing areas will disappear** from the image.

To see how the Fourier transform can be applied in this case, let's look at the data of Figure 8.8 once again. The grayness level of the pixels of **row 9** is repeated in Figure 8.9(a). A simple first-approximation Fourier transform of the gray values [4] was performed for the first four harmonic frequencies, and then the signal was reconstructed, as shown in Figure 8.9(b). Comparing the two graphs reveals that a digital, discrete signal can be reconstructed, even if its accuracy is dependent on the number of sines and cosines, as well as the method of integration, etc.

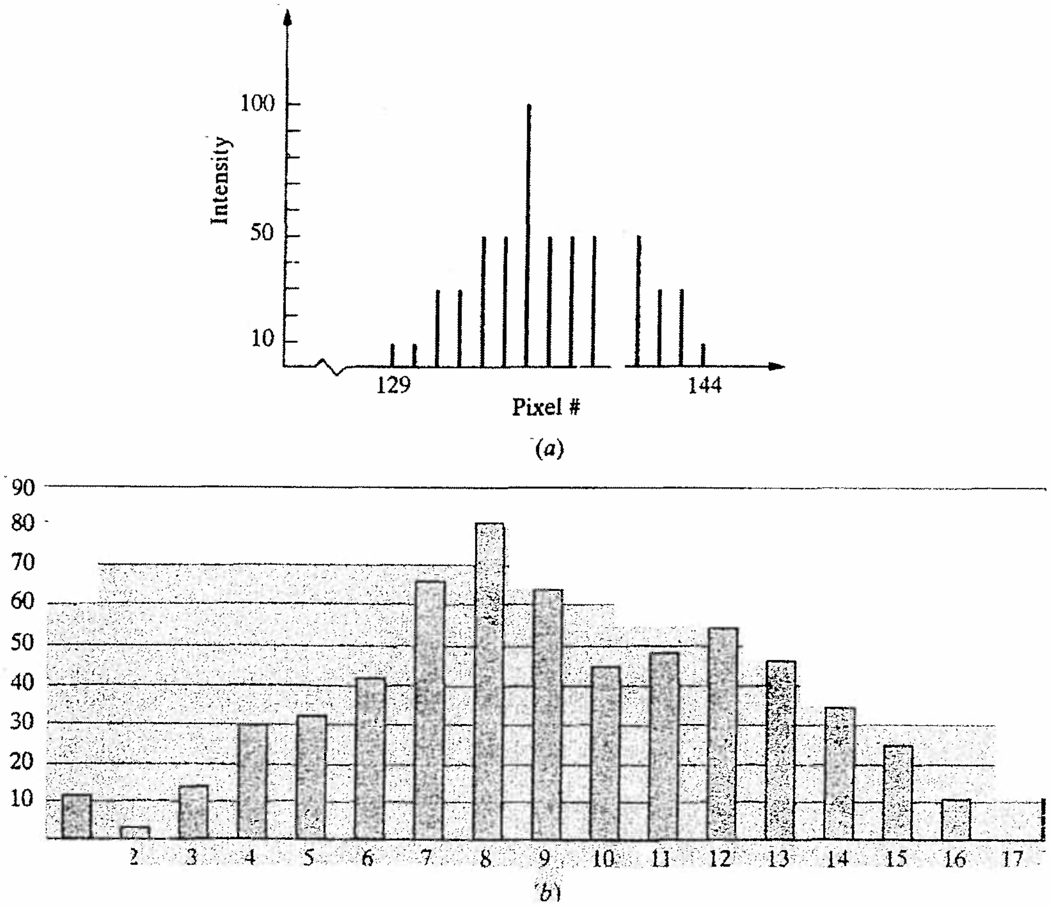

Figure 8.9 (a) Signal. (b) Discrete signal reconstructed from the Fourier transform of the signal in (a), using only four of the first frequencies in the spectrum.

Figure 8.9 (a) Signal. (b) Discrete signal reconstructed from the Fourier transform of the signal in (a), using only four of the first frequencies in the spectrum.