CHAPTER 2

In the manipulations of Robotic manipulators, control is applied on (a) forces and torques, etc. or (b) kinematic parameters such as displacement, velocity, acceleration, etc. of the end effector or the tool. Naturally, all of these have to be defined with respect to some inertial coordinate system - UNIVERSE COORDINATE SYSTEM which is a Cartesian frame.

2.2 DESCRIPTIONS : POSITIONS, ORIENTATIONS AND FRAMES

Once a coordinate system is established, we can locate any point in the space with a 3x1 POSITION VECTOR. In this course, the vector will be written with leading superscript which identifies the frame. For example a vector $\underline{P}$ or {P} will be expressed as

$$^{A}\underline{P} \text{ or } ^{A}\{P\}$$

In terms of the components we can write

$$^{A}\underline{P} = {}^{A}\{P\} = {}^{A}\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

fig

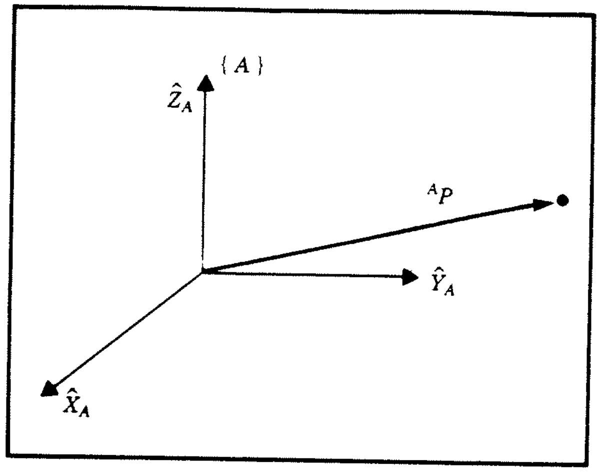


FIGURE 2.1 Vector relative to frame example.

ORIENTATION

The orientations of any of the axes of the system B are shown in the [R] matrices.

$$[R] = \begin{bmatrix} \hat{x}_B.\hat{x}_A & \hat{y}_B.\hat{x}_A & \hat{z}_B.\hat{x}_A \\ \hat{x}_B.\hat{y}_A & \hat{y}_B.\hat{y}_A & \hat{z}_B.\hat{y}_A \\ \hat{x}_B.\hat{z}_A & \hat{y}_B.\hat{z}_A & \hat{z}_B.\hat{z}_A \end{bmatrix} \quad (2.3)$$

$$^{A}_{B}[R] = \begin{bmatrix} ^{A}\hat{x}_B & ^{A}\hat{y}_B & ^{A}\hat{z}_B \\ & & \end{bmatrix} \quad (2.4)$$

Please note that all the vectors are unit vectors. Therefore, their magnitudes are equal to 1. The [R] has its elements, the direction cosines. It is an ORTHONORMAL MATRIX and the following relationships apply

$$ {}^{A}_{B}[R] = {}^{B}_{A}[R]^{-1} = {}^{B}_{A}[R]^{T} \qquad (2.7) $$

fig

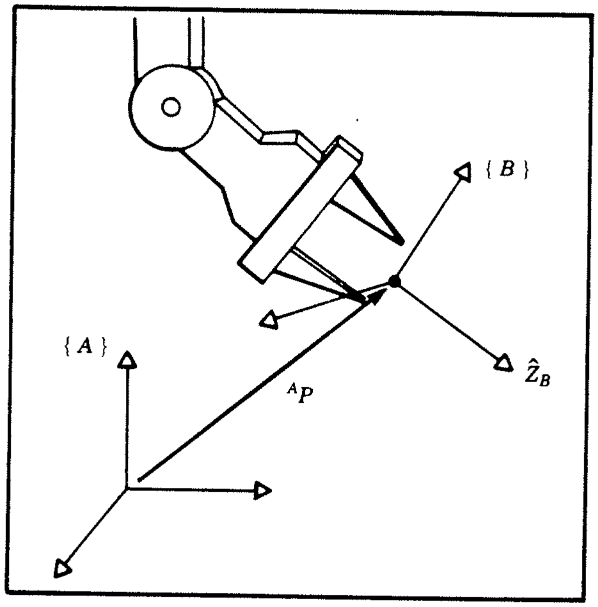


FIGURE 2.2 Locating an object in position and orientation.

DESCRIPTION OF A FRAME

A frame is described with respect to a REFERENCE FRAME by its [R] matrix and the position vector of its origin.

$$ {}^{U}_{A}[T] = [\ {}^{U}_{A}[R] \vdots\ {}^{U}P_{A\ ORIGIN}] $$

fig

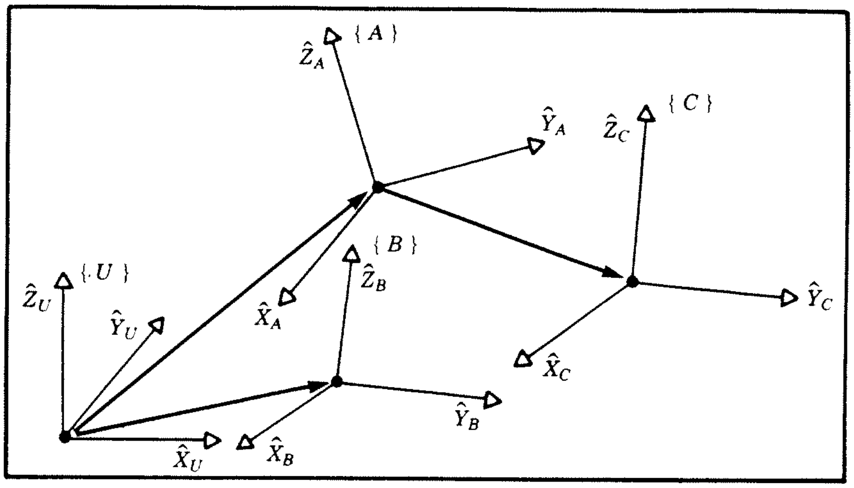


FIGURE 2.3 Example of several frames.

Naturally, ${}^U\underline{P}_{A\ ORIGIN}$ will have 3 components along the each of the axes of the reference frame U.

2.3 MAPPINGS: CHANGING DESCRIPTIONS FROM FRAME TO FRAME

A given point will have different position vector in different frames. In Fig 2.4, frames {A} and {B} have same orientation i.e., the corresponding axes are parallel. In this case {B} is displaced from {A} by a vector ${}^A\underline{P}_{B\ ORIGIN}$.

fig 2.4

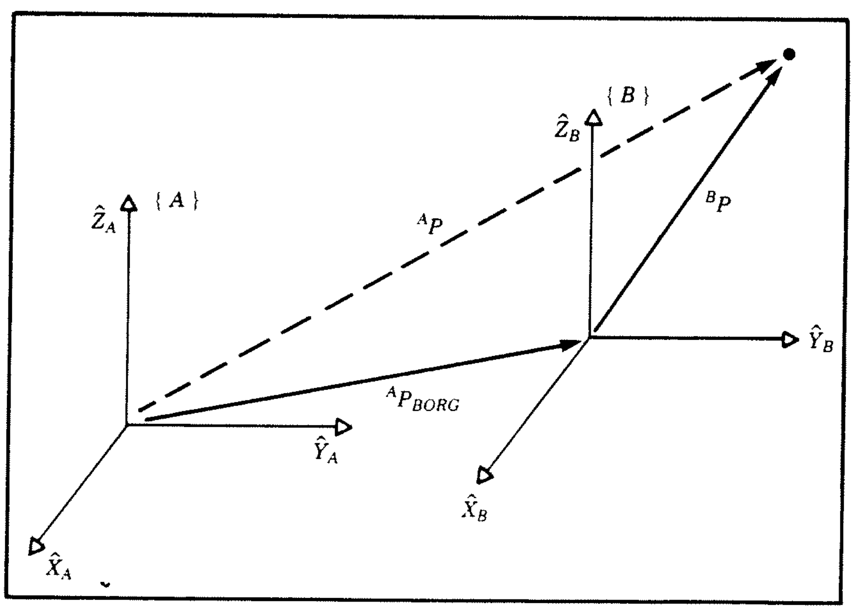


FIGURE 2.4 Translational mapping.

The unit vectors $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$ will be equal in magnitude. Therefore one can add vectors in system {A} and {B}.

$$
\begin{aligned}
{}^{A}\underline{P} &= {}^{B}\underline{P} + {}^{A}\underline{P}_{B\ ORIG} \\
{}^{A}\{P\} &= {}^{B}\{P\} + {}^{A}\{P_{B\ ORIG}\}
\end{aligned}
\qquad (2.9)
$$

Here we have mapped ${}^{B}\{P\}$ into ${}^{A}\{P\}$.

fig

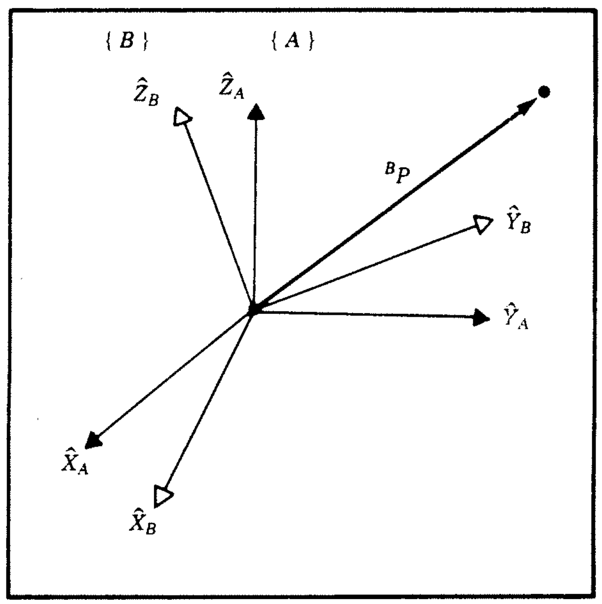


FIGURE 2.5 Rotating the description of a vector.

Here we say that ${}^{A}\{P_{B\ ORIGIN}\}$ defines this mapping. All the information necessary to perform the change in description is contained in this vector. A TRANSLATION VECTOR had sufficient information. On the other hand, as we will see next, when the ORIENTATION of two frames are different then we would need an ADDITIONAL ROTATIONAL MATRIX to completely define the mapping.

MAPPINGS INVOLVING ROTATED FRAMES

Fig 2.5 shows two frames {A} and {B} where the origins of the two systems coincide and there does exist an axis about which the frames {A} can be rotated to make it coincident with the frame {B}. It is also possible to rotate the frame {A} in three successive rotations about $\hat{x}_A$, $\hat{y}_A$, and $\hat{z}_A$ respectively to make it coincident with {B}. We will study these details later on. As shown in fig 2.5, we can write as columns, the direction cosines of $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$ to form a rotation matrix [R] as:

$$
{}^{A}_{B}[R] = \begin{bmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \\ & & \end{bmatrix} \tag{2.11}
$$

$$
= \begin{bmatrix} {}^{B}\hat{x}_A^T \\ {}^{B}\hat{x}_A^T \\ {}^{B}\hat{z}_A^T \end{bmatrix} = {}^{B}_{A}[R]^{-1} = [R]^T
$$

Suppose we are given ${}^{B}\{P\}$ and we want to know ${}^{A}\{P\}$. In compact notation, the solution is

$$ {}^{A}\{P\} = {}^{A}_{B}[R]\ {}^{B}\{P\} \qquad (2.13) $$

where

$$ {}^{A}P_x = {}^{B}\hat{x}_A \cdot {}^{B}P \qquad (2.12) $$

$$ {}^{A}P_y = {}^{B}\hat{y}_A \cdot {}^{B}P $$

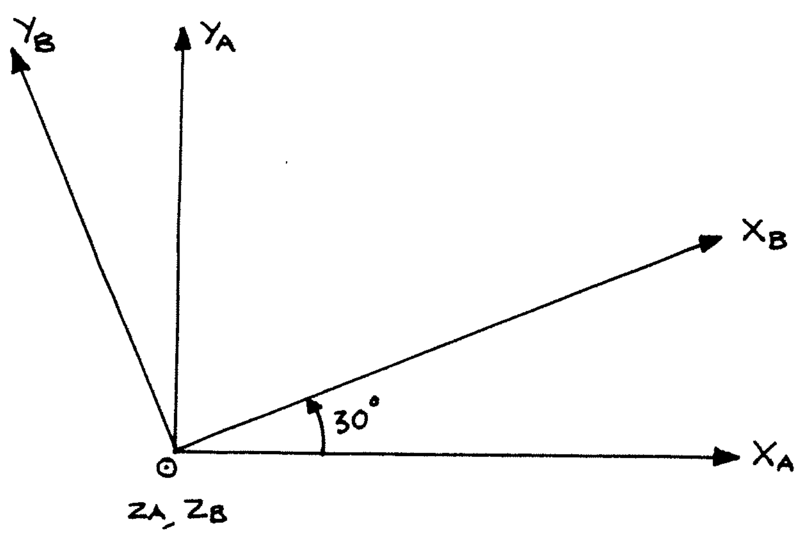
$$ {}^{A}P_z = {}^{B}\hat{z}_A \cdot {}^{B}P $$

Please note that for this mapping, <u>the origins</u> of the two systems were coincident. We should always remember the inverse relationship

$$ {}^{A}_{B}[R] = {}^{B}_{A}[R]^{-1} = {}^{B}_{A}[R]^{T} \qquad (2.10) $$

EXAMPLE

fig



$$
{}^{A}_{B}[R] = \begin{bmatrix} \text{Cos}30 & \text{Cos}120 & \text{Cos}90 \\ \text{Cos}300 & \text{Cos}30 & \text{Cos}90 \\ \text{Cos}270 & \text{Cos}270 & \text{Cos}0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{A}[R] = \begin{bmatrix} \text{Cos}330 & \text{Cos}60 & \text{Cos}90 \\ \text{Cos}240 & \text{Cos}330 & \text{Cos}90 \\ \text{Cos}270 & \text{Cos}270 & \text{Cos}0 \end{bmatrix}
$$

$$
{}^{B}_{A}[R]^{T} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

MAPPING INVOLVING GENERAL FRAMES

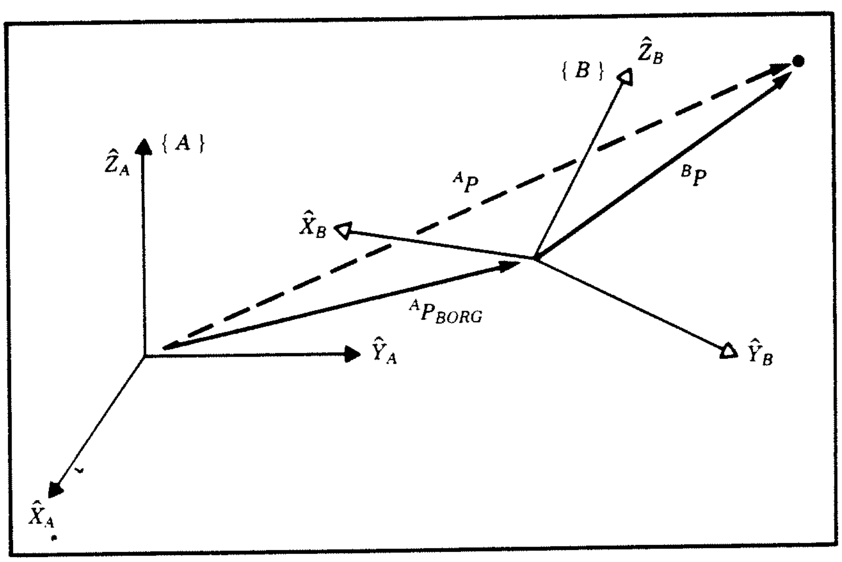


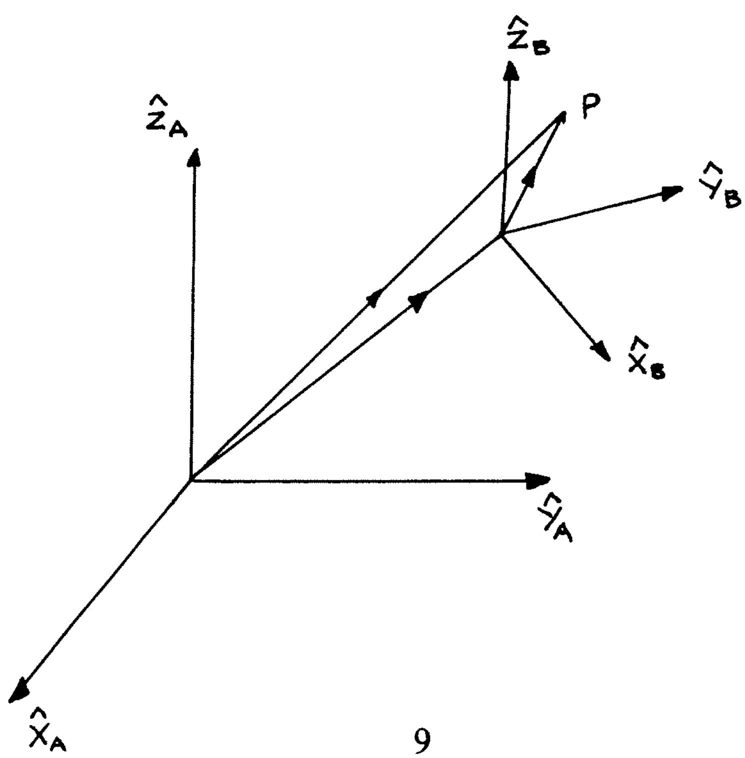
FIGURE 2.7 General transform of a vector.

In fig 2.7, the frame {B} has different orientation as well as the location of its origin is different from that of {A}. We are given the vector ${}^{B}\{P\}$ and we would like to know ${}^{A}\{P\}$. The formula is

$$\underset{4\times 1}{{}^{A}\{P\}} = \underset{4\times 4}{{}^{A}_{B}[T]} \ \underset{4\times 1}{{}^{B}\{P\}} \tag{2.18}$$

$$= \begin{bmatrix} [R] & | & {}^{A}P_{B\ ORIGIN} \\ 3\times 3 & | & 3\times 1 \\ -- & -- & -- \\ [0] & | & 1 \\ 1\times 3 & | & 1\times 1 \end{bmatrix} \begin{Bmatrix} P \\ -- \\ 1 \end{Bmatrix} \tag{2.18a}$$

fig



Suppose for the figure shown we have the following values

$$[R] = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^{A}\{P_{B\ ORIGIN}\} = \begin{Bmatrix} 1 \\ 3 \\ 4 \end{Bmatrix}$$

$$3 \times 3 \qquad\qquad 3 \times 1$$

$${}^{B}\{P\} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ ---- \\ 1 \end{bmatrix}$$

$$3 \times 3 \qquad\qquad 3 \times 1$$

$${}^{A}_{B}[T] = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 1 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$1 \times 3 \qquad\qquad 1 \times 1$$

${}^{A}_{B}[T]$ is made out of 4 sub matrices:

1) The [R] rotation matrix.

2) $\{P_{B\ ORIGIN}\}$ vector.

3) [0] 1×3 null matrix.

4) [1] 1×1 unit matrix.

Now we can multiply the submatrix in Eq.(2.18 a) and obtain

$$^{A}\{P\} = {}^{A}_{B}[R]\ {}^{B}\{P\} + {}^{A}\{P_{B\ ORIGIN}\} = \begin{Bmatrix} 0.866 \\ 5.23 \\ 7.00 \end{Bmatrix}$$

The first term(vector $\{{}^{A}X_1\}$) on the right hand side is nothing but projections of ${}^{B}\{P\}$ along $(\hat{x}_A - \hat{y}_A - \hat{z}_A)$ system. Therefore the matrix ${}^{A}_{B}[R]$ projects a vector in {B} parallel to the coordinate axes of {A}.

The second term ${}^{A}\{P_{B\ ORIGIN}\}$ is already expressed in frame {A}. Now, these two vectors can be added because they are expressed in the same frame.

CONCLUSION

If the vector is expressed in a frame which is (a) oriented differently, and (b) its origin also does not coincide with the reference frame then one has to do two things:

1) Project the vector parallel to the reference axes by pre-multiplying it with [R] matrix.

2) Add the vector joining the two origins but these one also expressed in the reference frame.

$$\begin{bmatrix} {}^{A}\{P\} \\ ----- \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}[R] & | & {}^{A}\{P_{BORIG}\} \\ -- & -- & -- \\ [0] & | & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\{P\} \\ ----- \\ 1 \end{bmatrix} \qquad (2.19)$$

If we expand the lower submatrices, we would get

$$[1] = \underset{1\times 3}{[0]}\ \underset{3\times 1}{{}^{B}\{P\}} + \underset{1\times 1}{[1]}\underset{1\times 1}{[1]}$$

$$= \underset{1\times 1}{[0]} + \underset{1\times 1}{[1]}$$

$$1 = 1$$

2.4 OPERATORS: TRANSLATION, ROTATIONS, TRANSFORMATIONS

The same mathematical forms which were used for mapping can also be used for translation of points, or rotation of vectors or both.

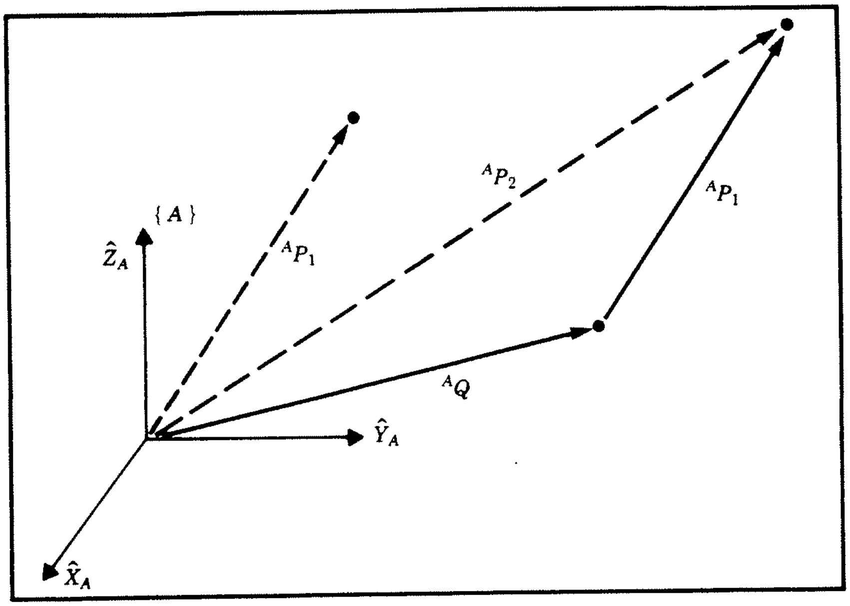


FIGURE 2.9 Translation operator.

A translation moves a point in space a finite distance along a given vector direction. In the fig 2.9, we would like to move a point P_1 along the direction of the vector $^A\{Q\}$. Since there is going to be only translation and no rotation i.e., $\theta = 0°$, we can write the transformation matrix D as

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & | & Q_x \\ 0 & 1 & 0 & | & Q_y \\ 0 & 0 & 1 & | & Q_z \\ -- & -- & -- & -- & -- \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

where the position vector of Q is written in the fourth column. The rotation matrix [R] is a unit matrix. The final position vector $\{P_2\}$ is obtained as

$$\underset{4\times1}{{}^{A}\{P_2\}} = \underset{4\times4}{[D_Q]}\ \underset{4\times1}{{}^{A}\{P_1\}} \tag{2.25}$$

ROTATIONAL OPERATORS

A rotation matrix [R] will rotate a vector by certain angle θ about certain axis in the three dimensional space. While operating on a vector, it pre-multiplies it.

$${}^{A}\{P_2\} = [R(\theta)]\ {}^{A}\{P_1\} \tag{2.27}$$

when $\theta = 0°$, [R] becomes a unit matrix with 1 along its diagonal and 0 elsewhere. [R] rotated about z axis is written as

$$[R_z(\theta)] = \begin{bmatrix} Cos\theta & -Sin\theta & 0 \\ Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.29}$$

The z axis is represented by a direction

$$[0 \quad 0 \quad 1]^T$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can have a general axis in space called 'k' axis and rotation about this axis whose directions are given by

$$^{A}\begin{Bmatrix} k_x \\ k_y \\ k_z \end{Bmatrix}$$

The corresponding rotation matrix is written as equation (2.80)

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}. \qquad (2.80)$$

Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and $^{A}\hat{K} = [k_x \; k_y \; k_z]^T$.

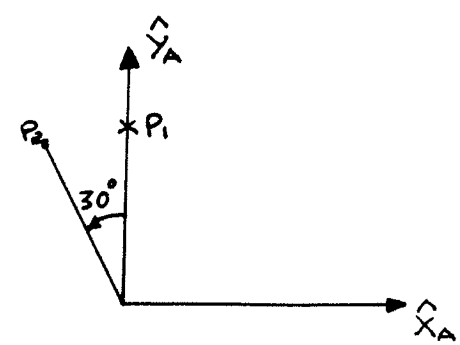
The expressions for $[R_y(\theta)]$ and $[R_x(\theta)]$ are

$$[R_x(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}\theta & -\text{Sin}\theta \\ 0 & \text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$[R_y(\theta)] = \begin{bmatrix} \text{Cos}\theta & 0 & \text{Sin}\theta \\ 0 & 1 & 0 \\ -\text{Sin}\theta & 0 & \text{Cos}\theta \end{bmatrix}$$

EXAMPLE

$$\text{GIVEN: } {}^{A}\{P_1\} = \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$$



Find ${}^{A}\{P_2\}$ which is obtained by rotating ${}^{A}\{P_1\}$ about $\hat{z}_A$ axis by 30°.

SOLUTION:

$$^{A}\{P_2\} = [R_z(30°)]\ ^{A}\{P_1\}$$

$$= \begin{bmatrix} Cos30 & -Sin30 & 0 \\ Sin30 & Cos30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$^{A}\{P_2\} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}$$

TRANSFORMATION OPERATORS

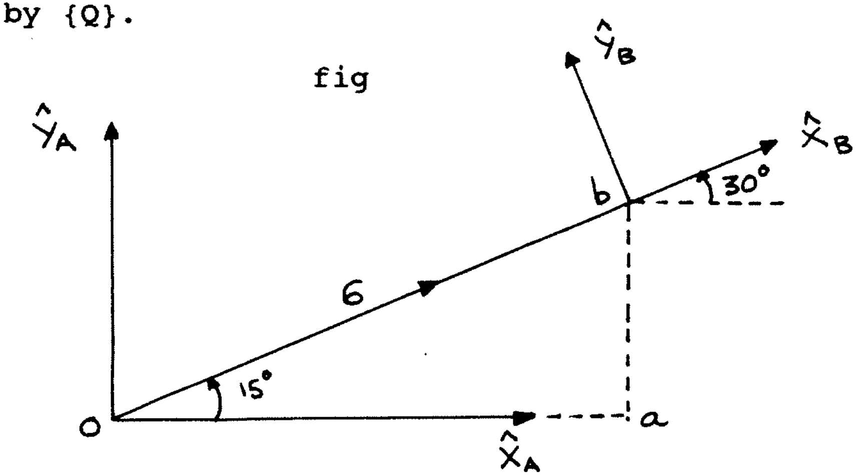
The combined operations of rotation and a translation are done using transformation operator [T] which has [R] and {Q} which are rotational and translational components as its sub-matrices.

$$[T] = \begin{bmatrix} [R] & | & \{Q\} \\ -- & -- & -- \\ [0] & | & 1 \end{bmatrix}$$

AN IMPORTANT THEOREM

The transform [T] which rotates by [R] and translates by {Q} is the same as the transform which describes a rotated frame by [R] and translated by {Q}.

fig



$$[T] = \begin{bmatrix} [R_z(30°) & | & o\ a \\ & | & a\ b \\ & | & 0 \\ -- & -- & -- \\ 0\ 0\ 0 & | & 1 \end{bmatrix}$$

2.8 MORE ON REPRESENTATION OF ORIENTATION

The rotation matrices are special in that all columns are mutually orthogonal which means their dot products with other are equal to zero. Furthermore, the determinant is always equal to +1. They are called <u>proper orthonormal matrices</u>. Proper orthonormal matrices have determinant = +1, and the non proper have equal to

-1.

Next question is, what or how many independent parameters are there in 3x3 rotation matrices which has 9 elements. The answer comes from Cayley's formula for orthonormal matrices which states that for every rotation matrix [R] there exists a skew-symmetric matrix, [S], such that

$$[R] = [[I_3] - [S]]^{-1} [[I_3] + [S]] \tag{2.56}$$

Where $[I_3]$ is an identity matrix and [S] is given by

$$[S] = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix} \tag{2.57}$$

(Skew Symmetric)

One can see that [S] in Equation (2.57) above has only three independent parameters. If we see the Equation (2.56), we see that $[I_3]$ being an identity matrix is completely known; therefore, the right hand side contains only three unknowns or three independent parameters. It shows that the left hand side of this equation must also contain only three independent parameters.

The other way would be to express [R] as three columns as

$$[R] = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

where each of $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors. Then we should also have the following equations of constraints:

$$|\hat{x}| = 1$$

$$|\hat{y}| = 1$$

$$|\hat{z}| = 1 \qquad (2.59)$$

$$\hat{x} \cdot \hat{y} = 0$$

$$\hat{x} \cdot \hat{z} = 0$$

$$\hat{y} \cdot \hat{z} = 0$$

To obtain the 9 elements of [R], we should have 9 equations which are subject to 6 equations of constraints. Therefore, there are only 3 independent parameters.

One should also remember that the products of rotation matrices are not commutative i.e.,

$$^{A}_{B}[R]\ ^{B}_{C}[R] \neq\ ^{B}_{C}[R]\ ^{A}_{B}[R]$$

EXAMPLE:

$$\text{GIVEN} \quad {}^{A}_{B}[R] = \begin{bmatrix} 0.866 & -0.5 & 0.0 \\ 0.5 & 0.866 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$${}^{B}_{C}[R] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.866 & -0.5 \\ 0.0 & 0.5 & 0.866 \end{bmatrix}$$

$${}^{A}_{B}[R]\ {}^{B}_{C}[R] = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.5 & 0.75 & -0.43 \\ 0.0 & 0.5 & 0.87 \end{bmatrix}$$

$${}^{B}_{C}[R]\ {}^{A}_{B}[R] = \begin{bmatrix} 0.87 & -0.5 & 0.0 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix} \qquad (2.62)$$

In view of the fact that one can represent [R] by three independent parameters, there are representations which require only three independent parameters and are discussed below:

1 X-Y-Z FIXED ANGLES

Here, we are given the Reference Frame {A} and we have to specify the {B}.

fig

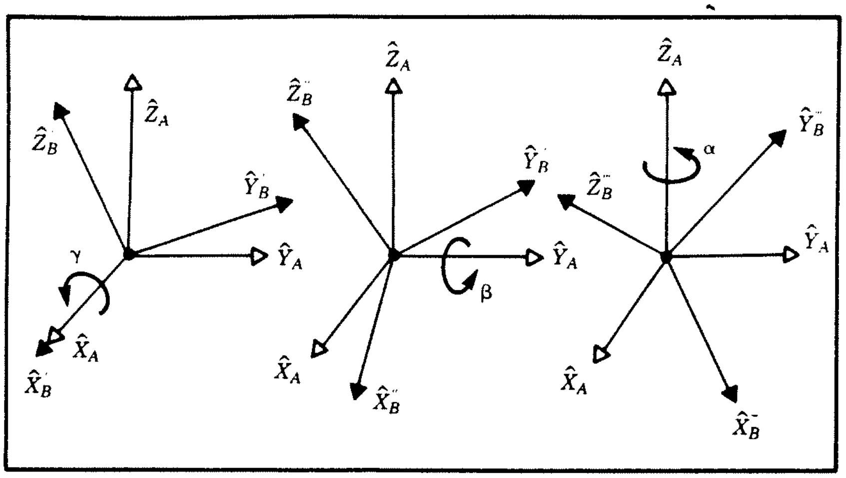


FIGURE 2.17 X-Y-Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

We start with a frame coincident with {A} and rotate this coincident but separate frame about various axes of {A}.

1) Rotate {B} about $\hat{x}_A$ by an angle γ.

2) Then rotate it about $\hat{y}_A$ by an angle β.

3) Finally, rotate it about $\hat{z}_A$ by an angle α.

It should be noted here that all the rotations were performed about the fixed or the Reference Axis. Representing the final matrix as

$$^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)]$$

the relationship between the various individual and the final matrix is written as

$$ {}^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)] = [R_z(\alpha)][R_y(\beta)][R_x(\gamma)] $$

In the equation above on the right hand side, the matrices have been pre-multiplied i.e., the rotation about the Y axis was performed after the X axis; so the rotation matrix corresponding to the Y axis rotations are pre-multiplied. It is an IMPORTANT RULE. Now we are in a position to write the complete matrices which are

$$ {}^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha)] = \underset{3}{\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{2}{\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}} \underset{1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}} $$

$$ {}^{A}_{B}[R_{xyz}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2.64} $$

$$ = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.65} $$

If we want to determine α, β, and γ from the matrices given in equation (2.65) then we can use the following formulas in the GIVEN SEQUENCE:

$$ \begin{aligned} \beta &= \mathrm{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \\ \alpha &= \mathrm{Atan2}(r_{21}/c\beta,\ r_{11}/c\beta) \\ \gamma &= \mathrm{Atan2}(r_{32}/c\beta,\ r_{33}/c\beta) \end{aligned} \tag{2.66} $$

where $\mathrm{Atan2}(y,x) = \tan^{-1}(y/x)$. Here signs of both x and y are used. It is 4 quadrant arc tangent function.

Z - Y - X EULER ANGLES

This involves rotations about {B} of the system B as follows:

1) Start with a frame {B} coincident with {A}, and rotate about $\hat{z}_B$ by an angle as shown in fig 2.18.

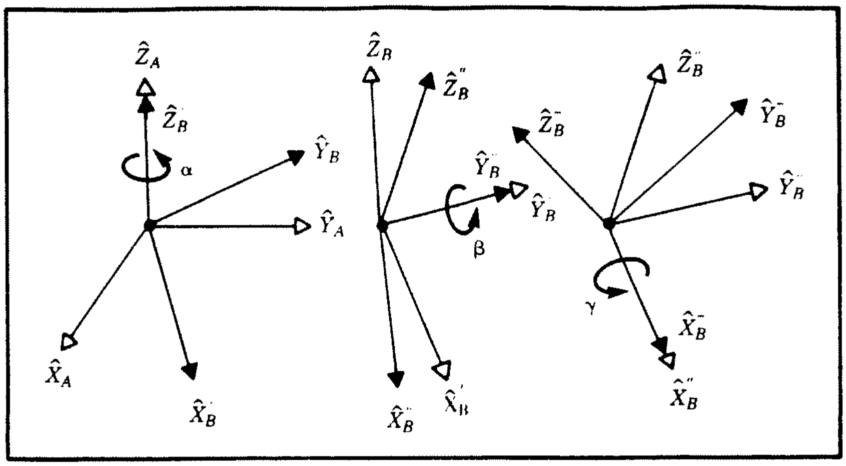


FIGURE 2.18 Z-Y-X Euler angles.

2) Then rotate about $\hat{Y}_B$ by an angle β.

3) Then rotate about $\hat{X}_B$ by an angle γ.

The final orientation matrix in this case will be

$$ {}^{A}_{B}[R_{z'y'x'}] = \underset{1}{[R_z(\alpha)]}\ \underset{2}{[R_y(\beta)]}\ \underset{3}{[R_x(\gamma)]} $$

$$ {}^{A}_{B}[R_{z'y'x'}] = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \tag{2.70} $$

$$ = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2.71} $$

Z - Y - Z EULER ANGLES

In this case, the final expression is

$$
{}^{A}_{B}[R_{\dot{z}\dot{y}\dot{z}}] = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} \quad (2.72)
$$

The formulas for extracting α, β, and γ from the matrix on the right hand side of equation (2.72) are

$$
\begin{aligned}
\beta &= \text{Atan2}((r_{31}^{2} + r_{32}^{2})^{1/2}, r_{33}) \\
\alpha &= \text{Atan2}(r_{23}/s\beta \;,\; r_{13}/s\beta) \qquad (2.74) \\
\gamma &= \text{Atan2}(r_{32}/s\beta \;,\; -r_{31}/s\beta)
\end{aligned}
$$

EQUIVALENT ANGLE AXIS

Instead of three successive rotations in these three cases, it is also possible to rotate about an axis in space, only once to reach to the final orientation.

Eq(2.80)

$$
R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}. \quad (2.80)
$$

Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^{A}\hat{K} = [k_x \; k_y \; k_z]^T$.

where $c\theta$ = Cos θ, $s\theta$ = Sin θ, $v\theta$ = 1 - Cos θ and ${}^{A}\hat{K} = [K_x \; K_y \; K_z]^T$. If the matrix [R] is given and one wants to find out θ and $\hat{K}$, then one has to use the formulas

$$\theta = ACos\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right] \qquad (2.81)$$

$$\hat{k} = \frac{1}{2Sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (2.82)$$

In Equation (2.81), θ should lie between 0 and 180° which is obvious from the Fig 2.19

fig

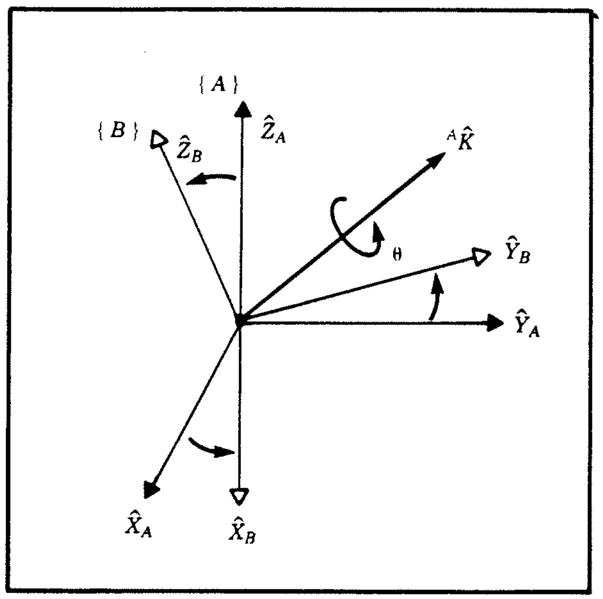


FIGURE 2.19 Equivalent angle-axis representation.

It would amount to a maximum of one complete rotation about the $\hat{K}$ axis.

TRANSFORMATION OF FREE VECTORS

So far we have discussed only the transformation of position vectors. However, there are other kinds of vectors such as velocity, force etc. These are transformed differently using only the rotation matrices.

1) Two vectors are said to be equal if they have (a) same dimensions, (b) same magnitude, and (c) same direction.

2) Two vectors are equivalent in certain capacity if each produces the very same effect in this capacity.

3) Vectors which are not equal may produce equivalent effects.

4) A line vector is one which has dependence on line of action besides having magnitude and direction. Force vector is an example.

5) A free vector is one which may be positioned anywhere in space without loss or change of meaning provided that magnitude and direction are preserved.

An example of this is a Moment Vector. Suppose we have a moment vector in frame B denoted by ${}^{B}\{N\}$. This vector in frame A will be

$$ {}^{A}\{N\} = {}^{A}_{B}[R]\ {}^{B}\{N\} \tag{2.93} $$

Similar relationships can be written about the velocity vector also

$$ {}^{A}\{V\} = {}^{A}_{B}[R]\ {}^{B}\{V\} \tag{2.94} $$