CHAPTER 4

INVERSE MANIPULATOR KINEMATICS

In the last chapter we considered the problem of calculating the position and orientation of the tool relative to the user's workstation given the joint angles of the manipulator. Of course, the other design parameters such as α_i, a_i, and d_i etc. are also known. However, in this chapter we will attempt and solve a more DIFFICULT problem of solving for the joint angles, given the position and orientation of the end effector. The reason it is more difficult is that the equations involved are many and also, they are nonlinear in nature which is quite clear by inspecting Eq. below. Here the matrix is

Eq(3.14)

$$ {}^{0}_{6}T = {}^{0}_{1}T \, {}^{1}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

where

$$ r_{11} = c_1 \left[c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6 \right] + s_1 (s_4 c_5 c_6 + c_4 s_6), $$
$$ r_{21} = s_1 \left[c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6 \right] - c_1 (s_4 c_5 c_6 + c_4 s_6), $$
$$ r_{31} = -s_{23}(c_4 c_5 c_6 - s_4 s_6) - c_{23} s_5 c_6, $$

$$ r_{12} = c_1 \left[c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6 \right] + s_1 (c_4 c_6 - s_4 c_5 s_6), $$
$$ r_{22} = s_1 \left[c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6 \right] - c_1 (c_4 c_6 - s_4 c_5 s_6), $$
$$ r_{32} = -s_{23}(-c_4 c_5 s_6 - s_4 c_6) + c_{23} s_5 s_6, $$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$$
$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$$
$$r_{33} = s_{23}c_4s_5 - c_{23}c_5,$$

$$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1,$$
$$p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1,$$
$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}. \quad (3.14)$$

$$C_{23} = C_2C_3 - S_2S_3 = Cos(\theta_2 + \theta_3)$$
$$S_{23} = C_2S_3 + S_2C_3 = Sin(\theta_2 + \theta_3)$$
$$S_2 = Sin\theta_2$$
$$C_2 = Cos\theta_2$$

$$ {}^{S}_{T}[T] = \begin{bmatrix} r_{11} & \cdot & \cdot & \cdot & p_x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} $$

is given. Naturally, the left hand side of Equation (3.14) is known but all θ_i, i = 1...6 are unknown. Since this set of simultaneous transcendental equations has more than one solutions, an obvious conclusion is that a given end effector orientation and position can be attained in more than one ways.

On the other hand, in certain situations when it is beyond the reach of the manipulator, then no solution can be found. It simply means that the location and the orientation are not in the WORKSPACE of the manipulator.

A DEXTROUS WORKSPACE is that volume of space where the manipulator can reach with all orientations.

A REACHABLE WORKSPACE is one where the manipulator can reach with at least one orientation.

Clearly, a dextrous workspace is a subset of the reachable workspace.

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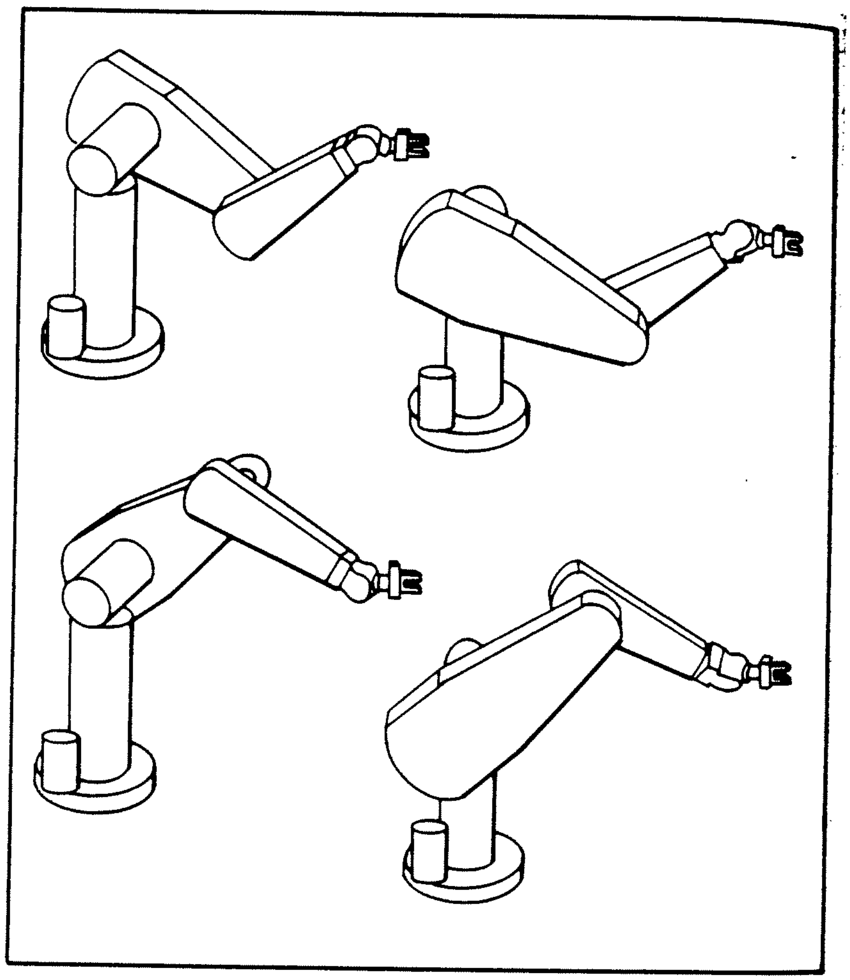

FIGURE 4.4 Four solutions of the PUMA 560.

METHOD OF INVERSE SOLUTION

There are no general methods available to solve these nonlinear set of equations. The solution methods are divided into two categories: (a) closed form, and (b) numerical.

Closed form solutions are preferred because the computational times are negligible as compared to the numerical solution (a) Newton Raphson Technique, or (b) Optimization Techniques.

Within the class of closed form solution are: (a) algebraic method, and (b) geometric method.

A major recent result is that all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series of chain are now solvable(Numerical Solution). In the practical design of manipulators a closed form solution is a must.

A sufficient condition that a manipulator with six revolute joints will have a closed form solution is that three neighbouring joint axes intersect at a point. Almost every manipulator with six degrees of freedom built today has three axes intersecting. For example, axes 4,5, and 6 of the PUMA 560 intersect.

4.3 The Motion of Manipulator Subspace When $n < 6$

The set of reachable goal frames for a given manipulator constitutes its reachable workspace. For example a description of the subspace for a three link manipulator (Planar) is given by

$$ {}^{B}_{W}[T] = \begin{bmatrix} c\phi & -s\phi & 0.0 & x \\ s\phi & c\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} $$

where x, and y give the position of the wrist and Φ, the orientation. As x,y, and Φ are assigned arbitrary values, the subspace is generated.

4.4 Algebraic Verses Geometric Solutions

For the three link manipulator the inverse solution (only a few steps) are:

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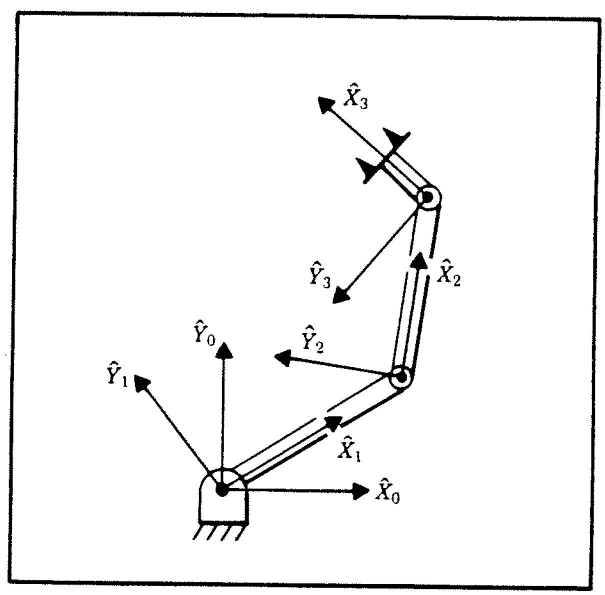


FIGURE 3.7 Link frame assignments.

PROBLEM

GIVEN THE MATRIX FIND θ_1, θ_2, θ_3?

$$ {}^{B}_{W}[T] = \begin{bmatrix} c\phi & -s\phi & 0.0 & x \\ s\phi & c\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \quad (4.7) $$

$$ = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1+l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1+l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \quad (4.6) $$

where $c_{12} = \text{Cos}(\theta_1+\theta_2)$ etc.

$$ c_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2} \quad (4.14) $$

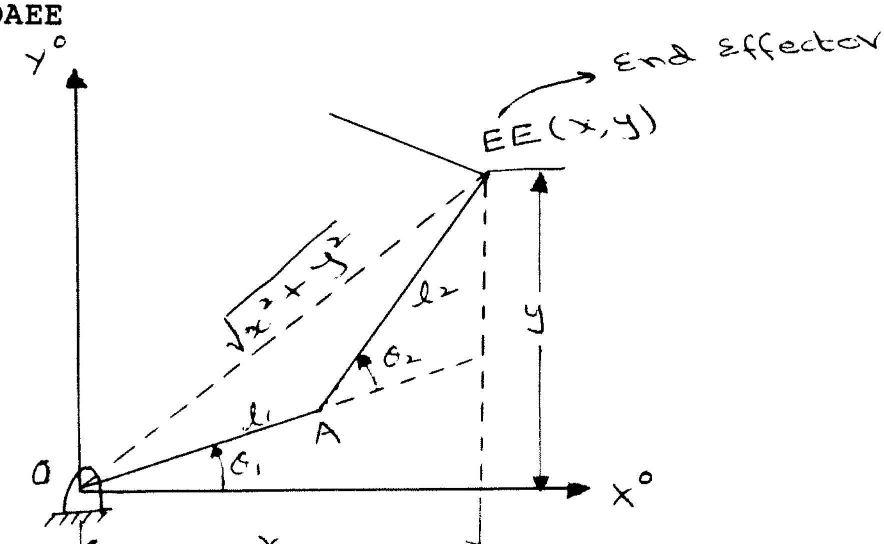
$$ s_2 = \pm\sqrt{1-c_2^2} \quad (4.15) $$

$$ \theta_2 = \text{Atan2}(s_2,c_2) \quad (4.16) $$

$$ K_1 = l_1+l_2c_2 \quad (4.19) $$

$$ K_2 = l_2s_2 \quad (4.19) $$

In triangle OAEE



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$$\mathrm{Cos}(180-\theta_2) = \frac{l_1^2+l_2^2-(x^2+y^2)}{2l_1l_2}$$

$$-\mathrm{Cos}\theta_2 = \frac{l_1^2+l_2^2-(x^2+y^2)}{2l_1l_2}$$

$$\mathrm{Cos}(\theta_2) = \frac{(x^2+y^2)-l_1^2-l_2^2}{2l_1l_2}$$

$$\theta_1 = \mathrm{Atan2}(y,x) - \mathrm{Atan2}(k_2,k_1) \qquad (4.27)$$

$$\theta_1 + \theta_2 + \theta_3 = \mathrm{Atan2}(s\phi,c\phi) \qquad (4.28)$$

from which we can solve for θ_3 because θ_2 and θ_1 are already known.In the Geometric Solution, we try to decompose the spatial Geometry into several plane geometry.

4.7 EXAMPLES OF INVERSE MANIPULATOR KINEMATICS - PUMA 560

Here we wish to solve

$$^0_6[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^0_1[T(\theta_1)]\ {}^1_2[T(\theta_2)]\ {}^2_3[T(\theta_3)] \ldots\ {}^5_6[T(\theta_6)]$$

Without going into the details,the sequence in which the solution can be obtained is as follows:

We are given

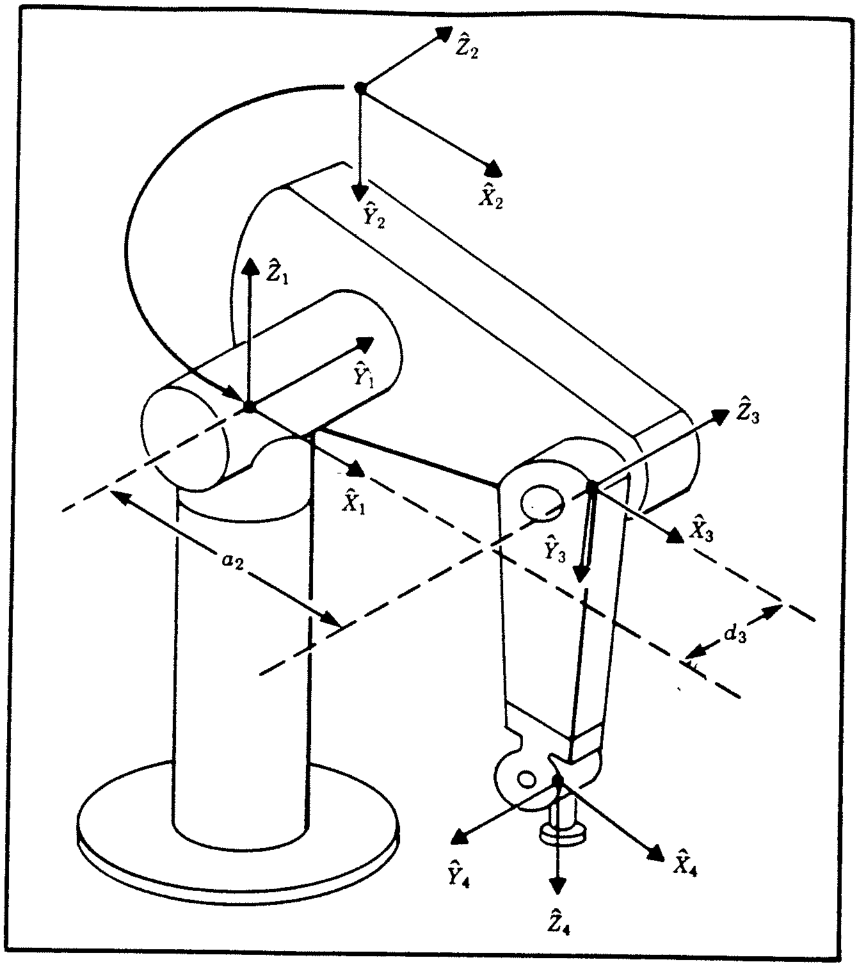

FIGURE 3.18 Some kinematic parameters and frame assignments for the PUMA 560 manipulator.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	$-90°$	0	0	θ_6

FIGURE 3.21 Link parameters of the PUMA 560.

1)

$$ {}^{0}_{6}[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2)

All α_i, d_i, a_i

$$ \theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right) \qquad (4.64) $$

$$ K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} \qquad (4.67) $$

$$ \theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}\left(k, \pm\sqrt{a_3^2 + d_4^2 - k^2}\right) \qquad (4.68) $$

$$ \theta_{23} = \text{Atan2}[(-a_3 - a_2c_3)p_z - (c_1p_x + s_1p_y)(d_4 - a_2s_3), (a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)] \qquad (4.73) $$

$$ \theta_2 = \theta_{23} - \theta_3 \qquad (4.74) $$

$$ \theta_4 = \text{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}) \qquad (4.76) $$

$$ -s_5 = r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) $$

$$ c_5 = r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) \qquad (4.79) $$

$$\theta_5 = \text{Atan2}(s_5, c_5) \qquad (4.80)$$

$$s_6 = -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4)$$

$$c_6 = r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] - r_{31}(s_{23}c_4c_5 + c_{23}s_5)]$$

$$\theta_6 = \text{Atan2}(s_6, c_6) \qquad (4.82)$$

NOTE:

1) Because of the + or - sign appearing in Equations(4.64) and Equations(4.68), these equations compute 4 solutions.

2) Additionally, ~~from~~ four more solutions are obtained by "flipping" the wrist which is expressed mathematically as

$$\left.\begin{aligned} \theta'_4 &= \theta_4 + 180^\circ \\ \theta'_5 &= -\theta_5 \\ \theta'_6 &= \theta_6 + 180^\circ \end{aligned}\right] \qquad (4.83)$$

After 8 solutions are computed, some or all of them may need to be discarded because joint limit limitations.