Lecture Notes for Engineering 6003

Ship Structures II

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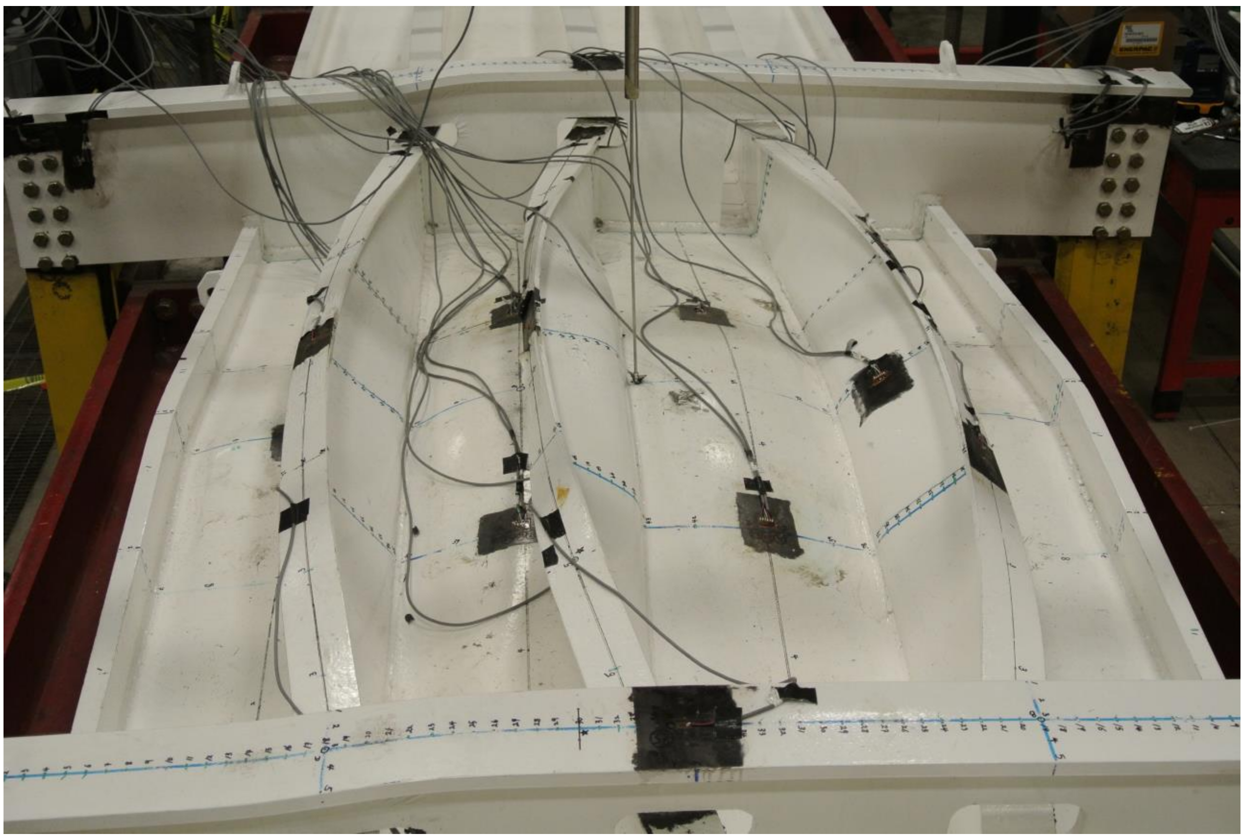

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Topic 1: Introduction

The course is intended to develop "advanced" knowledge of structures. The particular focus is on various types of structural failure in ships. Many of the topics presented in these lectures are the subject of ongoing research. This means that our understanding of these issues is still developing.

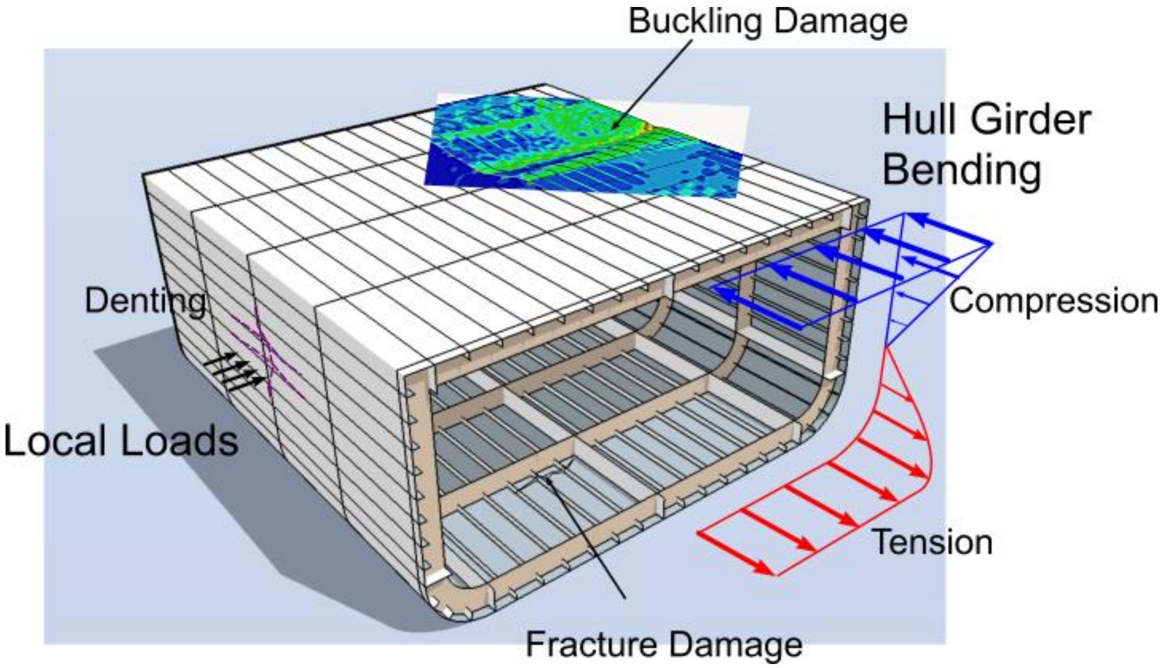
The photo shown below illustrates an important starting point about ship structures. Ships are a collection of stiffened panels (plates with frames attached). To understand the structural behavior of ships, it is necessary to understand the behavior of stiffened panels.

Kvaerner Masa-Yards Inc.

While there are many causes and symptoms of structural damage, there are three main types of damage that are essential to understand;

- Buckling due to compression (from global bending stress)
- Fracture (usually resulting from fatigue from global loads)
- Local denting (elasto-plastic bending from local loads)

The sketch below highlights these three damage types. In this course we will examine the mechanisms for these types of failure.



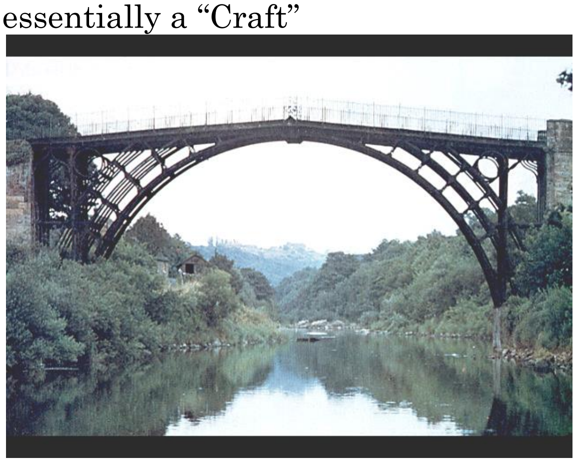
General Concepts and Trends

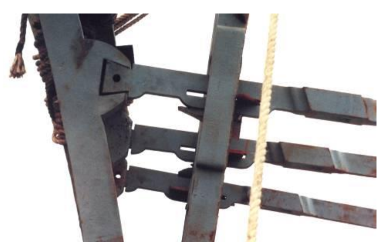
Humans have been constructing structures for a long time. A structure is a tool for carrying (carrying what is in or on the structure).
Ship structures have evolved like all other types of structures (buildings, aircraft, bridges ...). Design was once purely a craft. Design is evolving as we understand more about the structure itself and the environment that we subject it to.

Structural Design

Traditional Design

- Built by tradition (prior example)
- changes based primarily on experience (some analysis)
- essentially a "Craft"

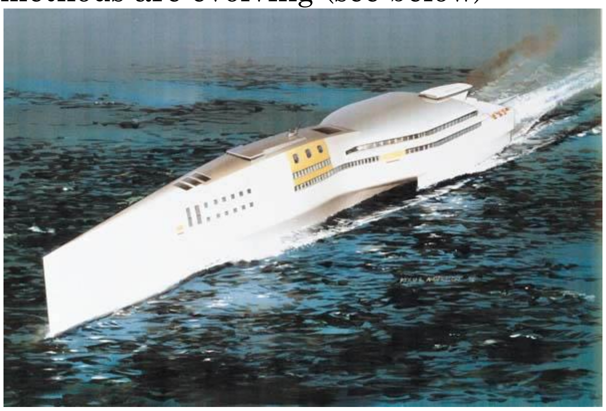




Iron Bridge, Coalbrookdale, UK
First iron bridge
The transition from tradition.

Engineering Design

- Incorporates analysis based on math/science
- Builds upon prior examples
- common designs are codified (building code, class rules..)
- new designs should follow the "Engineering Method"
- methods are evolving (see below)



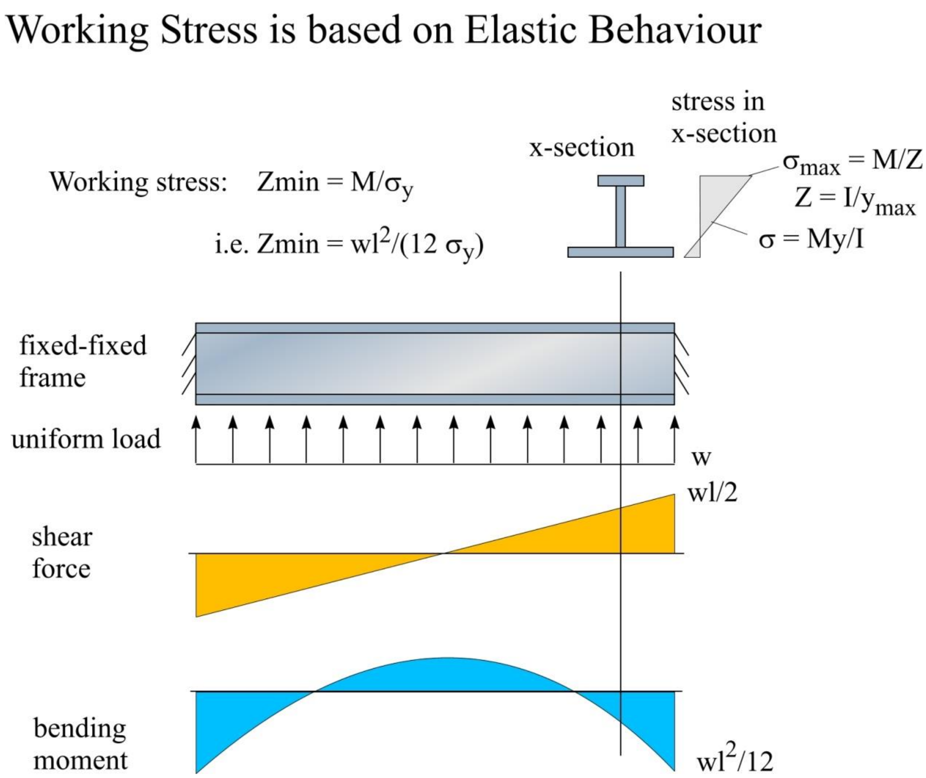
Superslender Monohull Ropax
Kvaerner Masa-Yards Inc.

Engineering Design: Working Stress Design

'**Working stress design**' refers to designs which employ an analysis of the intact structure, meaning the as-built structure in an elastic (unbuckled) state of stress. The design rules ensure that there is sufficient elastic capacity under the design load.

example : simple beam under uniform load:

we can solve the beam equation to determine the maximum stress, and require sufficient section modulus to ensure that the max. stress is less than the yield stress under the load.



There are several concerns/questions with working stress design:

- **what is so important about "yield at the extreme fiber"?**
  - first yield would be important if it led to unacceptable behaviour, such as a fracture, or an instability.
  - Brittle materials fracture without any ductile stretching.

- Very simple structures (single load path) have no way to re-distribute the load, and thus exhibit signs of structural failure at the first stage of material failure.
- Some high-precision structures (aircraft turbines, most 'machines') loose their functionality with even small permanent distortions.
- In most steel structures, and certainly in plated structures such as are found in ships, first yield remains a very localized behaviour, with no observable deterioration in the functioning of the structure.

- **can we observe "first yield at the extreme fiber" ?**
    - This is a very difficult thing to do. Structures are full of residual stresses even in the "unloaded" condition. A visual inspection will nether show the initial stresses, nor any local yielding. A system of installed strain gauges would need to be extremely extensive and costly to ever show local yielding. The gauge would have to be right over the point of yield (very difficult, as peak stresses occur at discontinuities), and functioning when the peak load occurs. We would need to get the gauge to read the initial stress, which is theoretically possible, but very difficult.
    - In practice, I don't know of anyone ever observing initial yield. We know it must happen a lot, but we only observe the distortions that happen much later.

- **will all structures designed using 'working stress' be equally satisfactory? (and what does *satisfactory* mean?)**
    - no. some will have no reserve, while some will have large reserve. If structures are "optimized" under working stress, they will tend to have the least reserve.

- **what about variable loads? flaws in the material?**
    - Variable loads, and many other uncertainties are handled with a single, standard, factor of safety.

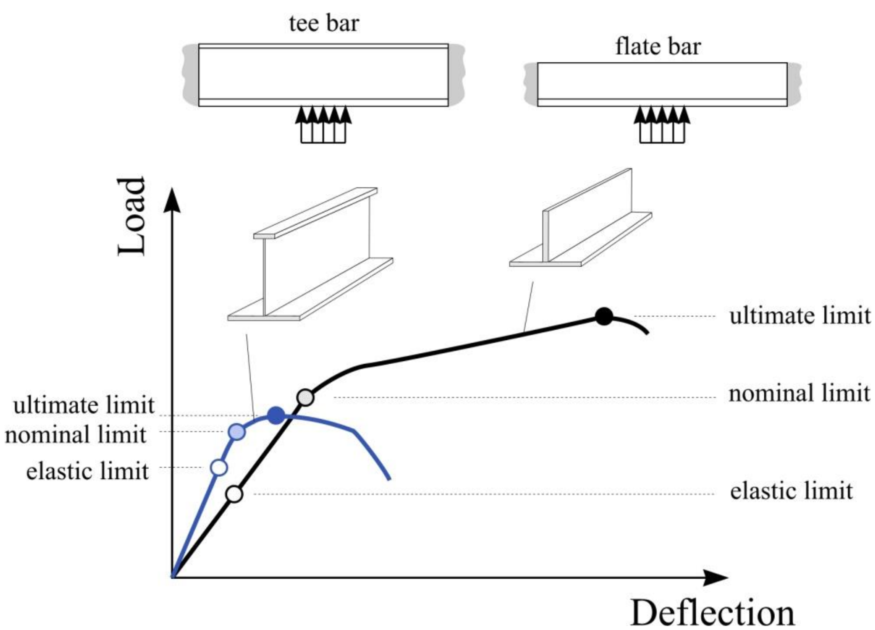
You should be able to explain the concept and weaknesses of "working stress design"

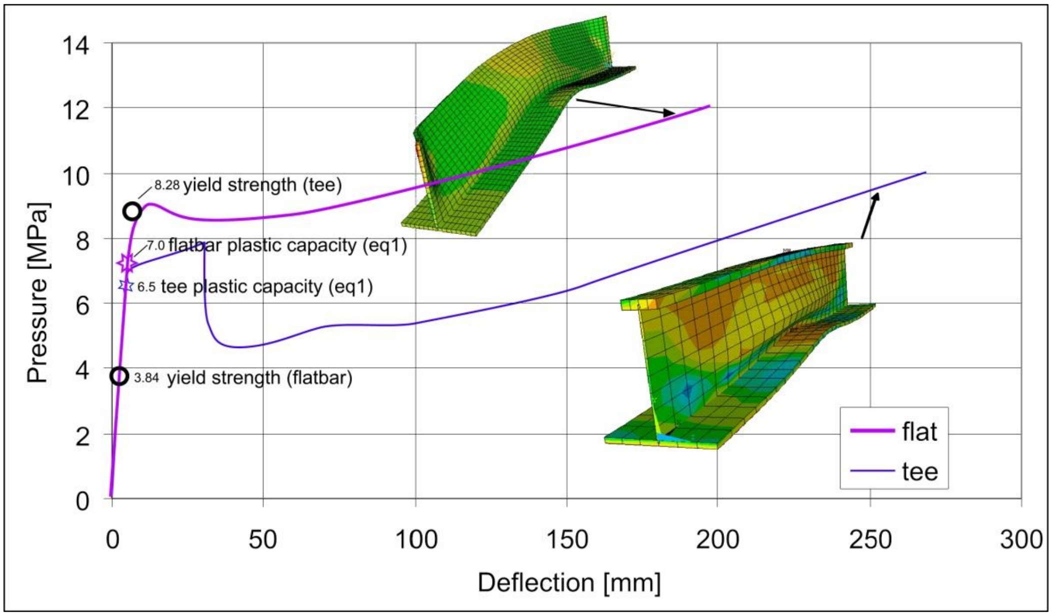
Engineering Design: Limit States Design

'**Limit states design**' refers to designs which employ an analysis of the failure of the structure, meaning the processes and mechanisms that cause unacceptable behaviour. The design rules ensure that there is resistance to various types of failure.

example: compare two simple beams under a patch load:

we can solve the non-linear structural behaviour (FE or experiment) to determine the various failure mechanisms, loads and deflections. The design rules would require sufficient geometry (shear area, plate thickness, plastic section modulus) to ensure that the member can carry the load without excessive deformation. Structures that are optimized for yield strength tend to be relatively tall and slender, with little reserve. 'Robust' structures, while having a lower yield point, can have a larger range of essentially linear behaviour, with significant reserve.
The two frames sketched below illustrate the different types of behaviour that frames can exhibit.





At a conceptual level, the figure below sketches a general load-deflection curve for a ductile steel structure. Note that yielding takes place only part way up the linear portion of the load deflection curve. Redistribution of stress occurs as the plastic strain field grows. The <essentially> linear potion of the curve ends when the strain field can redistribute no more and a mechanism is formed (e.g. a hinge or a fracture). After this the structure may continue to support further load increase, now in a new way (e.g. by membrane behavior). With more deflection, the new behavior may reach a new stress limit, start a new redistribution process and lead to another mechanism. One of the mechanisms will define the overall ultimate capacity. It is reasonable to set the design capacity at the first plastic mechanism, where the capacity is well above yield, but yet prior to large permanent deformations.

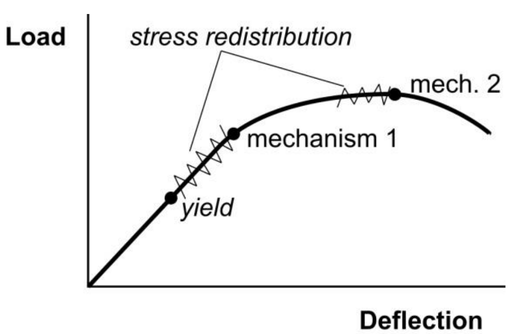
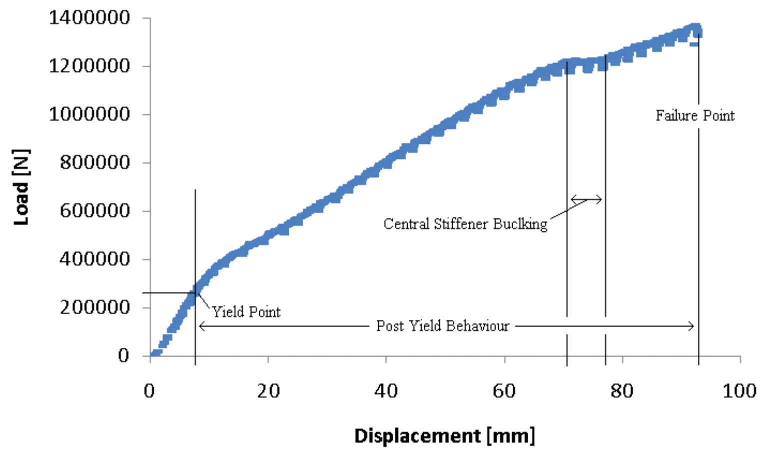


Figure 1 (alternate)

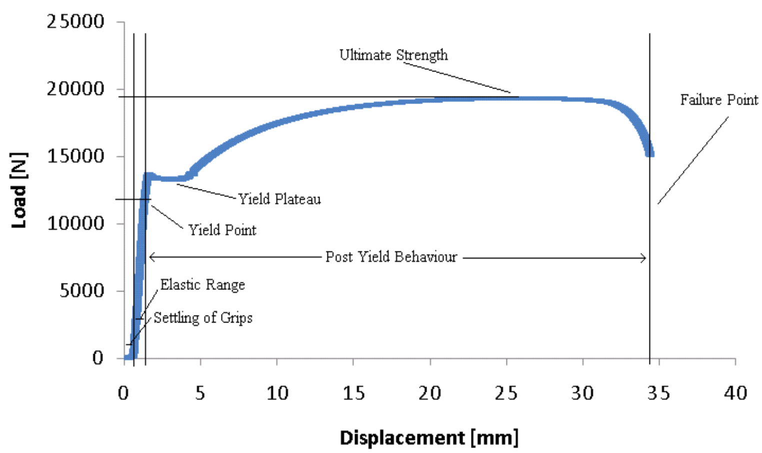
from ISSC 2003 – San Diego

Subtopic: Structural Yield vs Material Yield

Structural yield is point on the structure's load vs. deflection plot (see figure below) where non-linearity first occurs. Structural load-deflection curves for steel structures are notionally similar to the load-deflection material property curves for steel that we are used to seeing (see second figure below), however for a structure the load and deflection parameters are generally gross parameters; e.g. the deflection at the centre of a grillage vs. the load applied to the whole grillage.



STRUCTURAL LOAD VS DEFLECTION CURVE



MATERIAL LOAD VS DEFLECTION CURVE

There are several concerns/questions with Limit States design:

- are there equations (solutions) that adequately describe the behaviour?
  - In many cases we have no general solution of the limit state that will give an expression of the influence of various parameters.
  - There are potentially very many candidate limit states, which may interact in unknown ways. We may never be able to find algebraic solutions for all behaviours. In such cases we either

use exact solutions for similar behaviours, or use simulation (finite element analysis).

- can we corroborate (observe) the predicted limit states?
    - Yes. One of the key benefits of limit state analysis, is that the limit states are generally observable. We can see when collapse has occurred.
- In limit states design, a probabilistic assessment of loads and strength is often included. Can we describe the probabilities adequately?
    - There are generally two types of uncertainties, objective and subjective. Objective uncertainty refers to matters that can be measured and quantified with probability distributions. Subjective uncertainty deals with the many quantities for which little or no empirical data exists (worries and hunches).

Topic 2: Elastic Plate Behaviour - The basic ship structural element


steel frames inside the icebreaker MV TERRY FOX

Introduction

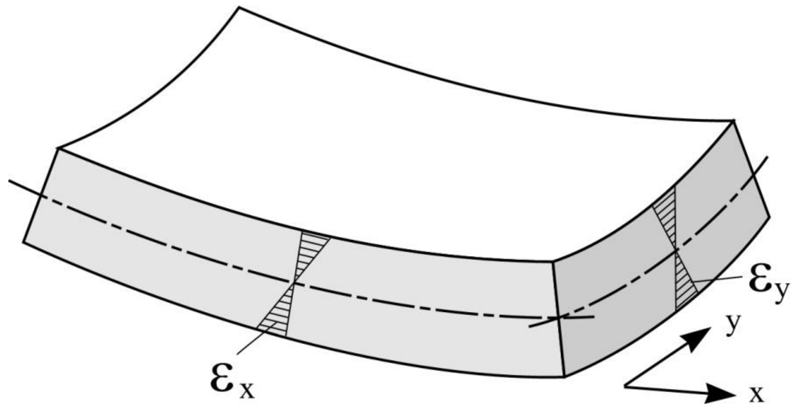
Ships are cut, almost entirely, from steel plate. Only a few structural elements are not made from plate (e.g. a few castings and some rolled sections). An understanding of plate behaviour is therefore crucial. Some components such as deck plate between supports, behave as ideal plates under simple boundary conditions and loads. Other components, such as the webs of large frames are subject to more complex loading. Still others such as deck/stiffener combinations behave somewhat like simple plates, but with special aspects.

In this lecture we will discuss elastic plate bending in general, simplify this for the case of long plates and derive the differential equation of a long plate (bending in one direction).

For now we will assume the following aspects of what is called "small deflection theory":

1. Plane sections remain plane, hence – strain is linear
2. Deflections are small ($\sin\theta \approx \theta$) , typically $w_{max}< ¾\ t$
3. Plating remains elastic

We start by sketching a small portion of plate bending in two directions, but with no in-plane stress. In other words the stress at the mid-plane will be zero, and (equivalently) the average stress in the x or y direction will be zero.



There is bending stress and strain in both x and y directions. We assume that $\sigma_z \cong 0$.

The general 2D elasticity relationships amoung stress and strain (Hooke's Law for 2D case) are;

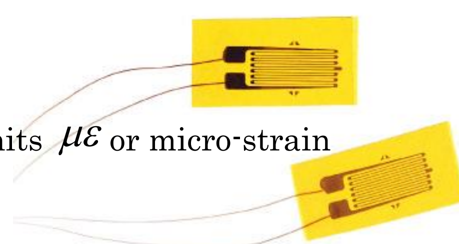
$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \qquad \nu\text{=poisons ratio}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

$$\varepsilon_z = \frac{-\nu}{E}\left(\sigma_y + \sigma_x\right)$$, which after substitution gives $\bar{\varepsilon}_z = 0$

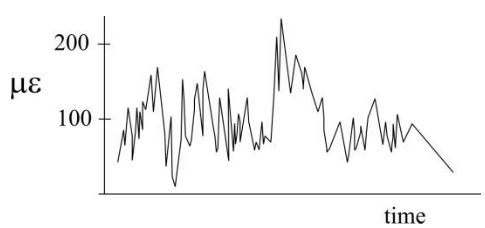
Aside: Note on units of strain

$\varepsilon \equiv$ strain = length/length



Strain is unit-less, but we typically use the units $\mu\varepsilon$ or micro-strain

$\mu\varepsilon \equiv 10^{-6}$ in/in or 10^{-6} m/m

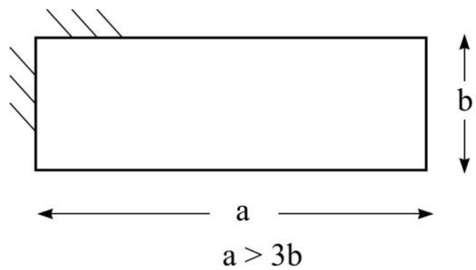


eg. typical range for elastic strain in steel in ships is 0 to +- 1200 $\mu\varepsilon$

corresponds to σ_y =250 MPa

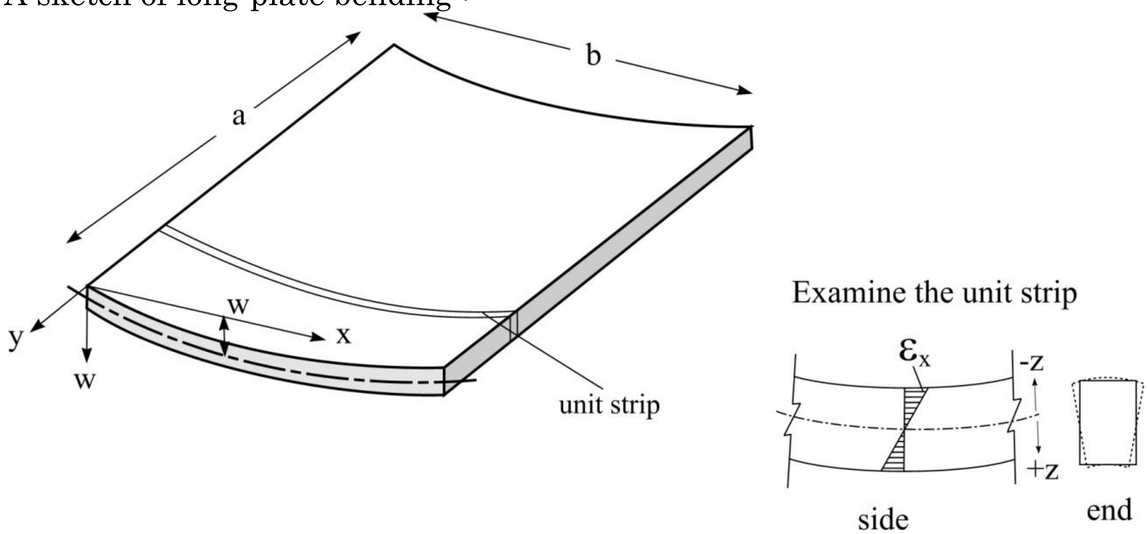
Long Plates

We will focus on the behaviour of "long" plates. A long plate generally has a length of at least 3x the width;



Long plates bend in one direction, meaning they form a cylindrical shape (locally). The cylindrical bending occurs towards the center of the plate (say at least 'b' away from the ends). There are end effects, but the maximum stresses and deflections occur in the center region, and are essentially identical to that of an infinitely long plate.

A sketch of long-plate bending :



in long plates, the natural symmetry that each unit strip has with each neighbouring strip prevents any strain in the y direction. This causes a non-zero stress in the y-direction:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \qquad (1)$$

$$\varepsilon_y = 0 = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \qquad (2)$$

(2) is re-arranged to: $$\frac{\sigma_y}{E} = \frac{\nu\sigma_x}{E} \quad (3)$$

(3) into (1) gives: $$\varepsilon_x = \frac{\sigma_x}{E} - \nu\left(\frac{\nu\sigma_x}{E}\right)$$

which is $$\varepsilon_x = \frac{\sigma_x}{E}\left(1-\nu^2\right) \quad (4)$$

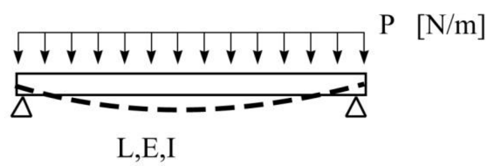
or $$\sigma_x = \frac{E}{\left(1-\nu^2\right)}\varepsilon_x$$

which can be written as $\sigma_x = E'\varepsilon_x$ where $E' = \dfrac{E}{\left(1-\nu^2\right)}$

This means that plates behave like beams, but with a modified modulus. We can use beam equations for long plates.

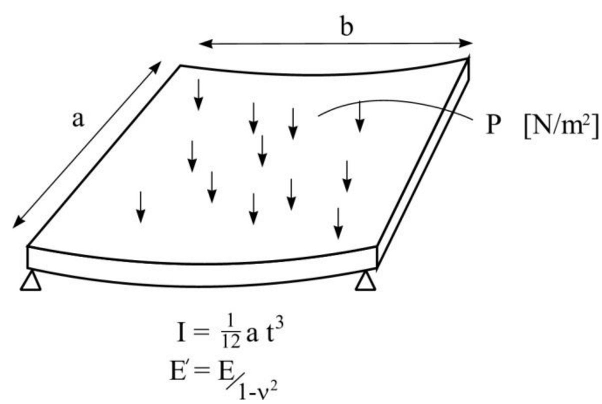
e.g. for a beam, length L, inertia I, load q (N/m)the central deflection is:

$$w_{\max} = \frac{5}{384}\frac{pL^4}{EI}$$



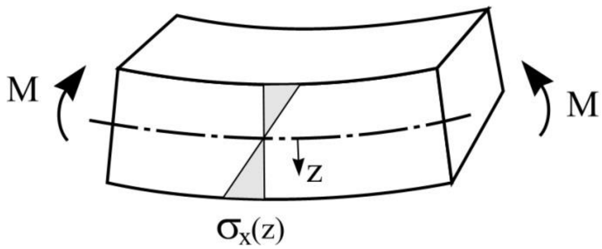
for a long plate of width b, length a, pressure p :

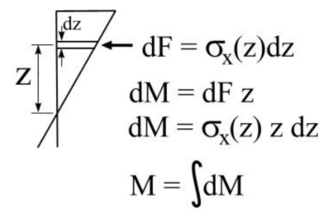
$$w_{\max} = \frac{5}{384}\frac{pab^4}{E'I}$$



Partly as a review, lets derive the flexural differential equation for a beam, which is equally valid for a long plate.

Take a unit strip of plate:

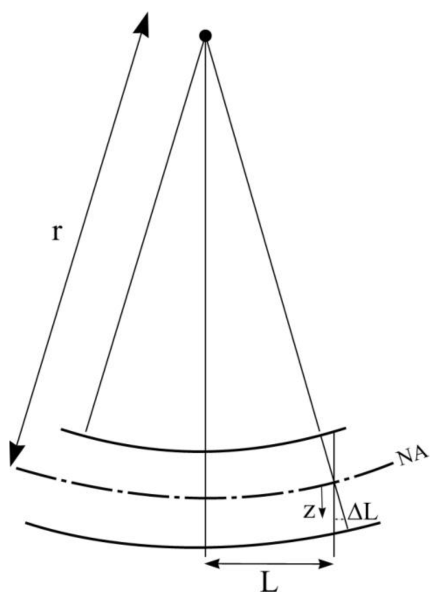




The external moment M balances with the internal moment created by the stress distribution $\sigma_x(z)$ (see sketch) :

$$M = \int_{-t/2}^{t/2} \sigma_x(z) \cdot z \cdot dz$$

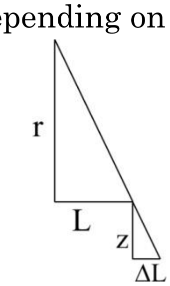
Note that we have sketched a shape for $\sigma_x(z)$, but we need to confirm this. To find $\sigma_x(z)$, we will use Hooke's Law, and a pattern of deformation. We assume that the plate deforms from straight into a circular curve of radius 'r' (at the NA).



The circular deformation results in a change in 'fibre' length, depending on z:

$$\frac{\Delta L}{L} = \varepsilon_x(z)$$

from similar triangles



$$\frac{z}{\Delta L} = \frac{r}{L} \quad \text{or} \quad \frac{z}{r} = \frac{\Delta L}{L} = \varepsilon_x(z)$$

which gives: $\varepsilon_x(z) = \dfrac{z}{r}$

Hence for long plates: $\sigma_x(z) = \dfrac{E}{1-\nu^2}\,\varepsilon_x(z) = \dfrac{E}{1-\nu^2}\,\dfrac{z}{r}$

We substitute $\sigma_x(z)$ back into the moment equation to get;

$$M = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}\frac{z}{r}\cdot z\cdot dz$$

$$= \frac{E}{r(1-\nu^2)}\int_{-t/2}^{t/2} z^2\cdot dz = \frac{E}{r(1-\nu^2)}\frac{z^3}{3}\Big]_{-t/2}^{t/2}$$

$$M = \frac{Et^3}{12(1-\nu^2)}\frac{1}{r}$$

note: M seems to have units of force (e.g. N). This is because M is a moment per unit length of plate (i.e. N-m/m).

We define: $D \equiv$ flexural rigidity

$$D = \frac{Et^3}{12(1-\nu^2)}$$

note:
for a beam: D=EI
For a plate : D=E' t^3/12

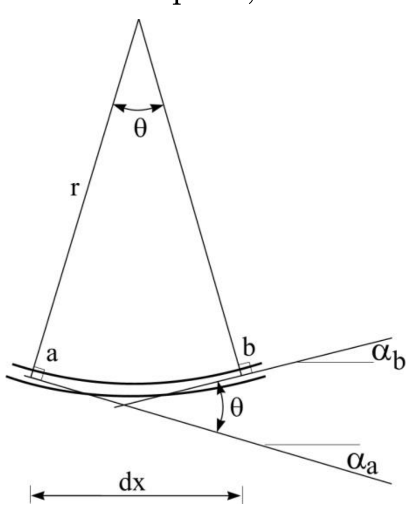
hence:

$$M = \frac{D}{r} = \frac{\text{flexural rigidity}}{\text{radius of curvature}}$$

Now we need to relate the radius 'r' with the plate deflection 'w'/

We define $\frac{1}{r} \equiv$ curvature (odd, but curvature = 1/radius of curvature)

For a curved plate, radius 'r':



$$\alpha_a \equiv slope \quad at \quad a$$

$$\frac{d\alpha}{dx} = \frac{\alpha_b - \alpha_a}{dx} = \frac{\theta}{dx} = \frac{dx/r}{dx} = \frac{1}{r}$$

if w(x) is the plate deflection, then

$$\alpha(x) = \frac{dw(x)}{dx} \qquad \frac{d^2w(x)}{dx^2} = \frac{d\alpha(x)}{dx} = \frac{1}{r}$$

consequently:

$$\Downarrow$$

$$\boxed{M = \frac{d^2w(x)}{dx^2}D}$$

This is the moment-deflection differential equation

Topic 3: more plate behaviour - add yielding


structural elements in the FPSO Tera Nova

Introduction

In this lecture we will review the general set of differential equations for plate bending. Then we will discuss yielding of the plate.

In the previous lecture we derived the moment-deflection relationship as:

$$M = D\frac{d^2 w(x)}{dx^2}$$

As for a beam there are a family of differential equations as follows;

$w(x)$=deflection

$w' = \alpha$ = slope

$w'' = \frac{1}{r} = \frac{M}{D}$ = moment / rigidity

$w''' = \frac{V}{D}$ = shear / rigidity

$w'''' = \frac{P}{D}$ = load / rigidity $\Rightarrow \frac{d^4 w(x)}{dx^4} = \frac{P}{D}$

1st Aside: dynamics

The equation w””(x)=P(x)/D is a static equation (no time!).

We can include time and dynamics quite easily. We could include time in a slowly varying situation, (called quasi-static) as follows;

$$w''''(x,t) = \frac{P(x,t)}{D}$$

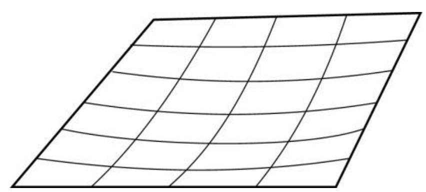
The above equation does not include inertial effects in the plate, (i.o.w. the acceleration of the plate is not considered in the force balance).

For a dynamic equation we need to add a term that gives the inertial force;

$$w''''(x,t) = \frac{P(x,t) + m(x)\cdot \ddot{w}(x,t)}{D}$$

Note that the primes refer to derivatives in x, while the dots refer to derivatives in time.

2nd Aside: bending in 2 directions



for a plate bending in 2 directions we can not assume that $\varepsilon_y=0$

we need to express the deflection as w(x,y). This results in a general equation for static equilibrium in terms of partial derivatives;

$$\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \frac{P(x,y)}{D}$$

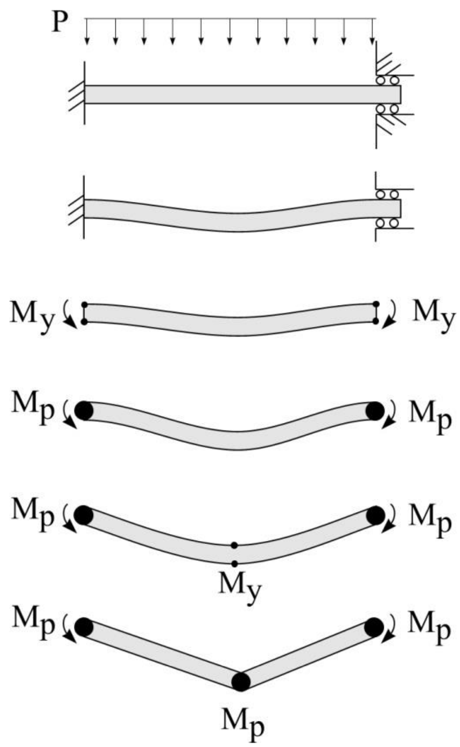
this is abbreviated as; $\nabla^4 w(x,y) = \dfrac{P(x,y)}{D}$

The analytical solutions for this equation can be complex, even for simple cases. These problems are more often solved numerically (e.g. FE method).

The equations given above all represent linear elastic behaviour. The solutions depend on the pattern of load and boundary conditions. There are many standard solutions tabulated in, for instance, Roark.

We will move on to the post-yield behaviour.

As an introduction, the behaviour of a plate under increasing uniform lateral load (e.g bottom plating), is shown;



The plate can resist end moments, but is free to pull in.

elastic deflections

yielding starts at the plate edges (location of highest stress)

plastic hinges form at edges

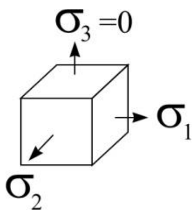
yielding starts at the center

plastic hinges form at the center, creating a mechanism -> collapse

Yield Criteria

There are many material failure criteria, typically dependent on stress or strain. For ductile materials, yield is the material behaviour of interest. It is commonly modeled with the **von-Mises yield criteria,** as described below;

For the case of a uniaxial stress, the stress reaches yield at σ_y.



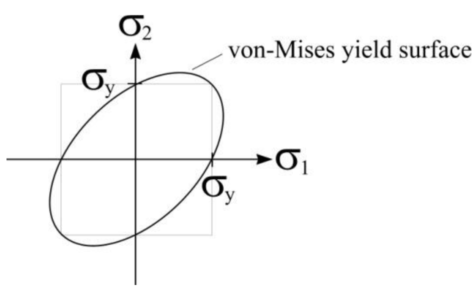
Biaxial stress is limited by an envelope of points along a curve. Consider a small cube of material, subject to biaxial stress. The three spatial coordinates are labeled 1,2,3. One of the stresses is zero.
We have: σ_1, σ_2 , σ_3=0. There are no shear stresses (these are principal stresses).

For this case the failure envelope is described by the equation;

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$$

where σ_y is the yield stress.

We can plot this (on σ_1, σ_2 coordinates) as;



if σ_1=0, $\sigma_2 = \pm\,\sigma_y$ $\leftarrow$ uniaxial in σ_2
if σ_2=0, $\sigma_1 = \pm\,\sigma_y$ $\leftarrow$ uniaxial in σ_1
if $\sigma_1 = \sigma_2$, $\sigma_1 = \sigma_2 = \pm\,\sigma_y$

The surface is an ellipse when viewed on the σ_3= 0 plane.
The maximum possible stress is:

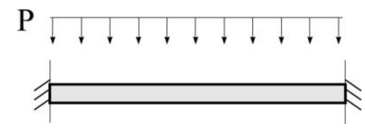
$$\sigma_2 = \sigma_1/2 \;\Rightarrow\; \sigma_1^2 - \frac{\sigma_1^2}{2} + \frac{\sigma_1^2}{4} = \sigma_y^2 \;\Rightarrow\; \sigma_1 = \sqrt{\frac{4}{3}}\sigma_y = 1.154\cdot\sigma_y$$

The von-Mises yield criteria is best explained as saying that yield occurs when the elastic energy stored in shear reaches a limit. Every point on the von-Mises failure surface represents the same amount of elastic potential energy in shear.

Strength of plates

We can use the von-Mises criteria to determine the load (pressure) that will case yielding to occur in a plate.

Assume: - long plate
- uniform pressure p
- plate clamped at edges



The stress in the x direction is σ_1.
The stress in the y direction is σ_2.

For long plates we know that $\sigma_2 = \nu\,\sigma_1$. If we use this in the von-Mises equation we get;

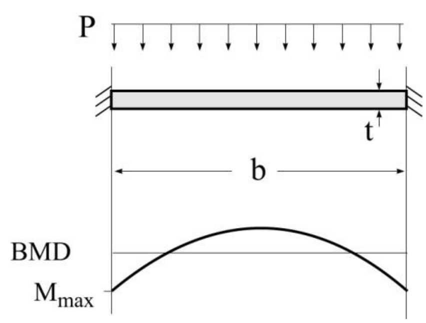
$$\sigma_1^2 - \sigma_1(\nu\sigma_1) + \nu^2\sigma_1^2 = \sigma_y^2$$

or

$$\sigma_1 = \frac{\sigma_y}{\sqrt{1-\nu+\nu^2}} = 1.125 \cdot \sigma_y$$

We see that the plate is apparently *stronger* than a beam, by 12.5%, due to the effects of y-direction restraint.

For a clamped plate:



$$M_{max} = \frac{pb^2}{12} \quad \text{(moment per unit width)}$$

at yield;

$$\sigma_x = \sigma_1 = \frac{Mc}{I}$$

$$= \frac{M\ {}^{t}\!/_{2}}{{}^{t^3}\!/_{12}} = \frac{6M}{t^2}$$

collecting terms we get:

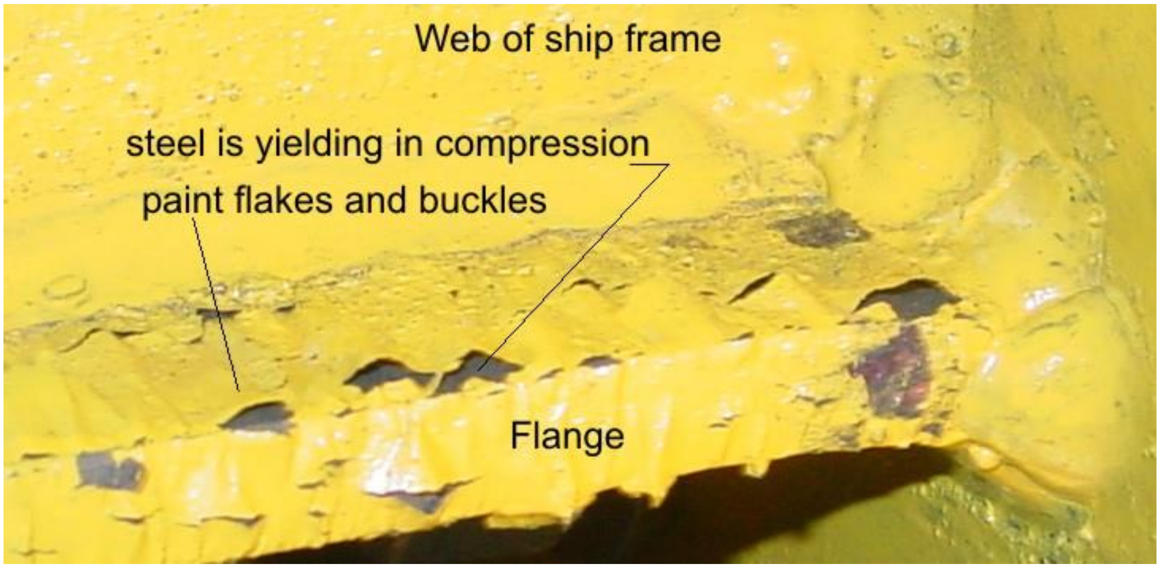
$$\sigma_1 = \frac{\sigma_y}{\sqrt{1-\nu+\nu^2}} = \frac{pb^2}{2t^2}$$

We can re-arrange this to find the pressure to cause yield;

$$p_y = \frac{2t^2\sigma_y}{b^2\sqrt{1-\nu+\nu^2}} = 2.25\sigma_y\left(\frac{t}{b}\right)^2$$

This equation can be used to determine the capacity of the plate if σ_y, t, b or to design the plate (determine t) if p, σ_y, b are known.

Topic 4: Mohr: State of Stress



close-up of ship structural test

Introduction

In this lecture we will

- review Mohr's Circle (stress on any plane)
- show von-Mises in terms of engineering stresses
- describe elasto-plastic behaviour

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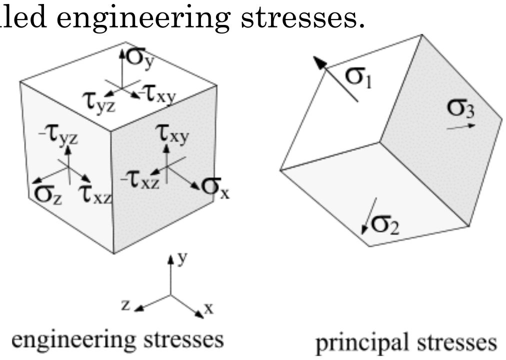
We defined the von-Mises failure criteria, and used it for a simple case. In that case we had no shear stresses (we ignored them).
We called $\sigma_1=\sigma_x$. This was a case in which σ_x was the maximum stress on any plane. Normally we use the notation σ_1, σ_2, σ_3 to refer to principal stresses.

We will review Mohr's circle of stress to show what principal stresses are and how they can sometimes be more useful than σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}, which are called engineering stresses.



Consider a small element of material with normal and shear stresses;
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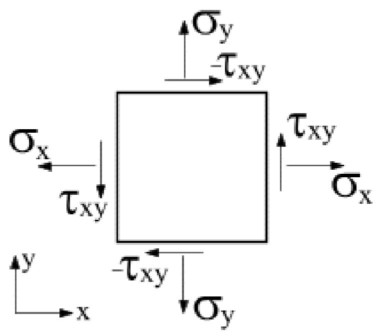
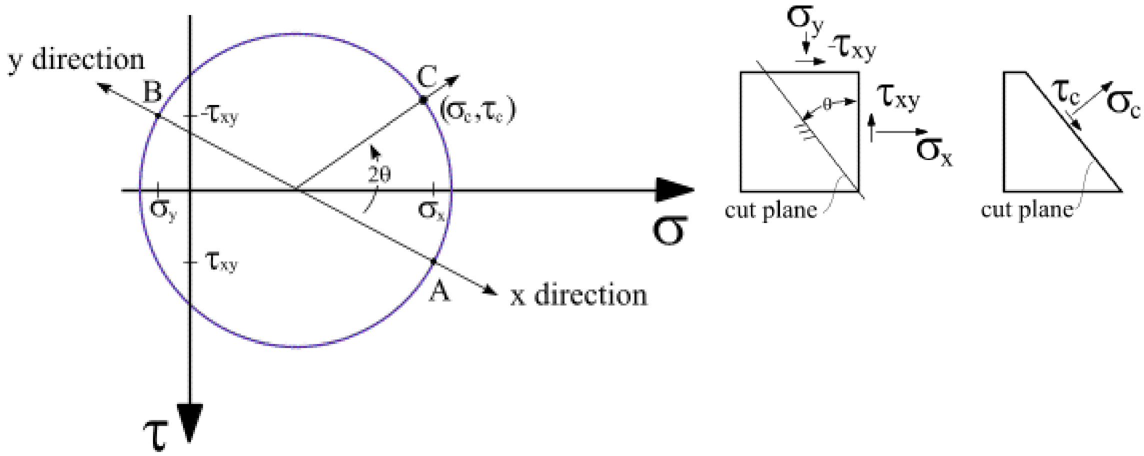


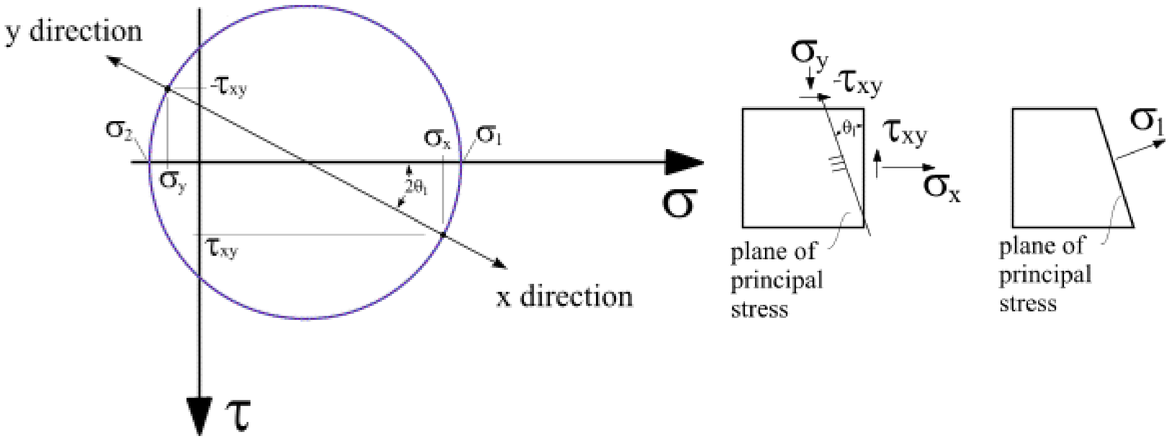
We have drawn the stresses on the y-z plane (σ_x, τ_{xy}) and on the x-z plane (σ_y, $-\tau_{xy}$). Mohr showed that the stresses on all planes, when plotted, will form a circle in τ vs σ coordinates:



The stresses on the y-z plane (the x-direction) are plotted on the Mohr-s circle (point A). The stresses on the x-z plane (the y-direction) are plotted at point B. These two planes are physically 90 degrees from each other, but are 180 degrees apart on the Mohr's circle.

The line joining A, B is a baseline. To find the stresses on a cut plane at angle θ from the y-z plane, we must move 2θ from the x direction around the Mohr's circle. This lands us at point C, where the stresses are σ_c and τ_c.

You can see from the drawing that the largest value of σ occurs where τ is zero. The largest and smallest values of σ are called σ_1 and σ_2. They are sufficient to define the circle, and are called the principal stresses.



The stress σ_1, occurs on a plane at an angle ϕ from the plane of σ_x.

Exercise:
$\sigma_x = 4$, $\sigma_y = -2$, $\tau = 4$. Using force vector equilibrium, find the stresses on a plane whose normal is 15 deg up from the x axis.
Use Rhino to do the vector addition. Check the answer with a Mohr's Circle.

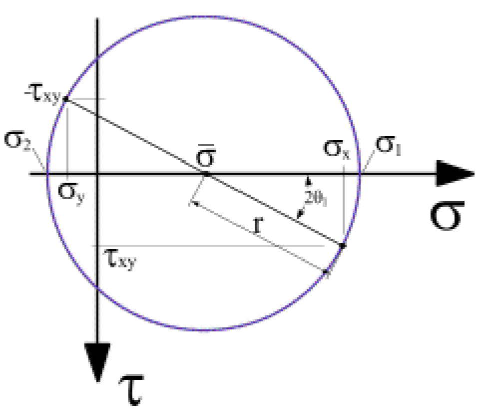
We do not need to solve for σ_1 and σ_2 graphically. We can use the following equations:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\sigma_x - \frac{\sigma_x + \sigma_y}{2}\right]^2 + \tau^2}$$

or

$$\sigma_1 = \bar{\sigma} + r$$

$$\sigma_2 = \bar{\sigma} - r$$



The von-Mises yield criterion is;

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{yield}^2$$

If we have σ_x ,σ_y and τ, we can either convert to principal stresses or we can use the von-Mises criterion expressed in engineering stress.

For the 2D case (σ_z=0) von-Mises is;

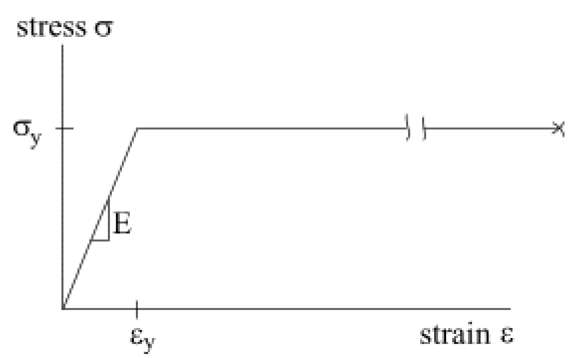
$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_{yield}^2$$

the most general form of von-Mises is;

$$\frac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 = \frac{\sigma_{yield}^2}{3}$$

Now that we have a criterion for yield stress we can examine what happened after yield occurs.

We will assume that we have ideal plasticity. In other words, we will assume hat the steel will stretch under constant stress. The stress –strain curve looks like;

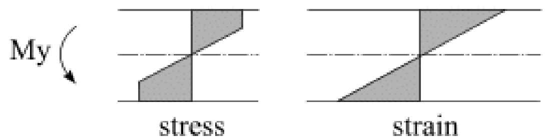


With this kind of elasto-plastic behaviour, the bending stress in a plate will develop as follows;

(1) up to yield, the stress and strain are linear;

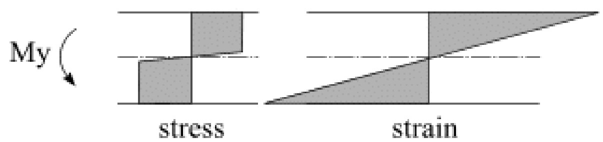
$$My = \sigma_{yield} \frac{t^2}{6}$$

(2) as yielding speads through the section;



the edges of the plate yield,
while the center stays elastic

(3) with very large rotations, the strain continues to grow
and the stress approaches a limiting pattern,
and the moment reaches Mp;



$$Mp = \sigma_{yield} \frac{t^2}{4}$$

the plastic moment is 1.5x the elastic moment

Topic 5: Plastic Bending Limit States in Laterally Loaded Plates

Introduction

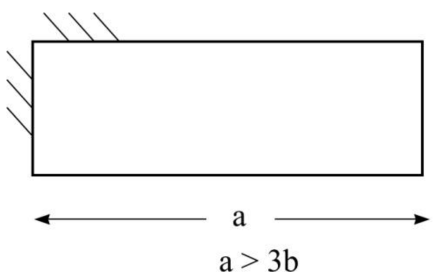
In this lecture we will

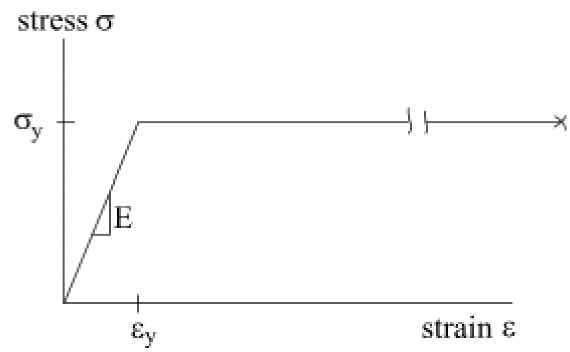
- Describe and derive equations for the plastic bending collapse of a long plate
- show the behaviour diagram in terms of a load-deflection plot

Long Plate Plastic Bending Collapse

Assumptions

- Long plate (i.e. $a \geq 3b$)
- *Elastic-perfectly plastic* material behaviour

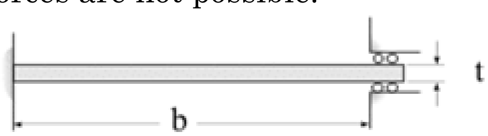




Boundary Conditions

Clamped boundary conditions – the plate is fixed in bending, but free to pull in.

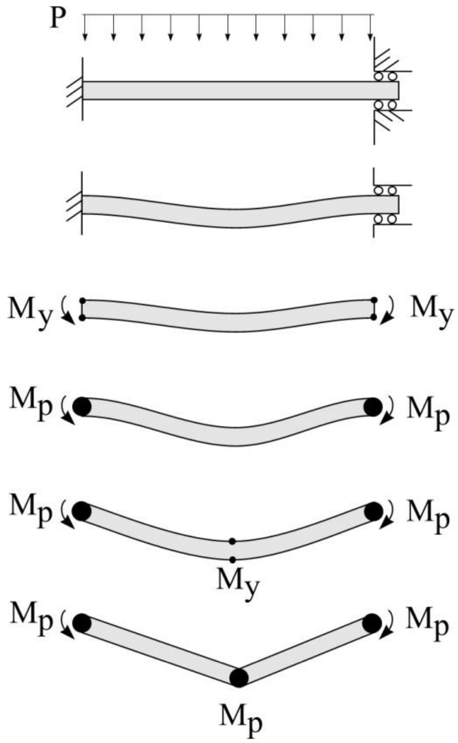
This implies that we can only have a *flexural* response (i.e. bending). There can be no *membrane*[1] behaviour because axial displacement is not restrained – and therefore axial forces are not possible.



[1] We will cover *membrane* behaviour in later lectures.

Review of Plate behaviour under increasing uniform lateral load

In Lecture 8, we introduced the following succession of behaviours for plates under increasing uniform lateral load. We review them here and further refine their definition:



Initial Condition: The plate can resist end moments, but is free to pull in.

Elastic Deflection: As load increases from zero we have *elastic deflections*.

First Yield: As load increases further yielding starts at the extreme fibre of the plate's edges.

Edge Hinges: As load increases further plastic hinges form at edges (i.e. *full yield* at edges).

Initial Yield at plate centre: As load increases further, yielding starts at the plate center.

3 Hinge Collapse: As the load increases further, a plastic hinge forms at the plate center, creating a mechanism -> *3 hinge collapse.*

Mechanics of Plate Behaviour under Increasing Uniform Load

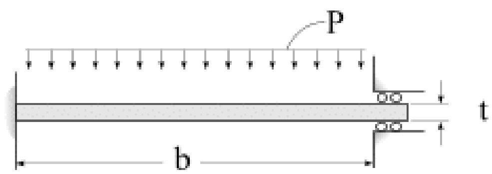
Remember that the following equations are for a *unit strip* of a *long plate*.

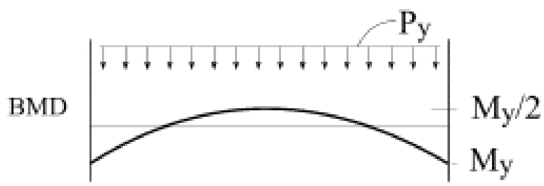
Definition: In Lecture 8 we found from von-Mises the effective yield strength for a *long plate*; which we shall define here as σ_{yp}:

$$\sigma_{yp} = \frac{\sigma_y}{\sqrt{1-\nu+\nu^2}} = 1.125\sigma_y \quad \text{(for } \nu = 0.3\text{)}$$

First Yield

As the pressure, P, increases to P_Y, the edges yield at the *extreme fibre*; all else remains *elastic*.





The bending moment at *first yield* is:

$$M_y = \sigma_{yp}\frac{I}{c} = \sigma_{yp}\frac{\frac{1}{12}t^3}{\frac{t}{2}} = \sigma_{yp}\frac{t^2}{6}$$

The pressure to cause *first yield* is:

$$P_Y = 2.25\cdot\sigma_y\left(\frac{t}{b}\right)^2 \text{ or}$$

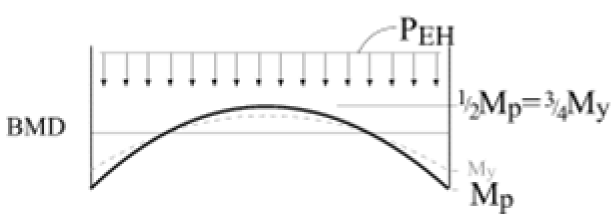
$$P_Y = 2\cdot\sigma_{yp}\left(\frac{t}{b}\right)^2$$

At this point the central deflection is:

$$\delta_Y = \frac{1}{384}\frac{P_Y b^4}{D}$$

Edge Hinges

As P increases from P_Y to P_{EH}, *edge hinges* form:



The moment to cause *full yield* (i.e. the plastic hinge moment) is:

$$M_p = 1.5M_y = \sigma_{yp}\frac{t^2}{4} \text{ or}$$

$$M_P = P_{EH}\frac{b^2}{12}$$

The pressure to cause *edge hinges* is:

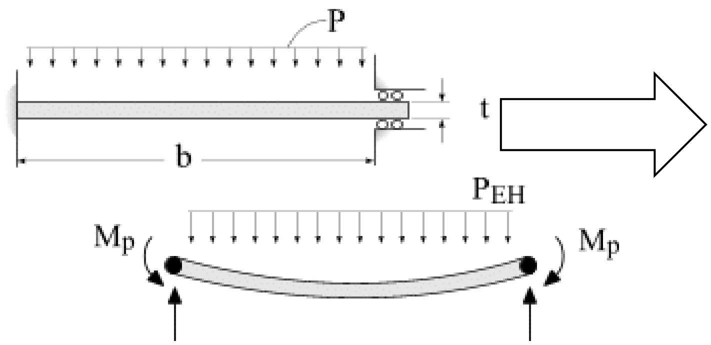
$$P_{EH} = 1.5P_Y \text{ or } P_{EH} = 3\cdot\sigma_{yp}\left(\frac{t}{b}\right)^2$$

At this point the deflection is:

$$\delta_{EH} = \frac{1}{384}\frac{P_{EH} b^4}{D}$$

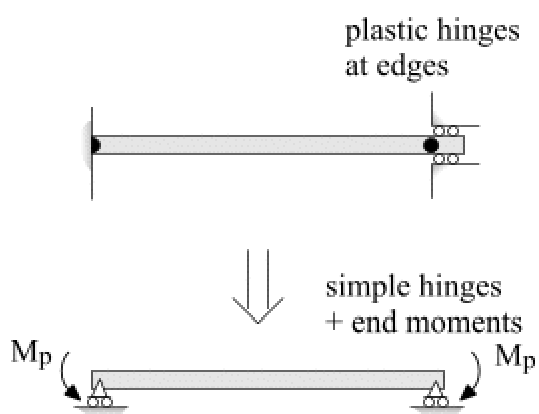
Problem change

Edge hinges are a *mechanism* (i.e. the behaviour of the system changes drastically when *edge hinges* form). When *edge hinges* form the problem changes from an *indeterminate* elastic plate with *fixed* supports, to a *determinate* elastic plate with *plastic hinges* as the supports.



Note: all structures lose their indeterminacy as they become damaged and move towards failure.

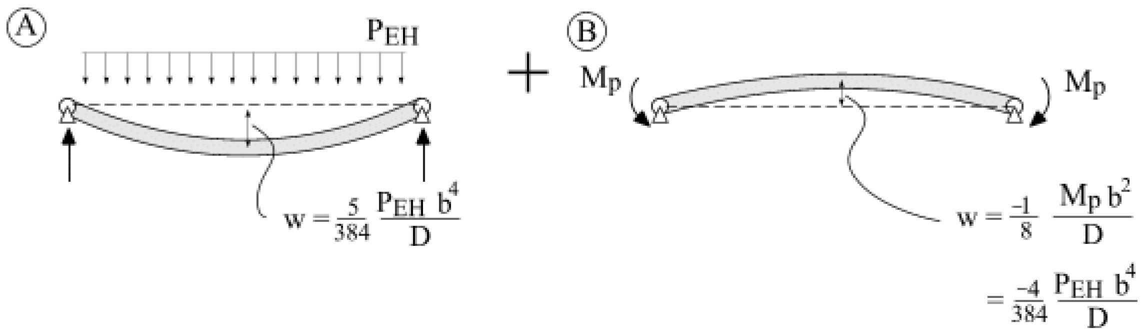
Note: We can think of a *plastic hinge* as a simple (frictionless) hinge with a constant applied moment.



To summarize: once the *edge hinges* (i.e. *plastic hinges*) have formed, we need to treat the plate as a new structure; which we can break into two parts.

Superposition of two Problems

We now divide the problem into the sum of two simpler problems; A + B:

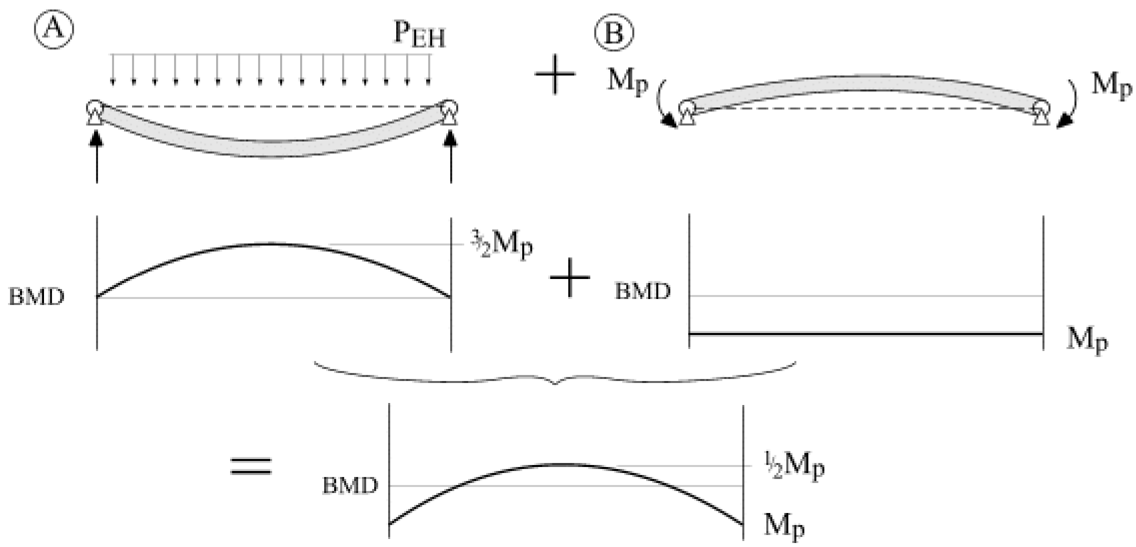


Problem A is simply supported and carries only the load, P_{EH}. Problem B is simply supported and has only a constant applied moment, M_P. Note that each of these two cases contains components of the real case, such that the cases can be added to give the real case (i.e. all load, reactions, stresses and deflections can be added).

Problem B stays unchanged (i.e. remains constant) as P increases above P_{EH}. Why? Because of the assumption of a *perfectly plastic* stress-strain relationship. The stress in a plastic hinge *cannot* go above σ_{yp} with this assumption.

Problem A, however, will continue to deflect elastically until a *central plastic hinge* forms.

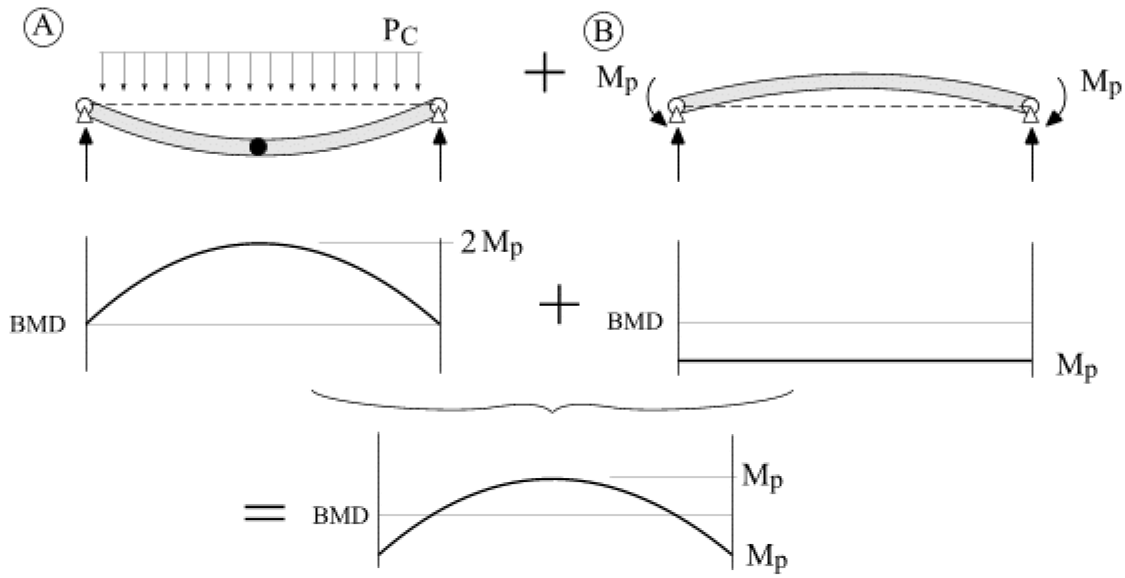
To find the load to cause the *central plastic hinge* we examine the bending moments in both A and B:



We see that in Problem A, the bending moment is $\frac{3}{2}M_P$, but in Problem B, it is $-M_P$; therefore the total bending moment (i.e. the sum of A and B) is $-M_P$ at the edges and $\frac{1}{2}M_P$ in the centre.

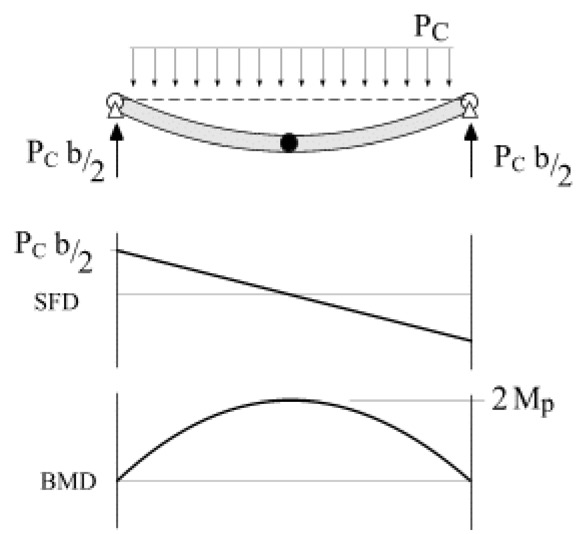
3 Hinge Collapse

When the P increases beyond P_{EH}, the values for Ⓐ increase until the central moment is $2M_P$. Ⓑ stays unchanged, so that the sum of the two cases gives edge *and* central moments of M_P. At this point we have *3 hinge collapse* and we call the load level P_C.



We can determine P_C by ratio:

$$P_C = P_{EH}\frac{2M_P}{1.5M_P} = \frac{4}{3}P_{EH}$$



Or we can determine P_C by solving the statically balanced beam:

The area of the shear force diagram (SFD) to the left of center will equal $2M_P$:

$$2M_P = P_C\frac{b}{2}\frac{b}{2}\frac{1}{2}$$

which is rearranged to give:

$$P_C = \frac{16M_p}{b^2} = 4\cdot\sigma_{yp}\left(\frac{t}{b}\right)^2 \text{ or}$$

$$P_C = 4.5\cdot\sigma_y\left(\frac{t}{b}\right)^2$$

For P_C the deflection is:

$$\delta_C = \frac{4}{3}\frac{5}{384}\frac{P_{EH}b^4}{D} - \frac{4}{384}\frac{P_{EH}b^4}{D} = \frac{8}{3}\frac{1}{384}\frac{P_{EH}b^4}{D} = \frac{2}{384}\frac{P_C b^4}{D}$$

Summary

Load

$$P_Y = 2.25 \cdot \sigma_y \left(\frac{t}{b}\right)^2 = 2 \cdot \sigma_{yp} \left(\frac{t}{b}\right)^2$$

$$P_{EH} = 3.375 \cdot \sigma_y \left(\frac{t}{b}\right)^2 = 3 \cdot \sigma_{yp} \left(\frac{t}{b}\right)^2$$

$$P_C = 4.5 \cdot \sigma_y \left(\frac{t}{b}\right)^2 = 4 \cdot \sigma_{yp} \left(\frac{t}{b}\right)^2$$

Deflection

$$\delta_Y = \frac{1}{384} \frac{P_Y b^4}{D}$$

$$\delta_{EH} = \frac{1}{384} \frac{P_{EH} b^4}{D}$$

$$\delta_C = \frac{2}{384} \frac{P_C b^4}{D}$$

If we normalize the loads by P_C and the deflections by δ_C we get:

Load

$$\tilde{P}_Y = \frac{P_Y}{P_C} = 0.5$$

$$\tilde{P}_{EH} = \frac{P_{EH}}{P_C} = 0.75$$

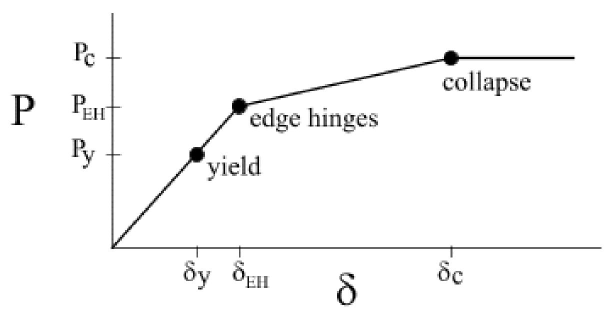
$$\tilde{P}_C = \frac{P_C}{P_C} = 1$$

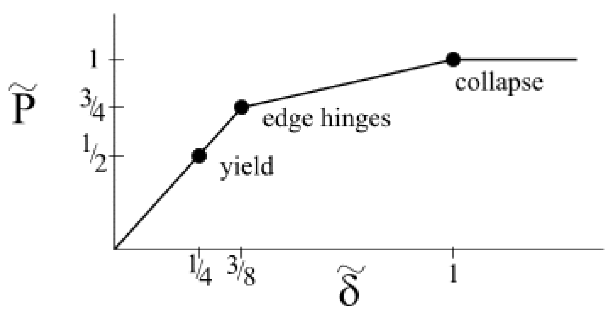
Deflection

$$\tilde{\delta}_Y = \frac{\delta_Y}{\delta_C} = \frac{1}{4}$$

$$\tilde{\delta}_{EH} = \frac{\delta_{EH}}{\delta_C} = \frac{3}{8}$$

$$\tilde{\delta}_C = \frac{\delta_C}{\delta_C} = 1$$





Thus we see that:
P_Y is 50% of P_C and P_{EH} is 75% of P_C; δ_Y is 25% of δ_C and δ_{EH} is 37.5% of δ_C

Topic 6: Elastic Bending in Restrained Laterally Loaded Plates

Introduction

In this lecture we will

- Describe and derive equations for the elastic bending of a laterally restrained long plate (with membrane stresses)
- show the behaviour diagram in terms of a load-deflection plot

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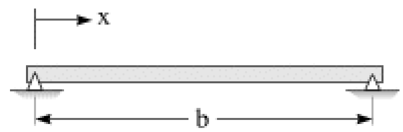
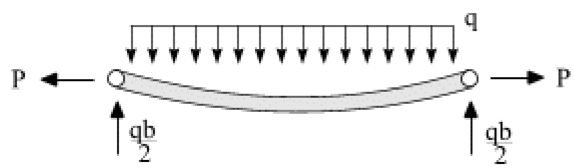
We start by considering a long plate, pinned at the supports but restrained from pulling in.

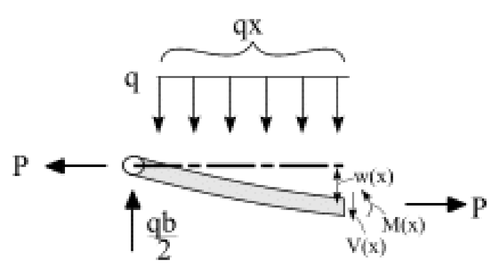
Initial situation:



Under uniform load q :



consider a free body diagram as part of the plate cut at x;



V: shear
M: moment
w : deflection
P: lateral reaction

Summing moments at the cut (at x); $\Sigma +\circlearrowright$

$$M(x) + P \cdot w(x) + q \cdot x \cdot \frac{x}{2} - \frac{qb}{2} x = 0$$

which gives a moment of;

$$M(x) = \frac{qb}{2} x - q \cdot x \cdot \frac{x}{2} - P \cdot w(x)$$

recall that;
~~~~~~

$$M(x) = -D\frac{d^2w(x)}{dx^2}$$

so that we can write the differential equation for an elastic restrained plate ;

$$D\frac{d^2w(x)}{dx^2} = P\cdot w(x) - \frac{q}{2}(bx - x^2)$$

The general solution for such a D.E. is;

$$w(x) = C_1\sinh\frac{2Ux}{b} + C_2\cosh\frac{2Ux}{b} + \frac{qb^4}{8U^2D}\left\{\frac{x}{b} - \frac{x^2}{b^2} - \frac{1}{2U^2}\right\}$$

where;

$$U^2 = \frac{Pb^2}{4D}$$

if we take the boundary conditions as;

$$w(0) = 0, \qquad w(b) = 0$$

we get the specific solution;

$$w(x) = \frac{qb^4}{16U^4D}\left\{\frac{\cosh\left[U\left(1-\frac{2x}{b}\right)\right]}{\cosh U} - 1\right\} + \frac{qb^2(bx - x^2)}{8U^2D}$$

unfortunately, this equation is recursive. The deflection w depends on U, which depends on the axial force P, which depends on the elongation of the plate, which depends on w. If not for this interdependence we would be done.

To get the membrane force, we use;

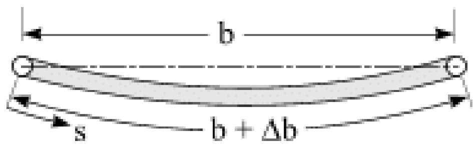
$$\begin{aligned} P &= \sigma_{axial}\cdot A \\ &= \frac{\varepsilon\cdot E\cdot A}{1-\nu^2} \\ &= \frac{\Delta b}{b}\frac{E\cdot t}{1-\nu^2} \end{aligned}$$

we find Δb, the elongation of the plate as follows;

$$\Delta b = \int_0^{b/2} \left(\frac{dw(x)}{dx} \right)^2 dx$$

~~~~~~~~~~~~~~~~~~

To derive this formula we consider the following;

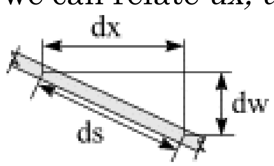


We define ‘s’ as the coordinate along the plate, while x is the purely horizontal coordinate.

x: 0 -> b

s: 0 -> b +Δb

we can relate *dx, ds* and *dw*; $ds = \sqrt{dx^2 + dw^2}$



we can say: $b = \int_{all} dx$ and $b + \Delta b = \int_{all} ds$

so we can write: $\Delta b = \int_0^b ds - \int_0^b dx = \int_0^b ds - dx = \int_0^b \sqrt{dx^2 - dw^2} - dx$

which we can re-arrange to give:

$$\Delta b = \int_0^b \left[\left(\sqrt{1 + \left(\frac{dw}{dx} \right)^2} \right) dx - dx \right]$$

and

$$\Delta b = \int_0^b \left[\left(\sqrt{1 + \left(\frac{dw}{dx} \right)^2} \right) - 1 \right] dx$$

we can make use of the approximation:

$\sqrt{1+a} \cong 1 + \frac{a}{2}$ for a<<1 (i.e. small deflections!)

to get: $\sqrt{1 + \left(\frac{dw}{dx} \right)^2} - 1 \cong \frac{1}{2} \left(\frac{dw}{dx} \right)^2$
~~~~~~~~~~~~~~~~~~

This allows us to write:

$$\Delta b = \frac{1}{2}\int_0^b \left(\frac{dw}{dx}\right)^2 dx$$

and by symmetry we can write:

$$\Delta b = \int_0^{b/2} \left(\frac{dw}{dx}\right)^2 dx$$

~~~~~~~~~~~~~~~

Returning to the problem, we have;

$$\boxed{w(x) = \frac{qb^4}{16U^3 D\tanh U}\left\{\frac{\cosh\left[U\left(1-\frac{2x}{b}\right)\right]}{\cosh U} - 1\right\} + \frac{qb^2(bx - x^2)}{8U^2 D}}$$

with $U = \sqrt{\frac{Pb^2}{4D}}$

The axial load P is;

$$P = \frac{\Delta b}{b}\frac{E\cdot t}{1-\nu^2} = \int_0^{b/2}\left(\frac{dw}{dx}\right)^2 dx \quad \frac{E\cdot t}{b(1-\nu^2)}$$

We can re-arrange this to give:

$$\int_0^{b/2}\left(\frac{dw}{dx}\right)^2 dx = \frac{Pb(1-\nu^2)}{E\cdot t}$$

Solving the left hand side and re-arranging will give a expression for U (which contains P) in terms of q (the applied load):

$$\boxed{\frac{E^2\cdot t^8}{q^2 b^8(1-\nu^2)} = -\frac{81}{16U^7\tanh U} - \frac{27}{16U^6\sinh^2 U} + \frac{27}{4U^8} + \frac{9}{8U^6}}$$

This equation (labeled #9 in Ratzlaff and Kennedy) can be solved for U, but only numerically (trial and error) (HINT: Use the GOALSEEK function in
~~~~~~~~~~~~~~~

Microsoft(R) Excel(R)).. The expression for w(x=b/2) will be the cental deflection and when plotted vs load will normally be very similar to the equation;

$\boxed{w_{central} = \frac{1}{384}\frac{qb^4}{D}}$, which is the elastic deflection equation for a pinned-pinned plate in the absence of axial loads.

Summary:

We have considered a pinned-pinned plate and included one small additional effect, the axial load that arises as the beam deforms. We have a solution, but as we see, it is quite complicated. The solution is elastic, so it is only really valid up to edge hinge formation. By this point the deflections are usually still so small that the axial force is too small to worry about. We would only want this solution for certain thin plates, of an intermediate thickness. Thicker plates have too little axial force, and very thin plates have negligible bending strength (i.e. all axial).

Topic 7: Membrane Behavior in Restrained Laterally Loaded Plates

Introduction

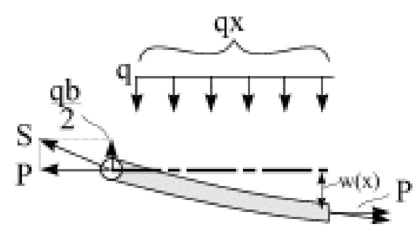
In this lecture we will

- Describe and derive equations for the membrane distortion of a laterally loaded long plate (with membrane stresses)
- show the behaviour diagram in terms of a load-deflection plot

~~~~~~

We will now look at a solution for plate with just membrane behavior. This means that there is a local axial force, but no bending moments. For this case we have the free-body diagram;

The force in a membrane is always aligned with the surface (no shear, no moment).



Taking moments at the cut:

$$P \cdot w(x) + \frac{q \cdot x^2}{2} - \frac{q \cdot b \cdot x}{2} = 0$$

$$P \cdot w(x) + \frac{q}{2}(x^2 - b \cdot x) = 0$$

$$q = \frac{2P \cdot w(x)}{x(b-x)}$$

Now for $x = b/2$ ($w(x)=w_{max}$);

$$q = \frac{2P \cdot w_{\max}}{b/2\,(b - b/2)} = \frac{8P \cdot w_{\max}}{b^2}$$

and we can write: $w_{\max} = \dfrac{q \cdot b^2}{8P}$ and $P = \dfrac{q \cdot b^2}{8w_{\max}}$

This gives us expressions relating load, membrane reaction and maximum deflection. If q and b are known, we still have two unknowns, P and w_{max}.
~~~~~~

From the sum of the reaction force vectors we get;

$$S = \sqrt{P^2 + \left(\frac{qb}{2}\right)^2}$$

using $P = \dfrac{q \cdot b^2}{8w_{\max}}$, inserted into the above we get;

$$S = \sqrt{\left(\frac{q}{8w_{\max}}\right)^2 b^4 + \left(\frac{qb}{2}\right)^2}$$

We can re-arrange this to get;

$$q = \frac{8w_{\max} S}{\sqrt{b^4 + (4w_{\max} b)^2}} \quad \text{(this is equation 16 in RK)}$$

This gives us load vs length, deflection and membrane force.

Now we need to find S from w_{max} and b.

Plastic Case (simplest)

When the plate is fully plastic;

$$S = \frac{\sigma_y \cdot t}{\sqrt{1 - \nu_p + \nu_p^2}} = 1.155\sigma_y \cdot t \quad \text{(note } \nu_p = 0.5\text{)}$$

hence;

$$\boxed{q = \frac{9.24 w_{\max} \cdot \sigma_y \cdot t}{\sqrt{b^4 + (4w_{\max} b)^2}}}$$

Note that this load – deflection is approximately linear.

Why is the plastic case the simplest?

Because the of the elastic-perfectly plastic assumption: when the plate is fully plastic, the maximum stress is σ_y. The fact that the stress cannot be higher than σ_y allows us to say that the membrane force is equal to σ_y for a plate (i.e. $\sigma_y / \sqrt{1 - \nu_p + \nu_p^2}$) times the cross sectional area of the plate (which is simply *t* for a unit strip of plate). In other words, *S* does not depend on w_{max}, it only depends on the plate being fully plastic.

~~~~~

Elastic Case (more complex)

In this case we need to derive S from w(x) because S depends on the stretch, which depends on w(x).

we derived the equation ;

$$q = \frac{2P \cdot w(x)}{x(b-x)}$$

Which we can re-arrange to give;

$$w(x) = \frac{q(xb - x^2)}{2P}$$

we will be using the formula for Δb that we used in the last lecture;

$$\Delta b = \int_0^{b/2} \left(\frac{dw(x)}{dx} \right)^2 dx$$

we find the derivative of w(x);

$$\frac{dw(x)}{dx} = \frac{q}{2P}(b - 2x)$$

substituting $P = \dfrac{q \cdot b^2}{8w_{\max}}$ we get;

$$\frac{dw(x)}{dx} = \frac{4w_{\max}}{b^2}(b - 2x)$$

This is the slope of the membrane in terms of b, w_{max} as a function of x.

we can now write;
~~~~~

$$\Delta b = \int_0^{b/2} \left(\frac{4w_{max}}{b^2}(b-2x) \right)^2 dx = \frac{16w_{max}^2}{b^4} \int_0^{b/2} \left(b^2 - 4bx + 4x^2 \right) dx$$

$$\Delta b = \frac{8}{3} \frac{w_{max}^2}{b}$$

so that the strain is;

$$\varepsilon = \frac{\Delta b}{b} = \frac{8}{3} \frac{w_{max}^2}{b^2}$$

Now we can

$$S = \sigma \cdot t$$

$$= \frac{\varepsilon \cdot E \cdot t}{1-\nu^2}$$

$$= \frac{8}{3} \left(\frac{w_{max}}{b} \right)^2 \frac{E \cdot t}{1-\nu^2}$$

We can now substitute S back into the membrane equation to get;

$$\boxed{q = \frac{64}{3} \left(\frac{w_{max}}{b} \right)^3 \frac{E \cdot t}{1-\nu^2} \frac{1}{\sqrt{b^2 + (4w_{max})^2}}}$$

This is the elastic membrane equation, and is approximately a cubic.

Behaviour Diagram

We have derived 4 different solutions for plates. One requires a trial and error solution. The other 3 are elastic-plastic bending, plastic and elastic membrane. We'll summarize the equations here and plot then for various example plates. For consistency, We'll use δ as the central deflection (instead of w_{max});

Elasto/Plastic bending:

Load

$$P_Y = 2.25 \cdot \sigma_y \left(\frac{t}{b}\right)^2$$

$$P_{EH} = 3.375 \cdot \sigma_y \left(\frac{t}{b}\right)^2$$

$$P_C = 4.5 \cdot \sigma_y \left(\frac{t}{b}\right)^2$$

Deflection

$$\delta_Y = \frac{1}{384}\frac{P_Y b^4}{D}$$

$$\delta_{EH} = \frac{1}{384}\frac{P_{EH} b^4}{D}$$

$$\delta_C = \frac{2}{384}\frac{P_C b^4}{D}$$

Plastic Membrane:

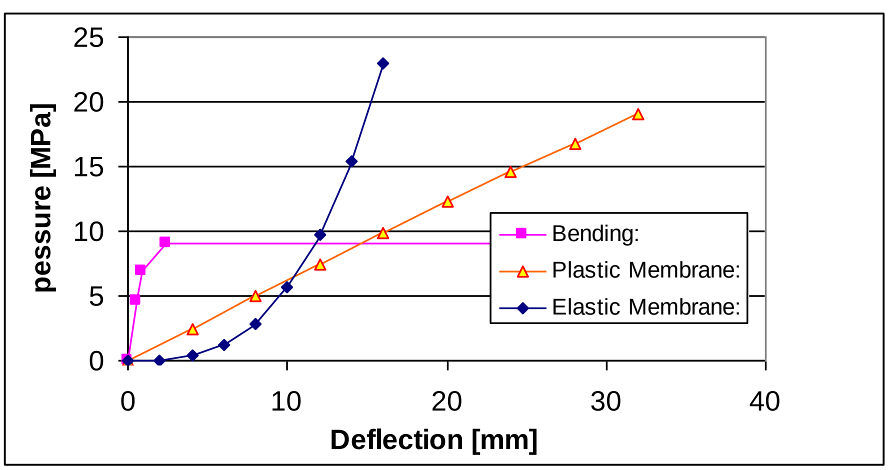
$$\boxed{q = \frac{9.24 \cdot \delta \cdot \sigma_y \cdot t}{\sqrt{b^4 + (4\delta \cdot b)^2}}}$$

Elastic Membrane:

$$\boxed{q = \frac{64}{3}\left(\frac{\delta}{b}\right)^3 \frac{E \cdot t}{1-\nu^2} \frac{1}{\sqrt{b^2 + (4\delta)^2}}}$$

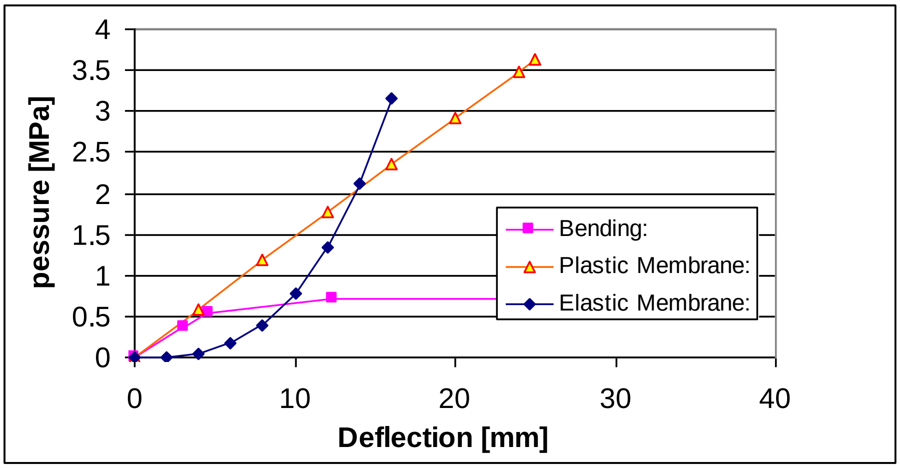
Case 1- a thick plate:

Parameters:		
b	400	mm
t	30	mm
E	207,000	MPa
ν	0.3	
σ	360	MPa
D	511,813,187	



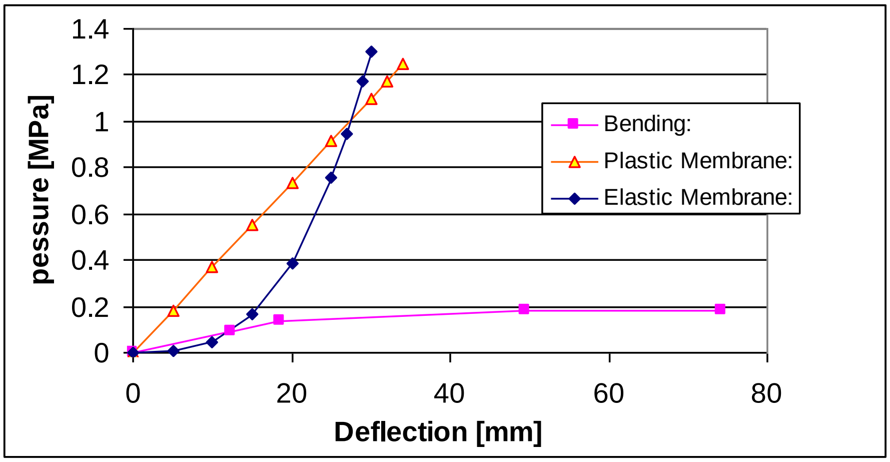
Case 2- an intermediate plate:

Parameters:		
b	500	mm
t	10	mm
E	207,000	MPa
ν	0.3	
σ	400	MPa
D	18,956,044	



Case 3- a thin plate:

Parameters:		
b	1000	mm
t	10	mm
E	207,000	MPa
ν	0.3	
σ	400	MPa
D	18,956,044	


pessure [MPa]
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
20
40
60
80
Deflection [mm]
Bending:
Plastic Membrane:
Elastic Membrane:

Topic 8: Low Aspect Ratio Plates, and Permanent Set

Introduction

In this lecture we will

- Define a normalized capacity for plates, and examine how to extend the capacity description to low aspect ratio plates (i.e. not long plates)
- examine the Clarkson method for designing plates for a specified permanent set.

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We define a normalized load Q;

$$Q \equiv \frac{qE}{\sigma_y{}^2}$$

For the yield condition in bending we have;

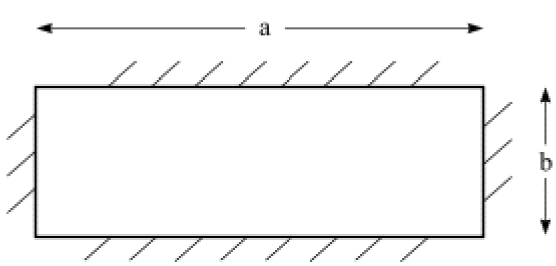
$$Q_y = \frac{q_y E}{\sigma_y{}^2} = \frac{2}{\sqrt{1-\nu+\nu^2}} \sigma_y \left(\frac{t}{b}\right)^2 \frac{E}{\sigma_y{}^2}$$

$$= \frac{2}{\sqrt{1-\nu+\nu^2}} \frac{t^2}{b^2} \frac{E}{\sigma_y}$$

We define $\beta = \frac{b}{t}\sqrt{\frac{\sigma_y}{E}}$, as a kind of slenderness ratio, which lets us write;

$$Q_y = \frac{2}{\sqrt{1-\nu+\nu^2}} \frac{1}{\beta^2}$$

This is for long plates, a>>b (say a>3b);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For low aspect ratio aspect ratio plates (a<3b), we can use the equation;

$$Q_y = \frac{2}{\sqrt{1-\nu+\nu^2}} \frac{1}{\beta^2}\left[1+0.6\left(\frac{b}{a}\right)^4\right]$$

The extra factor results in an increase of;

a/b	factor	increase
1	1.6	60%
1.2	1.29	29%
1.5	1.12	12%
2	1.04	4%
3	1.007	0.7%
4	1.002	0.2%

The above equation refers to the yield condition. When just brought to yield, there will be no permanent deformation (set). For load above Q_y there will be increasing levels of permanent set.

So far we have discussed load-deflection curves. We have approached plastic bending strength from the point of view of load capacity. Another approach to plastic design is to allow some level of permanent set, as a % of the plate thickness, or as a % of plate span.

Refer to pages 351-354 in Hughes. Hughes describes the experimental and analytical work of Clarkson, and develops a set of 'design' plots. Using the plots, you can determine capacity of scantlings to achieve a given level of permanent set (i.e. a given level of denting for the design load).

Figure 1 below, shows plate capacity Q, vs. the slenderness ratio β, for various aspects ratios (a/b) of plate. This particular plot gives values for a level of permanent set of;

$$\frac{w_p}{\beta \cdot t} = 1.0$$

This is equivalent to;

$$\frac{w_p}{b} = 1.0\sqrt{\frac{E}{\sigma_y}}$$

for typical shipbuilding steel (σ_y = 250 -> 400), this is approximately

$$\frac{w_p}{b} = 0.039$$

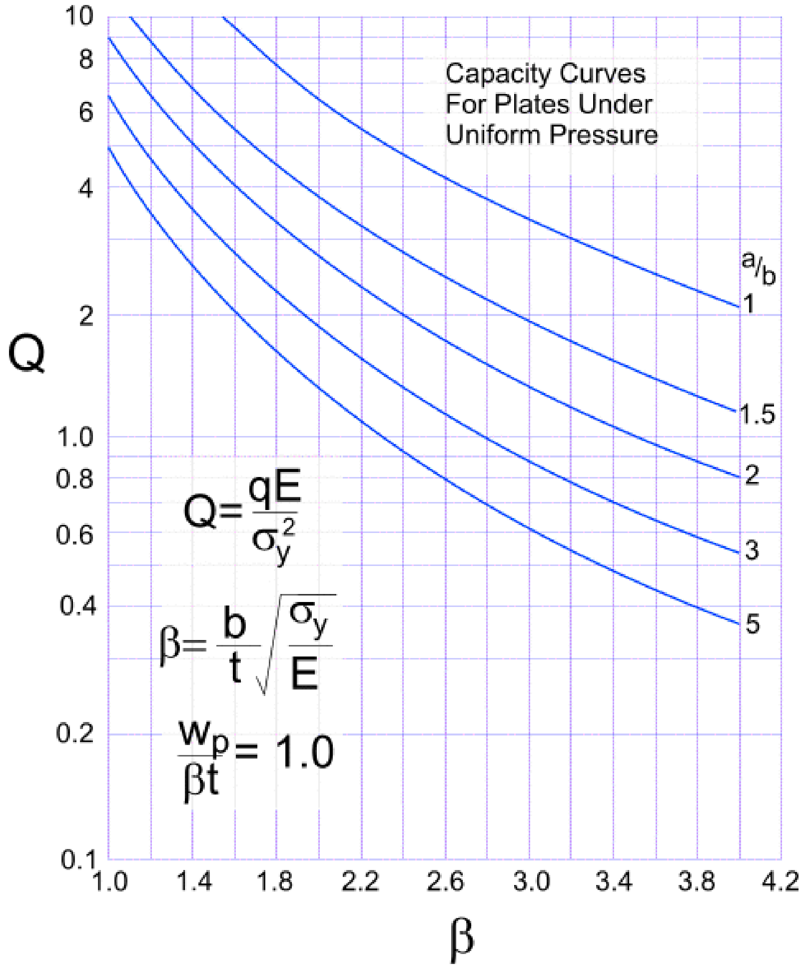

Fig 1.

<u>Example</u>: For a plate with the following parameters, what is the load capacity and permanent deflection according to Fig. 1?

a = 1000mm
b = 500mm
σ_y = 300MPa
t = 15mm

Step 1 : find β. $\beta = 500/15\sqrt{300/207000} = 1.59$
Step 2 : lookup Q on the a/b=2 curve. *Q=4.1*
Step 3 : find q. $q=4.1*300^2/207000= 1.78$ [MPa] <= ANS.
Step 4 : find w_p. $w_p=1.59*15=19mm$ <= ANS.

Note that $q_y = 2.25\, \sigma_y\, (t/b)^2=0.39$ MPa. The q at 19mm deflection is not only above yield, it is above the 3 hinge load q_c.

The plots in Figure 2 show four additional cases, for a range of permanent set. These plots can be used just like Figure 1.

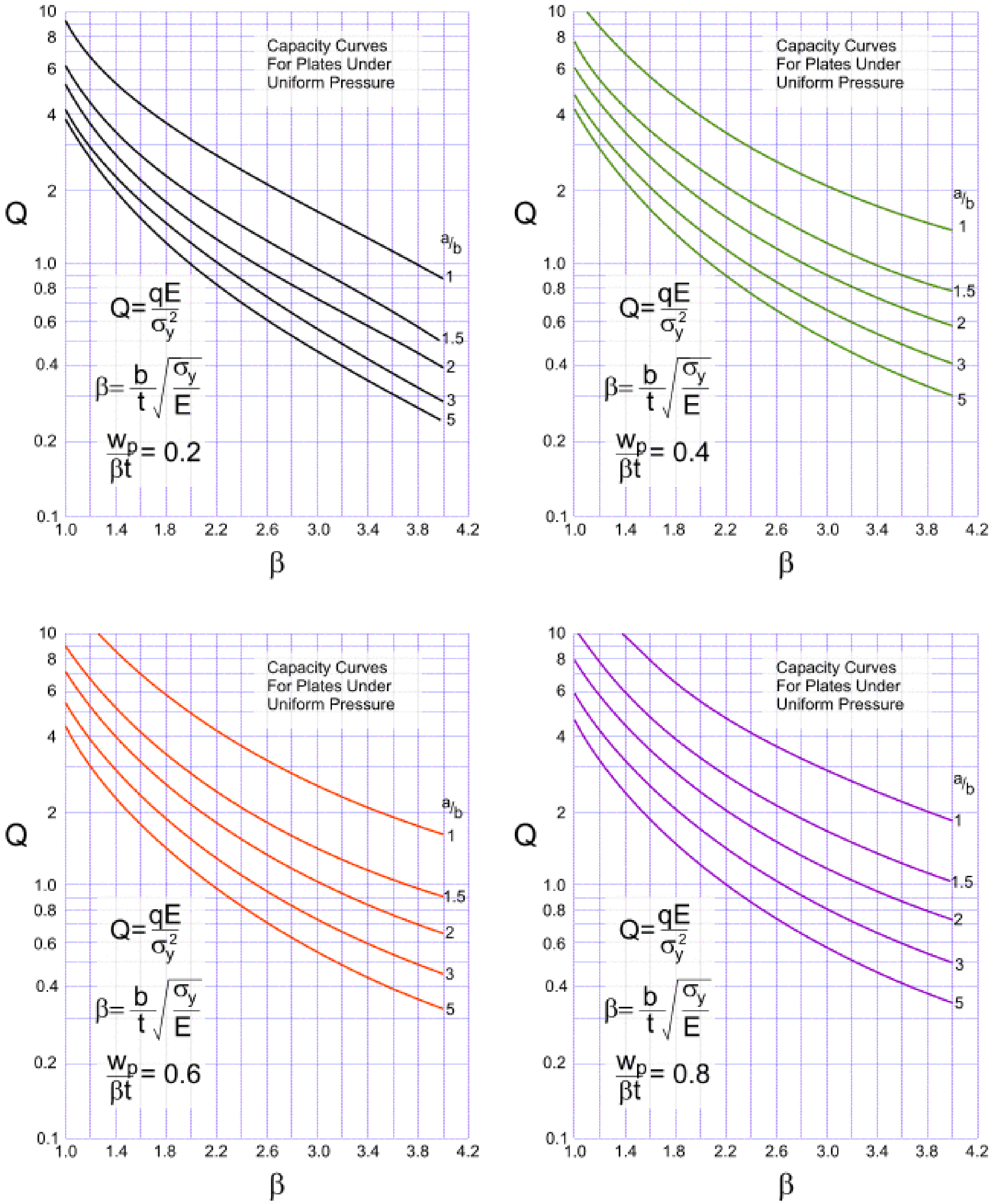

Fig 2. Capacity plots for $w_p/\beta t$= 0.2, 0.4, 0.6 and 0.8

We will now explore a variety of ways to use the above plots.

<u>**Case 1**</u>: find plate thickness and permanent deflection for $w_p/\beta t= 0.2$, with the given values;

a = 800 mm
b = 400 mm
σ_y = 250 MPa
q = 0.5 MPa

Step 1 : find Q. *Q=0.5 * 207000/250²=1.656*
Step 2 : lookup β on the a/b=2 curve (top left plot of Fig.2) . *β=1.9*
Step 3 : find t. *t=400/1.9*(300/207000)= 7.31* [mm] <= ANS.
Step 4 : find w_p. *w_p=0.2*1.9*7.31=2.8mm* <= ANS.

<u>**Case 2**</u>: find permanent deflection, with the given values;

a = 1800 mm
b = 600 mm
σ_y = 250 MPa
q = 0.5 MPa
t = 10 mm

Step 1 : find Q. *Q=0.5 * 207000/250²=1.656*
Step 2 : find β. $\beta = 600/10\sqrt{250/207000} = 2.083$
Step 3 : find w_p. We need to find a figure with a match between Q and β.
On the five figures, if we look up b=2.083 we get
$w_p/\beta t$= 0.2, Q=1.15
$w_p/\beta t$= 0.4, Q=1.35
$w_p/\beta t$= 0.6, Q=1.45
$w_p/\beta t$= 0.8, Q=1.65 ***
$w_p/\beta t$= 1.0, Q=1.80

***we can see that our Q and b correspond to a $w_p/\beta t$=0.8
this gives us:

$$w_p = 0.8 \cdot 600\sqrt{250/207000} = 16.7 \text{ mm}$$ <= ANS.

Topic 9: Plastic Plate Strength Based on Plate Folding

Introduction

In this lecture we will

7 Re-derive plate capacity at collapse based on energy methods

7 Extend the concept to low aspect ratio plates

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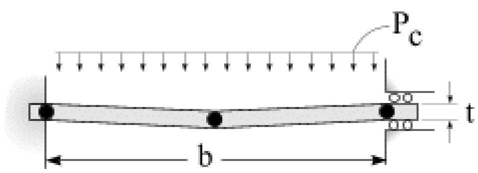
We derived the plastic collapse load for a long plate in lecture 10;



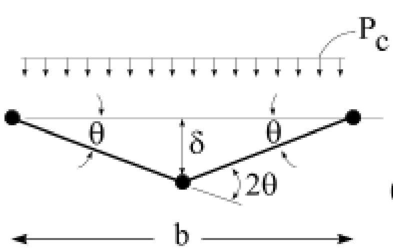
$$P_c \equiv \frac{16M_p}{b^2}, \quad M_p \equiv \frac{\sigma_y}{\sqrt{1-\nu+\nu^2}} \frac{t^2}{4}$$

Lets re-visit this with an energy approach.

Energy Solution Concept:

The applied load moves as the structure collapses. Consequently, the applied load does work (force x distance) to the structure. (work=energy expended). This is called the external work.
As it collapses, the structure forms plastic hinges, which rotate with a constant moment. Hence the hinges do work (moment x angle), and absorb energy. This is called the internal work.
We can equate the external and internal work. We need to find the angles in terms of the movement of the structure. We can solve the energy equation for the applied force.

We start by describing the pattern of collapse;



$$\theta = \frac{\delta}{b/2}$$

The work done by P_c is called external work (W_{EXT})

$$W_{EXT} = \int P_c \cdot \delta(x) \cdot dx$$
~~~~~~

In our case P_c is a constant so;

$$W_{EXT} = P_c \int \delta(x) \cdot dx$$

which is simply;

$$W_{EXT} = P_c \cdot b \frac{\delta}{2}$$

The internal work occurs at the plastic hinges (at discrete locations), and is the sum of the work done at each hinge;

$$W_{INT} = \sum M_p \cdot \theta = M_P \cdot 4\theta = M_P \frac{8\delta}{b}$$

When we equate external and internal work;

$$W_{EXT} = W_{INT}$$

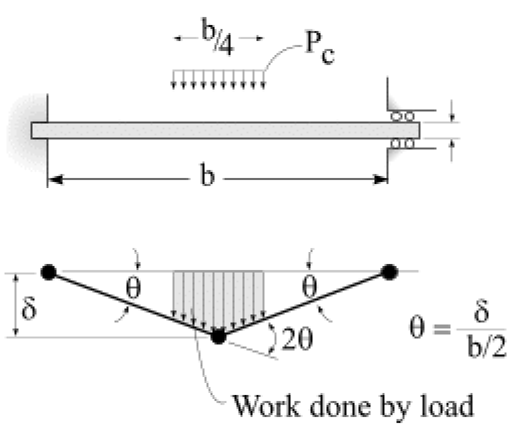
$$P_c \cdot b \frac{\delta}{2} = M_P \frac{8\delta}{b}$$

solving for P_c gives;

$$P_c = \frac{16 M_P}{b^2}$$

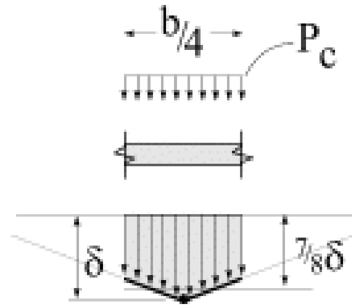
which is exactly the result we derived before, though without all the elastic analysis (differential equations) and the various steps.

We will now look at the real power of this method. We will find the plastic collapse load for a patch load. This analysis would be quite challenging if we were to try this using an elasto-plastic analysis.



The work done by P_c is called external work (W_{EXT})

$$W_{EXT} = \int P_c \cdot \delta(x) \cdot dx$$



In this case P_c only acts over b/4,
and the average deflection under the load is 7/8 δ;

$$W_{EXT} = P_c \cdot \frac{b}{4} \cdot \frac{7}{8} \delta$$

The internal work occurs at the plastic hinges (at discrete locations), and is the sum of the work done at each hinge, and is exactly as before;

$$W_{INT} = \sum M_p \cdot \theta = M_P \cdot 4\theta = M_P \frac{8\delta}{b}$$

When we equate external and internal work;

$$W_{EXT} = W_{INT}$$

$$P_c \cdot \frac{b}{4} \cdot \frac{7}{8} \delta = M_P \frac{8\delta}{b}$$

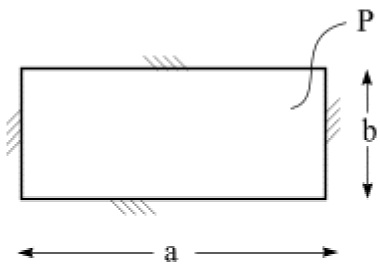
solving for P_c gives;

$$P_c = \frac{256}{7} \frac{M_P}{b^2} = 36.57 \frac{M_P}{b^2}$$

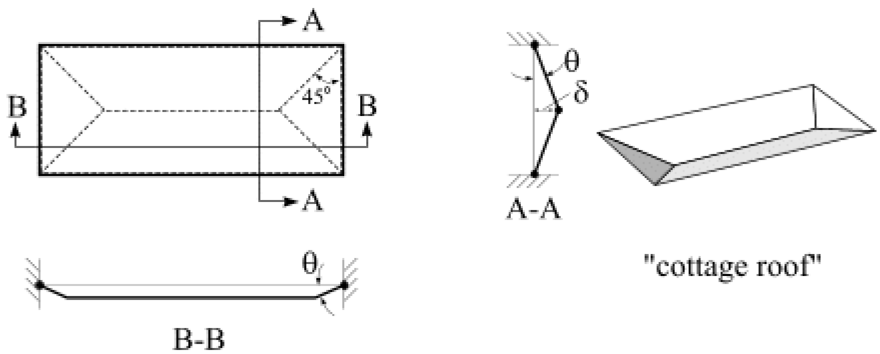
The pressure is 2.3x larger than the previous case, but the load length is ¼ as long, so the force is only 57% as much as in the uniform load case. In the limit case of a point load on the center, the force would be exactly half as much as in the uniform load case.

What pattern of load allows for the greatest load to be applied to a beam?

The above analysis applies to long plates. Now lets consider finite aspect ratio plates. Consider a plate a x b with a uniform load, fixed at all edges.



We assume that the plate collapses by folding along the lines shown below;

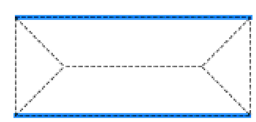


We assume that the deflections (i.e. δ) are very small, so that this folding pattern is kinematically possible. We again approach the problem as a balance of internal and external work. The total internal work is found from the sum of the work going into each hinge;

$$W_{INT} = \sum M_p \cdot l_n \cdot \theta_n$$

For the long edges;

$$\theta = \frac{2\delta}{b}, \; l = 2a$$

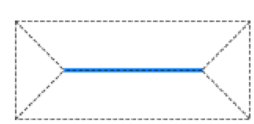


for the short edges;

$$\theta = \frac{2\delta}{b}, \; l = 2b$$

for the central hinge;

$$\theta = \frac{4\delta}{b}, \; l = a - b$$



and for the corner hinges;

$$\theta = 2\frac{\delta}{\sqrt{2}\, b/2}, \; l = 2\sqrt{2}b$$

$$= \frac{2\sqrt{2}\delta}{b}$$

This gives us a total internal work of;

$$W_{INT} = M_P\left(\frac{2\delta}{b}2a + \frac{2\delta}{b}2b + \frac{4\delta}{b}(a-b) + \frac{2\sqrt{2}\delta}{b}2\sqrt{2}b\right)$$

$$W_{INT} = M_P\frac{8\delta}{b}(a+b)$$

The external work is ;

$$W_{EXT} = \int P\cdot\delta\cdot da$$

which for a uniform pressure is;

$$W_{EXT} = P\int\delta\cdot da = P\cdot V$$

where V is the volume of the deflection (i.e. the volume of the cottage roof, or at least the volume of the cottage roof that is under the load). In this case we have;

$$W_{EXT} = P\left(\frac{b^2\delta}{3} + b(a-b)\frac{\delta}{2}\right)$$

or upon simplification;

$$W_{EXT} = P\cdot\delta\cdot b^2\left(\frac{a}{2b} - \frac{1}{6}\right)$$

When we equate internal and external work and solve for P_C, we get;

$$P_C = \frac{8\cdot M_P}{b^2}\frac{\left(1+\frac{a}{b}\right)}{\left(\frac{a}{2b}-\frac{1}{6}\right)}$$

for say a/b=4, we have;

$$P_C = \frac{8\cdot M_P}{b^2}\frac{(5)}{\left(\frac{11}{6}\right)} = \frac{21.8\cdot M_P}{b^2}$$

for a/b=1, we have;

$$P_C = \frac{8\cdot M_P}{b^2}\frac{(2)}{\left(\frac{2}{6}\right)} = \frac{48\cdot M_P}{b^2}$$. This is 4 x greater than the long plate solution (compared with 1.6 x for the elastic solution).

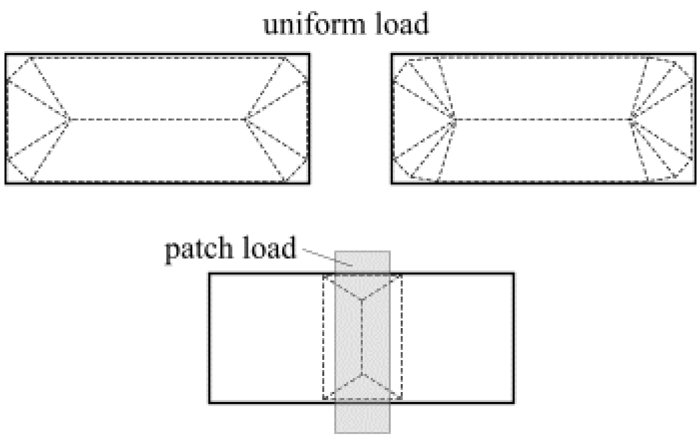
for a/b=10, we have;

$$P_C = \frac{8 \cdot M_P}{b^2} \frac{(11)}{\left(\frac{29}{6}\right)} = \frac{18.2 \cdot M_P}{b^2}$$

For very large a/b this converges to $\frac{16 \cdot M_P}{b^2}$ as before.

The above discussion shows the general idea of energy methods and plate folding method to find plate capacity. The method is very robust and can be handle a wide range of problems which would be very difficult with other analytical methods.
There are a few things to keep in mind when using these energy methods.

1) the method works by assuming a collapse pattern. There are an infinite number of patterns possible. The 'correct' pattern is the one that gives the lowest load. So the method often involves trying several patterns, and choosing the smallest. In the case given above, we assumed that the corner angle was 45°. We could have chosen another angle such as 40°. Or we could have tried the various patterns shown below.



2) the method is called an 'upper bound' method. This means that the method will theoretically bound the strength from above. If the strength is overestimated, this is non-conservative. In practice, there are issues that tend to both raise and lower the estimates, so that in practice the result may be conservative. The strength of the method is that it tends to be close, and is a handy way to check a value. Work/energy considerations are extremely strong principles, so that the results are seldom far from reality.

Entrepreneurial idea: develop a software tool to quickly solve this class of problems. It will find wide use as structural design moves increasingly to plastic behavior. (User defines geometry of structure, folds and load.

Software solves for internal and external work, and determines collapse load. Software solves for optimum parameters in pattern.)

Topic 10: Introduction to Elastic Buckling

Introduction

In this lecture we will

- Discuss the concept of buckling
- review column buckling

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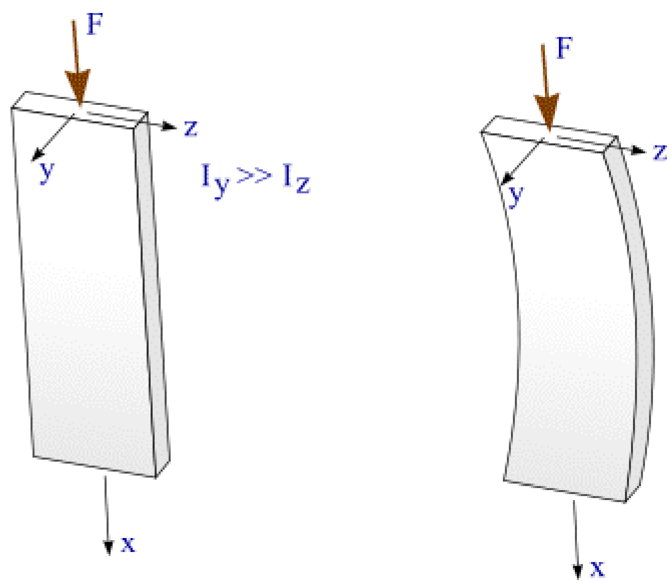
Buckling is a type of structural behavior.

- It can often lead to sudden loss of strength and collapse
- Elastic buckling may be acceptable, if it does not lead to plasticity
- Buckling tends to be more important as steel strength increases
- Buckling often dominates design, because buckling control results in geometric limits

Column Buckling

Consider a column of length L, area A, modulus E, moment of inertia Iz. Iz is the weak axis of bending, as Iy>Iz. The column will bend about the z axis, moving laterally in the y direction.

We assume that the column is initially straight, that the load F acts along the centroid (neutral axis). In the unbuckled state the column is in pure compression (no bending).



Under low load levels the column stays straight, and slightly shortens;
~~~~~~

The compression stress is $\sigma = \dfrac{F}{A}$, the strain is $\varepsilon = \dfrac{F}{A \cdot E}$.

The change in length is $\Delta L = \varepsilon \cdot L = \dfrac{F \cdot L}{A \cdot E}$.

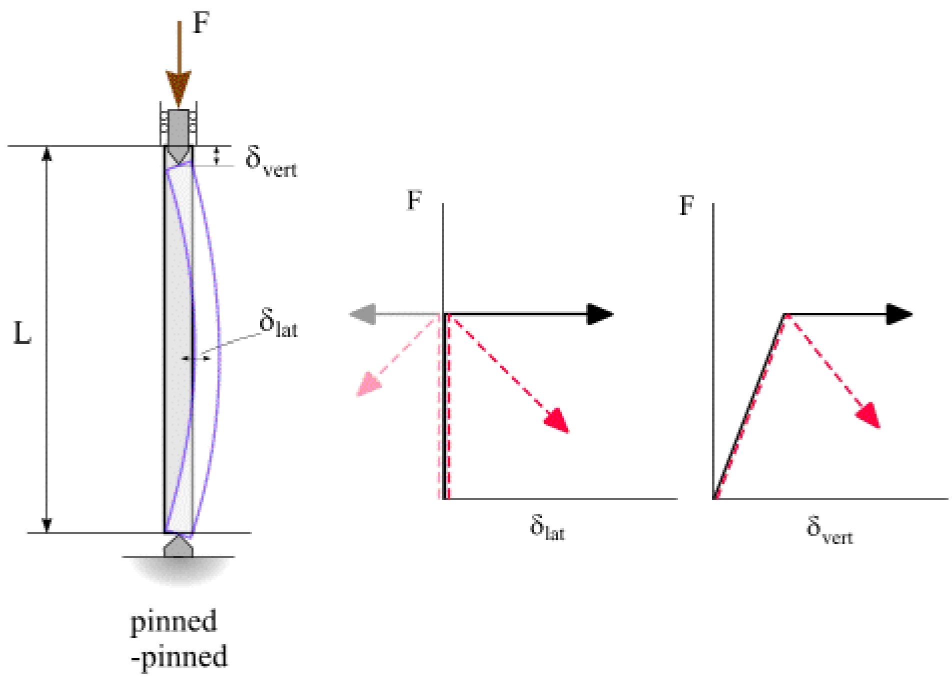
Observations show that at some critical load level F_{crit}, the column will suddenly deform sideways!

Plot Load vs Deflection.

As the load is steadily increased, the column deflects vertically (compresses). Prior to buckling there is no lateral deflection of the column. At the point of buckling the lateral deflection increases (and grows large) under no increase in load. The vertical deflection will also grow large under no additional load. This case is plotted with the solid lines in the plots below.

There is another case with slightly different post-buckling behavior. Instead of having a static load (like static weights), we might apply the load by imposing a vertical deflection on the top of the column. This would be called 'displacement control' (vs. 'load control'). In this case the load-deflection curves (plotted with dashed lines) would be the same up to buckling, but the load would fall after buckling. The drop in load is called the 'unloading' portion of the curve.

This type of instability is sometimes called 'Euler' buckling or 'bifurcation' buckling.



The buckling load can be found in various ways;

1) solve the differential equation:

$$EI_z \frac{d^2 y(x)}{dx^2} = -F \cdot y(x)$$

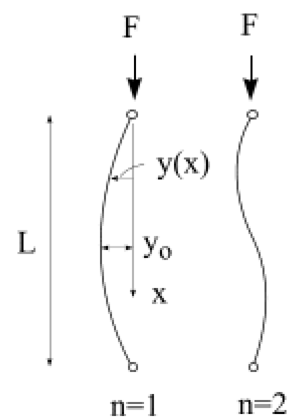
2) Assume a shape y(x), and solve for F (shortcut to solution of the D.E.;

3) Use energy methods.

~~~~~~~~~~

Lets try no. 2).

Assume a deflected shape:



$$y(x) = y_o \sin(\frac{n\pi}{L} x)$$

$$y'' = -\left(\frac{n\pi}{L}\right)^2 y_o \sin(\frac{n\pi}{L} x)$$

$$y'' = -\left(\frac{n\pi}{L}\right)^2 y(x)$$

From this the differential equation becomes;

$$EI_z \left(-\left(\frac{n\pi}{L}\right)^2 y(x) \right) = -F \cdot y(x)$$

$$F = \frac{n^2 \pi^2 EI_z}{L^2}$$

for n=1, we have;

$$\boxed{F_{crit} = \frac{\pi^2 EI_z}{L^2}}$$

which is called the Euler buckling formula.
~~~~~~~~~~

Use of High Strength Steel:

Notice that the buckling formula has no mention of strength. The buckling strength seems not to depend on strength. However, buckling strength does depend on I, which for a plate, depends on plate thickness, which does depend on strength.

Example: take a plate designed for a lateral load, but which experiences some in-plane compression. The plate is originally intended to be build with regular mild steel (σ_y=225 MPa). The plate was sized using the standard formula;

$$t = 0.67 \cdot b \cdot \sqrt{\frac{P}{\sigma_y}}$$

The critical buckling force is;

$$F_{crit} = \frac{\pi^2 E I_z}{L^2}, \; I_z = \frac{a \cdot t^3}{12} \text{ or } F_{crit} = \frac{\pi^2 E \cdot a \cdot t^3}{12 L^2}$$

If we double the steel strength to 450 MPa, we can decrease the thickness by $1/\sqrt{2}$ (= 0.707) with no loss of bending strength. Unfortunately the buckling strength will decrease to $.707^3$=.353. In other words the buckling strength of the high strength steel may be only 35% of what it would be with mild steel. Of course, it may be that the buckling strength (at 35%) is still high enough. This example just shows that increase of yield strength does not increase buckling strength. If buckling is the critical failure mechanism, then no increase in material strength will help.

Support Conditions and Critical Length

The buckling pattern, and the critical load, depend on the support conditions for the column. When the ends are pinned, the column can form a simple half-sine wave between the supports. If both ends of the column are fixed the column will buckle into a shape with two inflection points. The center half will form a half-sine curve. The inflection point will signify that there is no bending moment at the quarter points. In this case the middle half of the beam will act like an Euler column. The critical buckling load is;

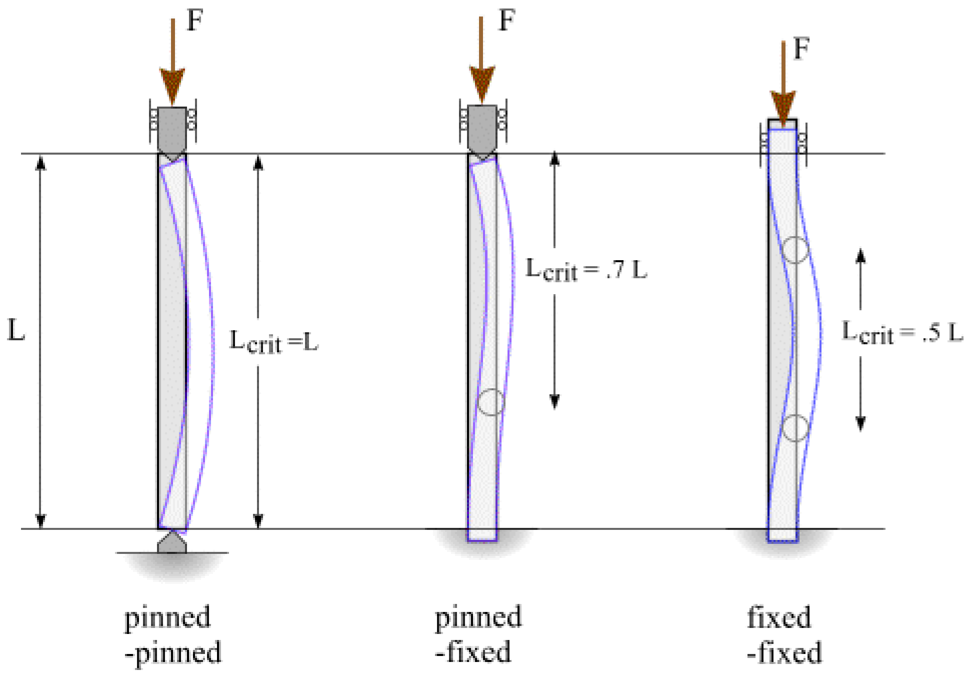
$$F_{crit} = \frac{\pi^2 E I_z}{(kL)^2}$$

where k=0.5. kL is called the critical length.

If one end is fixed and the other end is pinned, the column will buckle into a shape with one inflection point. The column between the inflection point (which is like a pin) will buckle according to the Euler formula, with k=0.7;

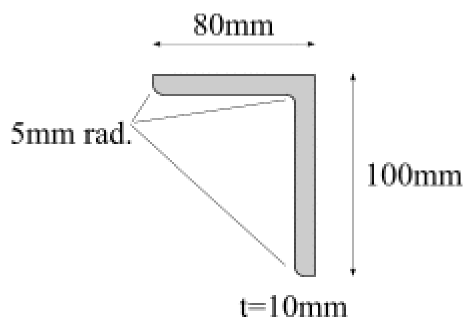
$$F_{crit} = \frac{\pi^2 E I_z}{(kL)^2}$$

These various behaviors are shown below.



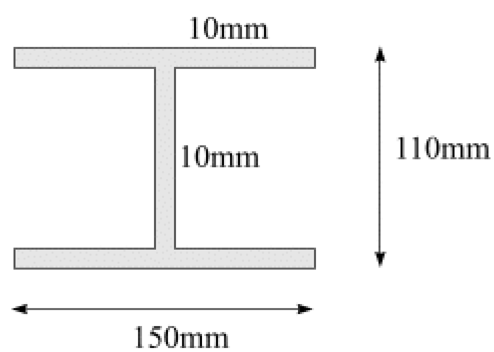
T.10 – Problems.

1. A column is constructed with an angle section (100x80x10), 2 m. long. The steel has a yield strength of 250MPa. The ends of the column are considered free to rotate. What is the largest load that can be applied to the column (without yielding or buckling)?



2. You want to use a steel pipe with a 5mm wall thickness as a column to support a deck edge. The base of the column is fixed, while the top is free to rotate. The column is 2.5m long. The steel has a yield strength of 230MPa. You want to carry a load of 150kN at failure (yield or buckling), and keep the pipe diameter as small as possible. What is the outside diameter of the pipe?

3. What is the buckling load for a column made from a wide I beam (110 tall x 150wide x 10 thick), that is 4m long with pinned ends? When braced at the center in the weak direction, what is the new buckling load?



Topic 11: Elastic Buckling by Energy Methods

Introduction

In this lecture we will
- Discuss buckling as an energy balance
- derive column buckling in 2 different ways

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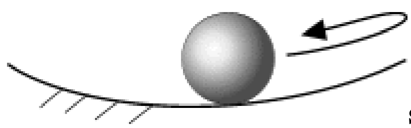
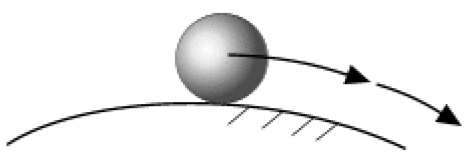
Energy Methods

IDEA: As a force is applied to any elastic body, the force does work (expends energy) which is stored in the body as elastic potential energy. When the force is low, and the system is stable, the body stores energy by distorting in the direction of the force. At this point any further movement of the body uses more energy than it releases. The body will return from any small disturbances.



stable state

When the load is sufficiently high, the elastic body is highly strained, and is storing a large amount of potential energy. The body may find that a lateral distortion releases more energy than is required to cause the lateral distortion. In a column, this means that a lateral bend will release more axial energy than is needed to bend the column. At this point the body is elastically unstable.



unstable state

We can examine any form of elastic buckling in this way.
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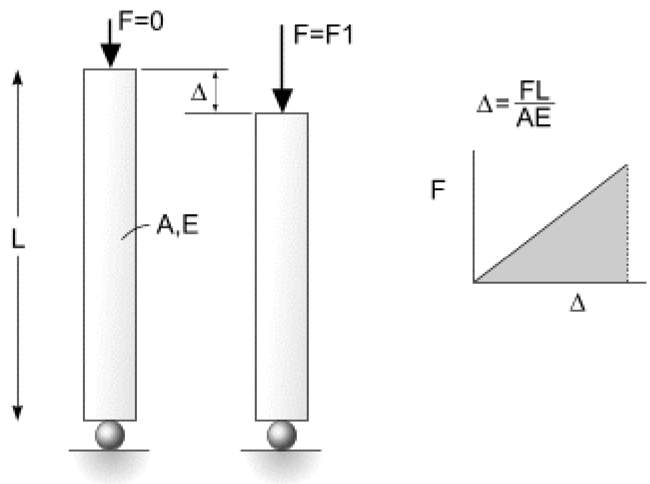
Consider the energy in a column;



The External Work done $= 1/2\, F \cdot \Delta$

$$= \frac{F^2 L}{2AE}$$

The internal Strain Energy is;

$$U_A = \frac{1}{2}\int_0^L EA\varepsilon^2 dx$$

for a simple column $\varepsilon = \dfrac{\sigma}{E} = \dfrac{F}{AE}$

$$U_A = \frac{1}{2}\int_0^L EA\left(\frac{F}{AE}\right)^2 dx = \frac{1}{2}\frac{F^2}{AE}\int_0^L dx$$

hence;

$$U_A = \frac{F^2 L}{2AE}$$ <= same as EW

Note:

$InternalWork$

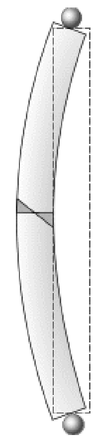
$$= \int \frac{1}{2} F \cdot \Delta x$$

$$= \frac{1}{2}\int \sigma A \cdot \varepsilon dx$$

$$= \frac{1}{2}\int E\varepsilon A \cdot \varepsilon dx$$

$$= \frac{1}{2}\int EA\varepsilon^2 dx$$

The column will distort in any way possible, so that it is storing the least amount of energy. One way to distort is to bend sideways. Buckling is bending sideways.



The elastic strain energy in bending is;

$$U_B = \frac{1}{2} EI \int_0^L \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad \text{(see derivation at end)}$$

- The bent column must store U_B
- The bent column is longer than the straight column
- The bent column has released axial energy and gained bending energy.
- If more energy is released axially than needed to bend the additional surplus will become kinetic energy (velocity)

Lets assume the buckled shape is;

$$y(x) = y_o \sin \frac{\pi x}{L}$$

which means;

$$y'' = -\frac{\pi^2}{L^2} y_o \sin \frac{\pi x}{L}$$

$$U_B = \frac{1}{2} EI \left(\frac{y_o \pi^2}{L^2} \right)^2 \int_0^L \sin^2 \frac{\pi x}{L} dx \quad \text{(integral part = L/2)}$$

$$U_B = \frac{\pi^4 EI \cdot y_o^2}{4L^3}$$

~~~~

If the structure bends into a sinusoidal shape with amplitude y_o, it will require an input of energy U_B.
Where will it get this energy?
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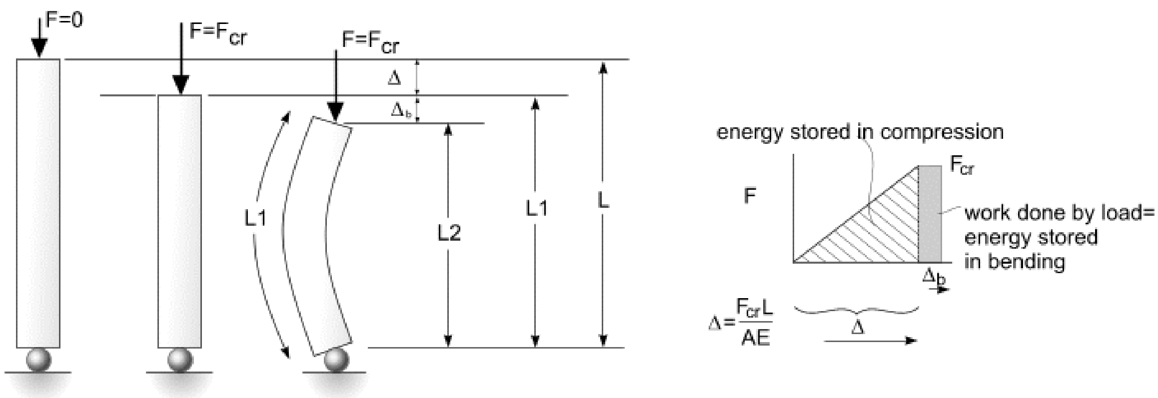
The energy comes from the end load.

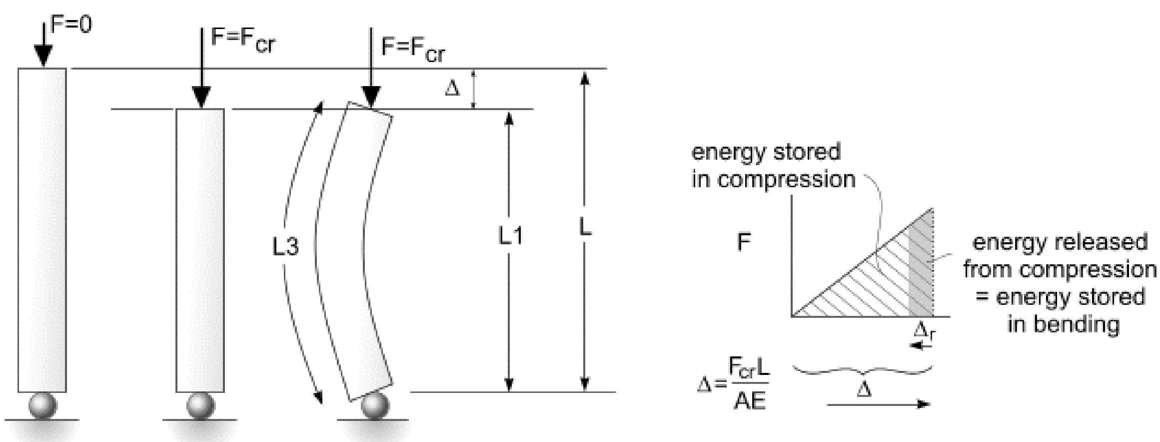
The are two ways to view the available energy (available to be converted to bending energy;

1) As the column buckles the load on the end of the column stays constant, and the end of the column moves.

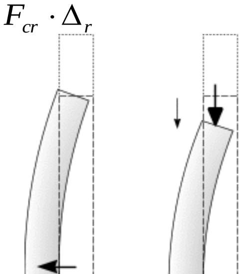


At the point of buckling the column has the length L1/ As it buckles the load stays applied, the column stays compressed, and the column stays L1 long. The straight-line length is L2, and the load drops by Δ_b = L1-L2. The available energy is: $U_A = F_{cr} \cdot \Delta_b$

2) As the column buckles the length of the column stays constant, and the compression of the column decreases.

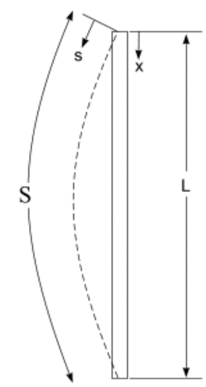


At the point of buckling the column is L1 long. As it buckles the ends don't move so that the straight line length remains L1. The bending results in a longer length, L3, along the column axis. The column stretches by Δ_r (axial rebound) where Δ_r = L3-L1. The available energy is $U_A = F_{cr} \cdot \Delta_r$



Note: While it may not seem so at first, the two cases above are identical. This is because we assume all incremental buckling deflections are vanishingly small, so that $\Delta_r = \Delta_b = L_{curved} - L_{straight}$.

Either way we need to find out the difference in length between the straight and curved paths. In other words we want to know The difference between the path length S and the straight length L.



$$S = L + \Delta = \int ds$$

$$\Delta = \int_0^S ds - \int_0^L dx$$

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Delta = \int_0^L \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1\right) dx$$

again we use the approximation: $\sqrt{1+a} \cong 1 + \frac{a}{2}$ for small a. Therefore we can write;

$$\Delta = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx = \frac{1}{2}\int_0^L (y')^2\, dx$$

To find D we need to integrate the square of the slope of the deflected shape. We will again assume

$$y(x) = y_o \sin\frac{\pi x}{L}$$

$$(y')^2 = \left(y_o \frac{\pi}{L} \right)^2 \cos^2 \frac{\pi x}{L}$$

$$\Delta = \frac{1}{2}\left(\frac{y_o \pi}{L} \right)^2 \int_0^L \cos^2 \frac{\pi x}{L} dx = \frac{y_o^2 \pi^2}{4L}$$

Which give the available energy as;

$$U_A = F_{cr} \cdot \frac{y_o^2 \pi^2}{4L}$$

We can now equate the available energy to the energy required for bending;

$$U_A = U_B$$

$$F_{cr} \cdot \frac{y_o^2 \pi^2}{4L} = \frac{\pi^4 EI \cdot y_o^2}{4L^3}$$

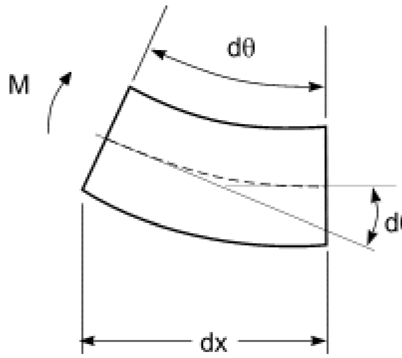
Solving for F_{cr} we get;

$$F_{cr} = \frac{\pi^2 EI}{L^2}$$ <= the Euler formula

<u>Summary</u>: At the point of buckling, the energy required to deform the structure laterally is just equal to the available energy. The available energy comes either from 1) the load doing work as the end of the column moves in response to the bending, or 2) the release of elastic potential energy as the column elongates as it bows out, while the ends remain fixed. Both mechanical concepts are identical in the limit, and produce identical results.

Annex - derivation of work done in elastic bending;

each part of the beam dx long bends under a constant moment;



$$\theta = \frac{dy}{dx} \Rightarrow \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$d\theta = \frac{d^2y}{dx^2}dx$$

The work done by the moment M is dU_B

$$dU_B = \frac{1}{2}M \cdot d\theta = \frac{1}{2}M\frac{d^2y}{dx^2}dx$$

recall that $M = EI\frac{d^2y}{dx^2}$

which gives;

$$dU_B = \frac{1}{2}EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

The total bending energy is the integral over the whole length;

$$U_B = \frac{1}{2}EI\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$$

T.11 – Problems.

1. Using the energy method, and assuming that the buckled shape of a column is y=4 yo ((x/L)^2 –x/L), {this is a parabola} find a formula for the critical buckling force. Compare the result to the Euler formula for a solid round steel column with a diameter of 20mm, 2 m. long.

2. Using the energy method, and assuming that the buckled shape of a column is an arc of a circle, can you find the critical buckling force?

Topic 12: Buckling of Transverse Plate Panels

Introduction

In this lecture we will

- Discuss plate buckling, analogous to column buckling
- Discuss the slenderness ratio and the transition to yielding
- Examine the effect of elastic boundary conditions
- Show origins of rule buckling checks

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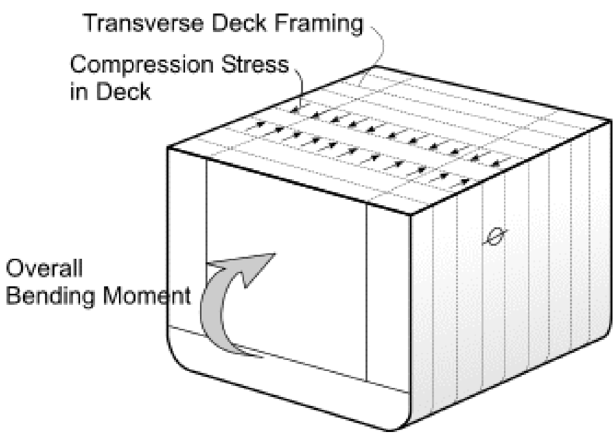
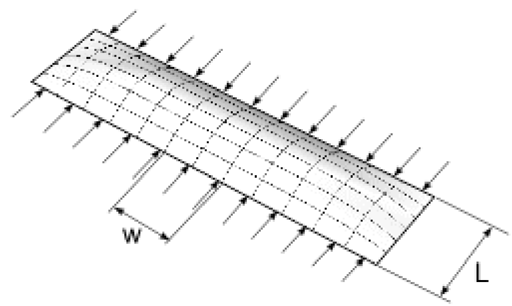
Long Plates/Transverse Buckling

When we consider a long plate loaded in the in-plane transverse direction. This will typically occur in the deck of a ship with transverse framing.





When we load a plate in this way, the buckling is simple. Each strip of plate (of some width w) buckles like each neighboring strip. We can write the Euler buckling load as;

$$F_{cr} = \frac{\pi^2 EI}{L^2}$$

we can re-write this in terms of stress in the plate;

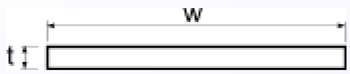
$$\sigma_{cr} = \frac{F_{cr}}{A} = \frac{\pi^2 EI}{L^2 A} = \frac{\pi^2 E\rho^2 A}{L^2 A}$$

which gives us;

$$\sigma_{cr} = \pi^2 \left(\frac{\rho}{L}\right)^2 E$$
~~~~~~

Note: for a plate of width w, $I = \frac{1}{12} w \cdot t^3 = \rho^2 A$

$\rho^2 = \frac{t^2}{12} \Rightarrow \rho = 0.29t$



Normally we try to prevent buckling before the stresses reach yield by limiting the slenderness of the structural member. We will ensure that yield will occur first when we set;

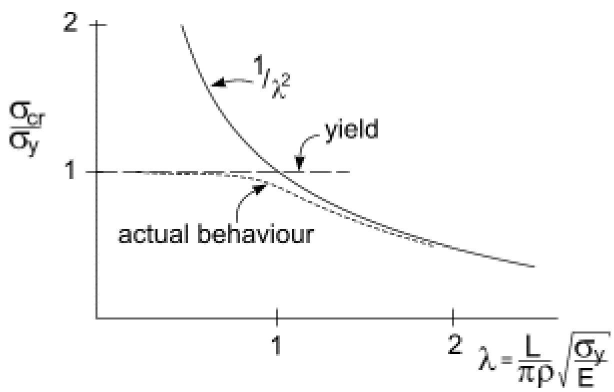
$$\sigma_{cr} > \sigma_y \text{ or } \sigma_{cr} > k \cdot \sigma_y \text{ for k>1}$$

The ratio of buckling to yield stress is;

$$\frac{\sigma_{cr}}{\sigma_y} = \pi^2 \left(\frac{\rho}{L}\right)^2 \frac{E}{\sigma_y} = \frac{1}{\lambda^2}$$

where λ is a slenderness ratio; $\lambda \equiv \frac{L}{\pi\rho}\sqrt{\frac{\sigma_y}{E}}$

This ratio is the Euler non-dimensional buckling stress. We can plot the Euler curve vs slenderness. The curve is invalid for stresses above yield. Due to imperfections (stress and geometry) the actual behavior tends to smoothly join the Euler curve and the yield stress limit.



We can effectively prevent buckling by setting λ <1;

$$\frac{L}{\pi\rho}\sqrt{\frac{\sigma_y}{E}} \leq 1$$

or

$$\frac{L}{\pi\rho} \leq \sqrt{\frac{E}{\sigma_y}}$$

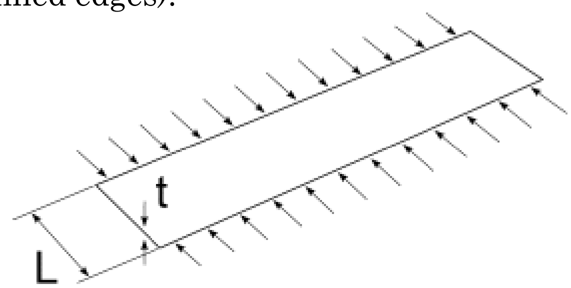
For plate this becomes;

$$\frac{L\sqrt{12}}{\pi \cdot t} \leq \sqrt{\frac{E}{\sigma_y}}$$

or

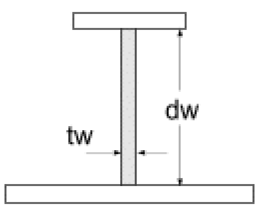
$$\boxed{\frac{L}{t} \leq .907\sqrt{\frac{E}{\sigma_y}}}$$

We would use this formula to limit the buckling of a wide plate in compression (for pinned edges).

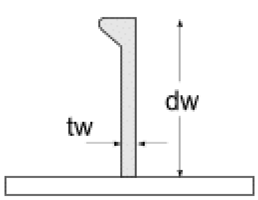


Classification society rules have very similar formulae to prevent buckling in various plate elements.

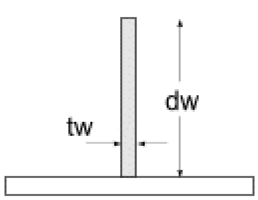
For example see ABS rules Part 5 App.5/2 AB for buckling restrictions for longitudinals;



tee stiffeners : $\frac{dw}{tw} \leq 1.5\sqrt{\frac{E}{\sigma_y}}$



bulb stiffeners: $\frac{dw}{tw} \leq .85\sqrt{\frac{E}{\sigma_y}}$

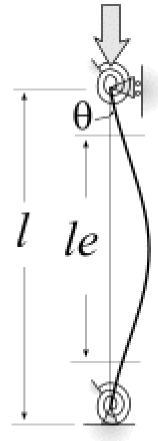


flat bar stiffeners: $\frac{dw}{tw} \leq .5\sqrt{\frac{E}{\sigma_y}}$

Q? Why do the constants range from 0.5 to 1.5?
Ans: Because of the boundary conditions and the effective length.

We've seen that the effective length for fixed ends is half that of pinned ends. When the ends are elastically restrained from rotating, we get effective lengths somewhere between 0.5 and 1 x the length.

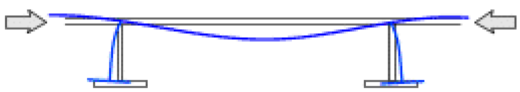
$n=\frac{c\cdot l}{EI}$	$\frac{le}{l}$	$\frac{P_{CR}}{P_E}$	Remarks
0	1	1	Pinned
1	.86	1.35	
2.5	.75	1.78	
5	.66	2.3	typical of
10	.59	2.85	ship structures
∞	.5	4	fixed

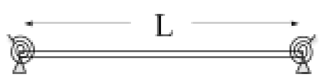


$c = \frac{M}{\theta}$

Plates supported by frames or bulkheads are closer to fixed than to pinned.

For example : a deck plate





Springs replace the effect of frames and neighboring plate

Assume that $n=\frac{c\cdot l}{EI}=10$, $\frac{P_{CR}}{P_E}=2.85$, and $\frac{le}{l}=.59$

This gives us a rule: $\frac{.59L}{t}\le .907\sqrt{\frac{E}{\sigma_y}}$, which can be written as

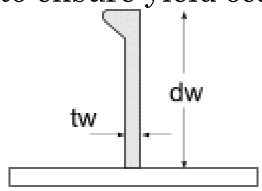
$$\frac{L}{t}\le 1.53\sqrt{\frac{E}{\sigma_y}}$$ similar to ABS rule.

We can also write this as;

$$\frac{L}{t}\le\frac{698}{\sqrt{\sigma_y}}$$ which is similar to the format used in DnV rules.

T.12 – Problems.

1. For a bulb stiffener with a height (dw) of 200mm, what is the minimum web thickness to ensure yield occurs prior to buckling.



2. What would be the value of cl/EI for a 20mm plate supported by deep frames every 600mm. Assume the frames act as simple supports and consider only one neighbor plate. (hint: this can be found from the rotary stiffness term k33 for a beam).

3. Find λ for a plate 15mm on 650mm span. (350MPa steel).

Topic 13: Buckling of Longitudinal Plate Panels

Introduction

In this lecture we will

- Discuss plate buckling
- Derive the general response based on strain energy

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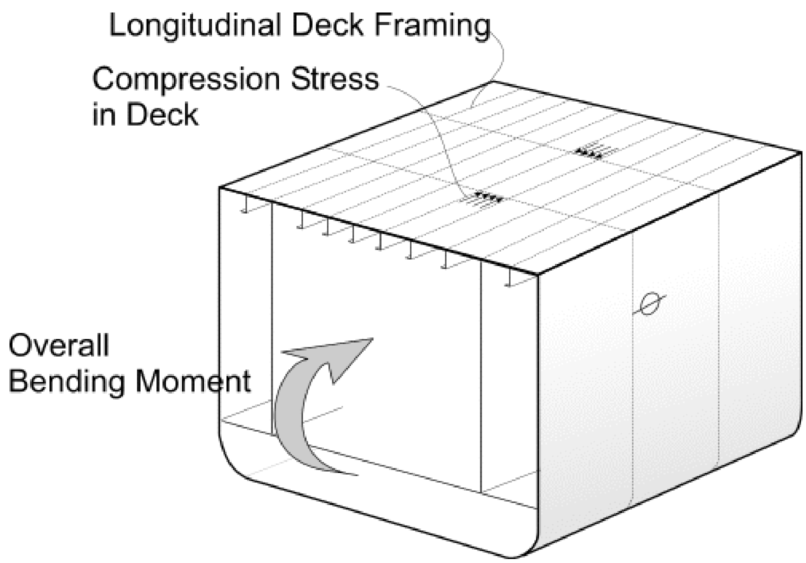
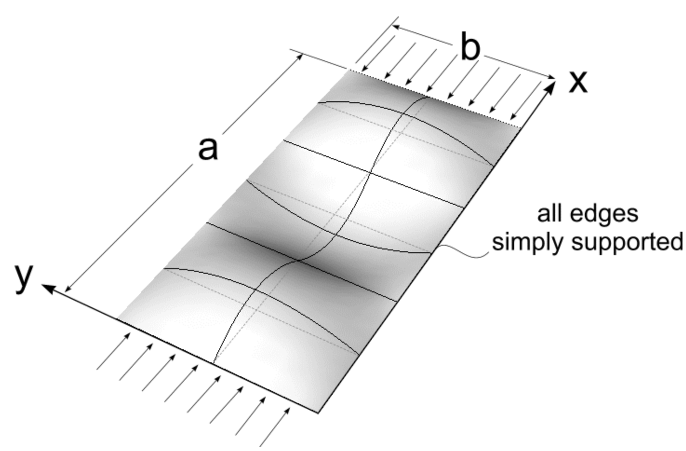
Long Plates/Transverse Buckling (edges simply supported)

When we consider a long plate loaded in the in-plane longitudinal direction. This will typically occur in the deck of a ship with longitudinal framing. In this configuration, the plate can buckle into a rippled pattern as shown below. We can't use the simplifying assumption that we used with transverse plating. We have to consider the whole plate.

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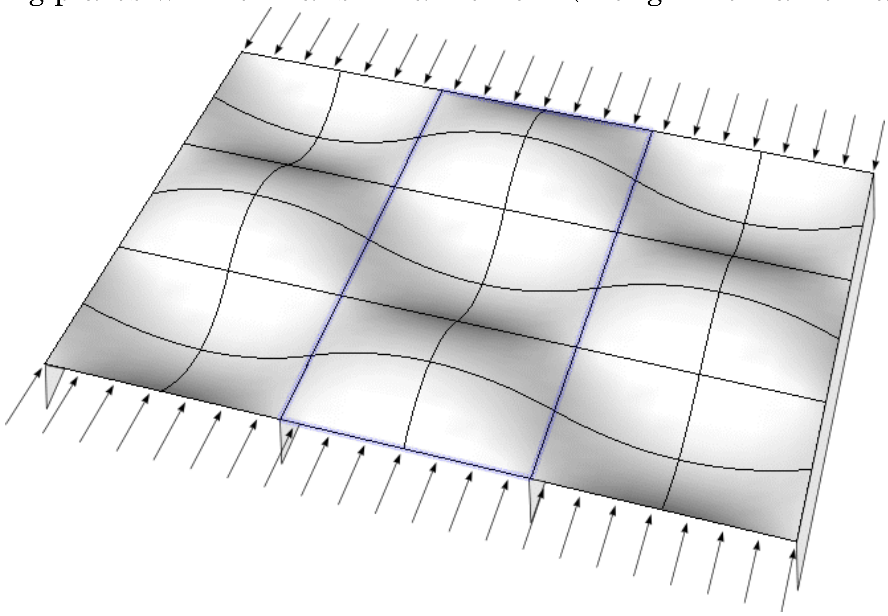
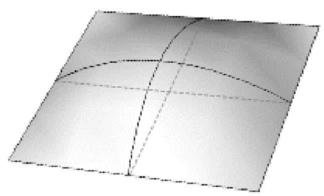
The assumption of pinned edges is reasonable in light of the anti-symmetry that is likely to develop in the buckling pattern. This beans that the neighboring plates will not transmit a moment (though the frame may).

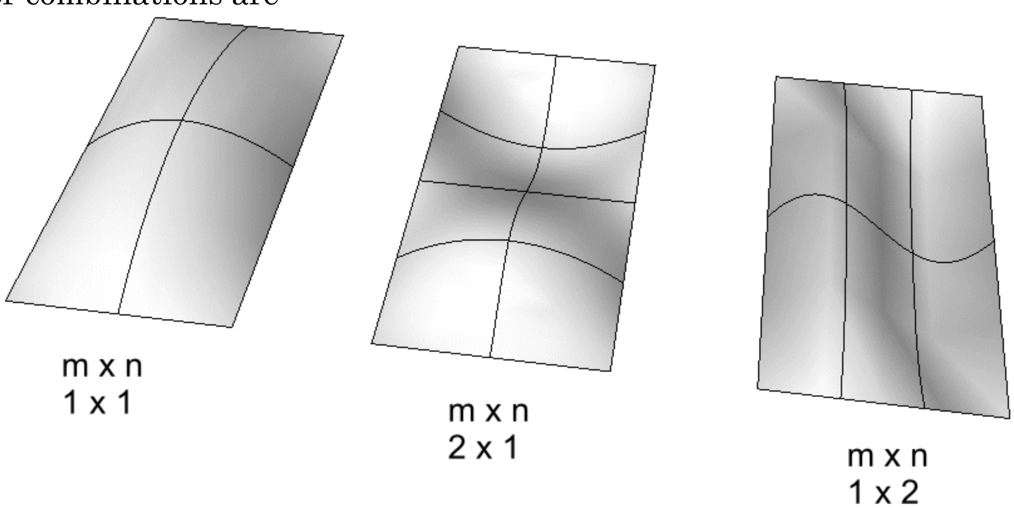


Plates buckle in two dimensions. We will us the variables m and n to give the number of half sine waves in the x and y directions.

1 half wave in x : m=1
1 half wave in y : n=1



Other combinations are:



The values of m and n (the buckling shape) will depend on which shape has the lowest elastic potential energy. The shape will depend on the aspect ratio (a/b) of the plate.

We can start the analysis by describing the deformed shape. We will assume the shape to be of the form;

$$w = \sum_m \sum_n A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

(this represents the superposition of many valid wave patterns, and could represent almost any deformation pattern for any load.)

For buckling, we will assume that the plate will take up only one sine-sine form, or in other words, only one value or m or n. This lets us drop the summation and write;

$$w = w_o \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (1)$$

Equation (1) satisfies all boundary conditions ay all four edges, which are;

$$w = 0 \ , \ \frac{\partial^2 w}{\partial x^2} = 0 \ , \ \frac{\partial^2 w}{\partial y^2} = 0 \ , \ \text{at any of} \begin{cases} x = 0 \\ x = a \\ y = 0 \\ y = b \end{cases}$$

and is symmetrical or anti-symmetrical about x=a/2, y=b/2. With this deformation pattern we can now determine the elastic potential energy (strain energy). For a plate (also termed an elastic shell), the strain energy U (internal work) in bending and twist[2] is;

$$U = \frac{1}{2} D \int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right] dxdy \qquad (2)$$

where $$D = \frac{Et^3}{12(1-\nu^2)}$$

[2] Hughes Ch. 12, pg. 404.

We can integrate the 2nd term over the plate and find that it equals zero;

$$-2(1-\nu)\int_{o}^{a}\int_{0}^{b}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x^2}-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]dxdy=0$$

This is illustrated in the extract from Maple™ shown below. Note that m and n can be any two integers and the result will be zero (Exercise: try this).

Maple™ file:

```
> restart;
> f(x,y):=(diff(w(x,y),x,y))^2-diff(w(x,y),x$2)*diff(w(x,y),y$2);
```

$$f(x,y) := \left(\frac{\partial^2}{\partial x\, \partial y} w(x,y)\right)^2 - \left(\frac{\partial^2}{\partial x^2} w(x,y)\right)\left(\frac{\partial^2}{\partial y^2} w(x,y)\right)$$

```
> int(int(f(x,y),x=0..a),y=0..b);
```

$$\int_0^b \int_0^a \left(\frac{\partial^2}{\partial x\, \partial y} w(x,y)\right)^2 - \left(\frac{\partial^2}{\partial x^2} w(x,y)\right)\left(\frac{\partial^2}{\partial y^2} w(x,y)\right) dx\, dy$$

```
> m:=2;n:=3;
```

$$m := 2$$

$$n := 3$$

```
> w(x,y):=wo*sin(m*Pi*x/a)*sin(n*Pi*y/b);
```

$$w(x,y) := wo \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right)$$

```
> f(x,y):=(diff(w(x,y),x,y))^2-diff(w(x,y),x$2)*diff(w(x,y),y$2);
```

$$f(x,y) := \frac{36\, wo^2 \cos\left(\frac{2\pi x}{a}\right)^2 \pi^4 \cos\left(\frac{3\pi y}{b}\right)^2}{a^2 b^2} - \frac{36\, wo^2 \sin\left(\frac{2\pi x}{a}\right)^2 \pi^4 \sin\left(\frac{3\pi y}{b}\right)^2}{a^2 b^2}$$

```
> int(int(f(x,y),x=0..a),y=0..b);
```

$$0$$

we can expand the terms;

$$\frac{\partial^2 w}{\partial x^2}=-\left(\frac{m\pi}{a}\right)^2 w, \quad \frac{\partial^2 w}{\partial y^2}=-\left(\frac{n\pi}{b}\right)^2 w$$

which allows us to write;

$$\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2=\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 w^2$$

we can now write;

$$U=\frac{1}{2}D\pi^4\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\right]\int_o^a\int_0^b w^2 dxdy$$

The integral of w² becomes;

$$\int_o^a\int_0^b w^2 dxdy=\frac{w_o^2 ab}{4}$$

So we can write;

$$U=\frac{\pi^4 ab}{8}D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 w_o^2$$

This is the stored elastic potential energy.

Recall that for a column the external work done is given by;

$$W=F_{cr}\Delta=F_{cr}\,\frac{1}{2}\int_0^L\left(\frac{dy}{dx}\right)^2 dx$$

Similarly, for a plate with an end stress, the external work is ;

$$W=\sigma_{cr}t\cdot\Delta=\frac{\sigma_{cr}t}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy$$

This evaluates as;

$$W=\frac{\pi^2 b\sigma_{cr}t}{8a}w_o^2 m^2$$

As before, we equate internal and external work (W=U), to get;

$$\frac{\pi^2 b\sigma_{cr}t}{8a}w_o^2 m^2=\frac{\pi^4 ab}{8}D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 w_o^2$$

solving for the buckling stress, we get;

$$\sigma_{cr} = \frac{\pi^2 a^2 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}{t \cdot m^2}$$

It is obvious that σ_{cr} will always be a minimum when $n=1$. In other words, we will only have one half wave across the short dimension of the plate. We can re-arrange σ_{cr} to give;

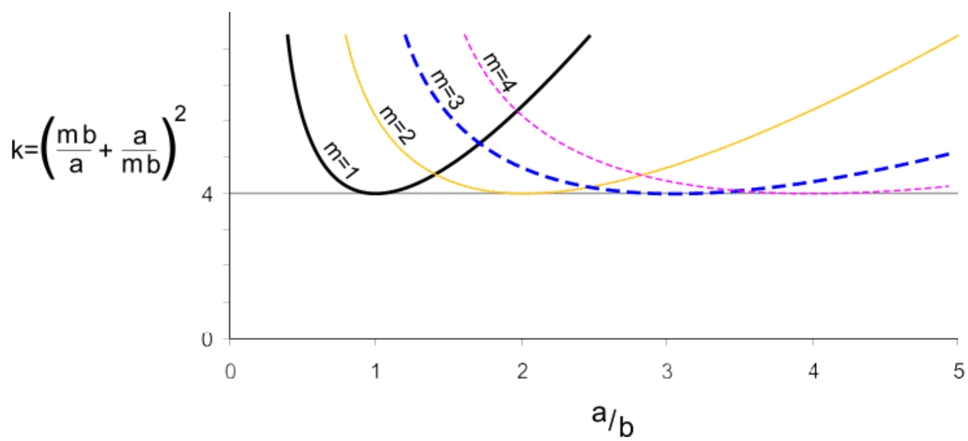
$$\sigma_{cr} = \frac{\pi^2 D}{b^2 t}\left(\frac{mb}{a}+\frac{a}{mb}\right)^2 = k\frac{\pi^2 D}{b^2 t}$$

where

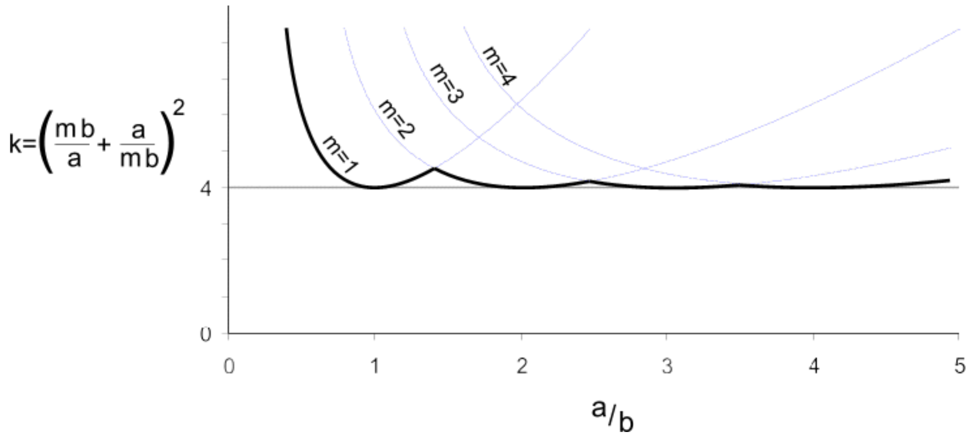
$$k = \left(\frac{mb}{a}+\frac{a}{mb}\right)^2$$

When we plot k as a function of a/b, for various values of m, we see that the minimum value of k is always 4, and always occurring where a/b=m.

This means that for a plate with an aspect ratio of, say, 3 we will get the lowest buckling stress when we have m=3 (3 half-waves).



The minimum buckling stress depends on the minimum k. It is reasonable to use k=4 for any aspect ratio.



This lets us write the critical buckling stress as;

$$\sigma_{cr} = 4\frac{\pi^2 D}{b^2 t}$$

We should note that although the plate is loaded in the longitudinal direction, we are using 'b', the transverse plate dimension in the formula. The 'length' of the plate 'a' doesn't really matter. The plate will deform into a pattern of buckles, each approximately 'b' x 'b'.

Using

$$D = \frac{Et^3}{12(1-\nu^2)}$$

we can write;

$$\sigma_{cr} = \frac{4\pi^2}{b^2 t}\frac{Et^3}{12(1-\nu^2)} = 3.62E\left(\frac{t}{b}\right)^2$$

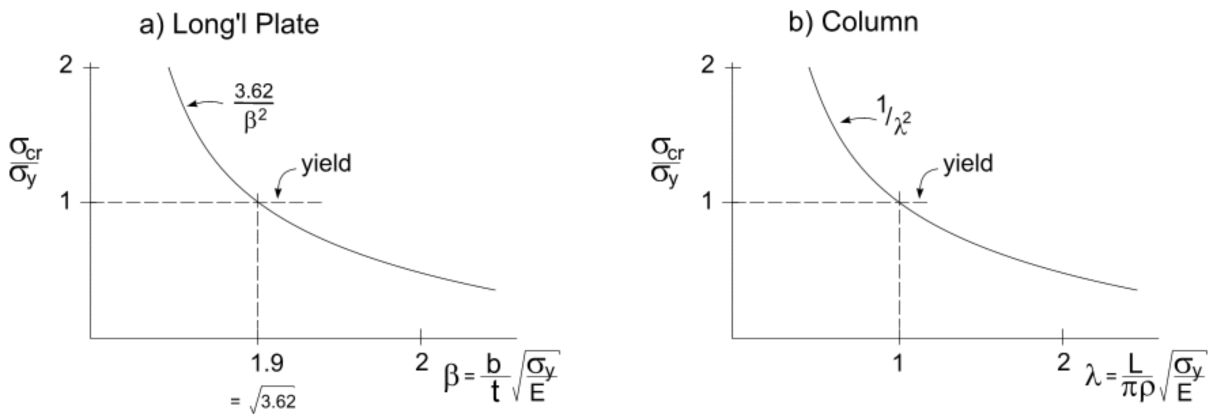
Now we can normalize the buckling stress as we did for columns;

$$\frac{\sigma_{cr}}{\sigma_y} = \frac{3.62}{\beta^2}$$

where

$$\beta = \frac{b}{t}\sqrt{\frac{\sigma_y}{E}}$$

The longitudinal plate buckling equation is similar to the column buckling equation:


a) Long'l Plate
2
3.62/β²
σcr/σy
yield
1
1.9
= √3.62
2
β = b/t √(σy/E)
b) Column
2
1/λ²
σcr/σy
yield
1
1
2
λ = L/πρ √(σy/E)

T.13 – Problems.

4. For a longitudinal plate with an aspect ratio of 1.5, what is the minimum value of k? For this k, what is the value of m?

5. Use Maple to evaluate the integral $\int_o^a \int_0^b w^2 dxdy$ for $w = w_o \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

6. What is the buckling stress for a longitudinal steel deck plate of 600mm x 1200mm x 12mm?

7. What is the maximum yield strength that you should use for the plate in 3) above, to be sure that the plate yields prior to buckling?

Topic 14: Post-Buckling Strength of Long Plates

Introduction

In this lecture we will

- Discuss von-Kármán's concept of post buckling behavior
- Compare to Faulkner's and Paik's equations

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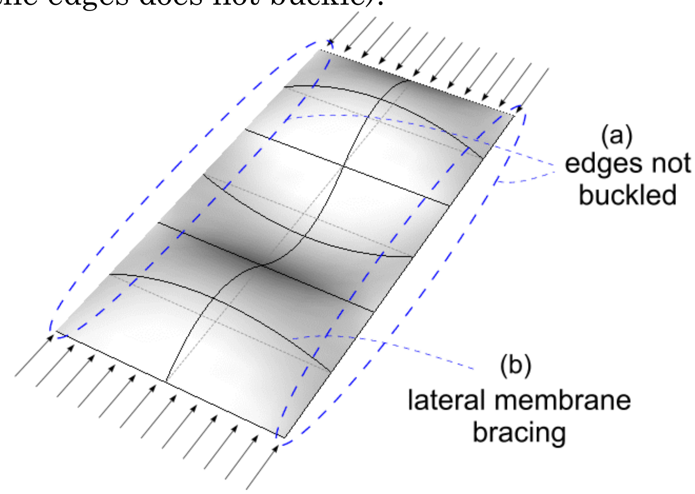
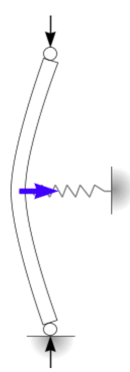
Long Plates/Post Buckling (edges simply supported)

While columns have very little post-buckling strength, and certainly less than their buckling strength, plates (between frames) can exhibit higher strength after buckling than before. This is due to two factors;

a) The long edges remain straight and intact due to the frames (i.e. the part of the plate at the edges does not buckle).



b) There is a stabilizing effect of the membrane stresses across the short dimension of the plate.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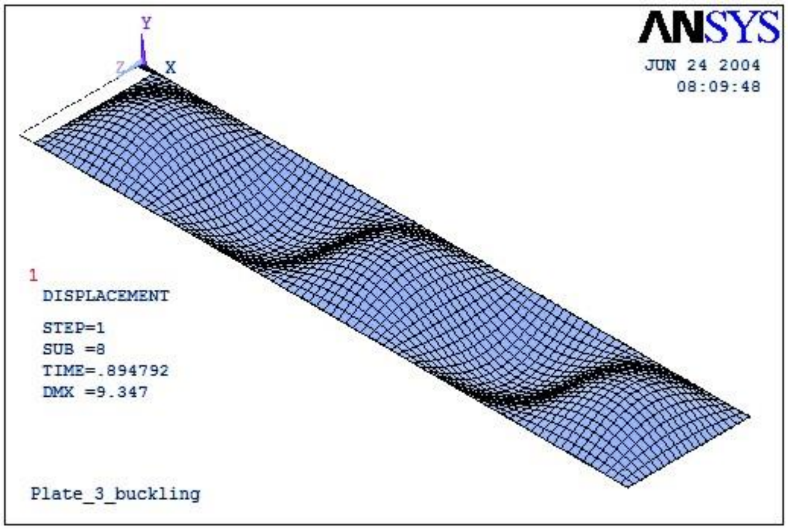
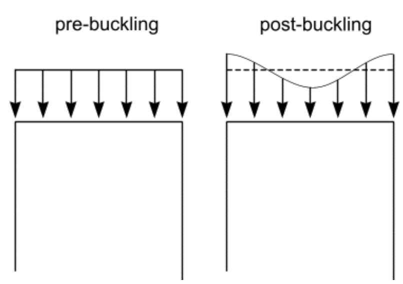
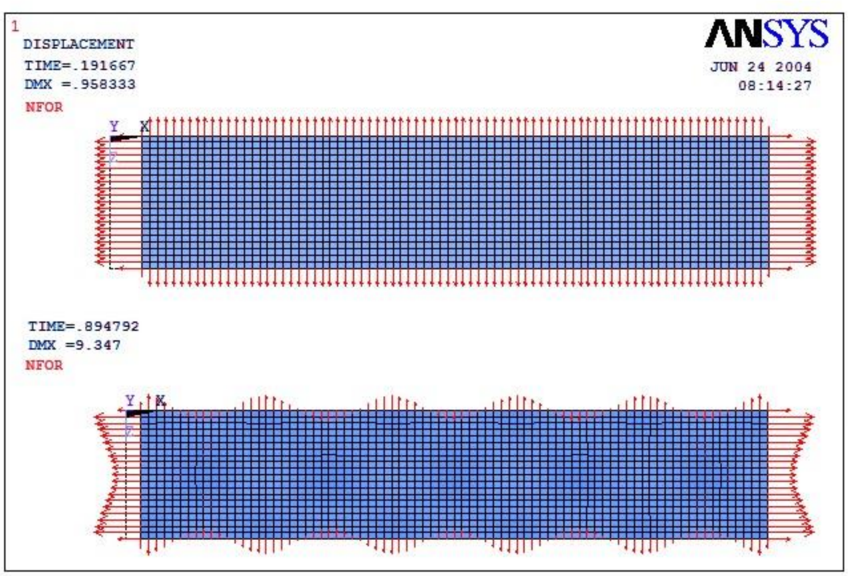


Finite element plot of the buckled shape of a 500mm x 2500mm x 9mm plate. The model was a non-linear analysis, with an imposed end displacement, solved using the arc-length method. The term TIME=0.894792 refers to 4.475mm (max imposed deflection was 5mm), which is well beyond the onset of buckling.

After buckling occurs the stresses re-distribute themselves across the plate. This is shown in a sketch below, and in a finite element plot below that. The ANSYS plot is actually showing the forces at each node pushing on the surrounding boundary (which is why they are pushing out for a state of compression)





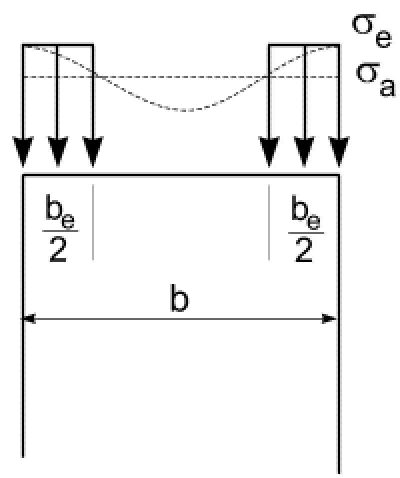
before buckling

after buckling

Effective Width

The redistribution of stresses can be treated by the 'effective width' concept.

- True stresses are replaced by two uniform zones, where the total zone width is b_e (note: the next plate is similar so the load patch is b_e wide, centered on the frame.)



The true edge stress σ_e is stress in b_e.

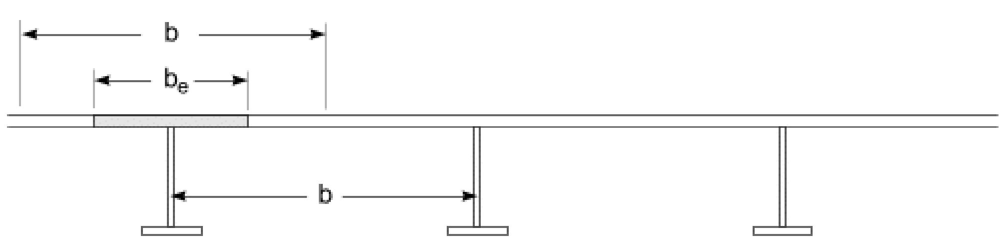
The total force can be expressed in a variety of ways;

$$F = \int_0^b \sigma dy = \sigma_e \cdot b_e = \sigma_a \cdot b \tag{1}$$

where σ_a is the average post-buckling stress. We can write;

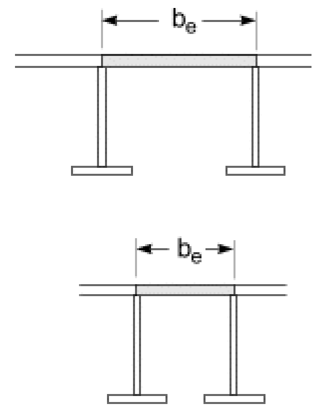
$$\frac{b_e}{b} = \frac{\sigma_a}{\sigma_e} \tag{2}$$

Why use this effective width concept? The reason is that it will allow us to determine the edge stress, which is also the axial stress on the frame, and thus we will be able to determine the total post-buckling force.



The way that we can find both σ_e and b_e is with a method proposed by von-Kármán (1924).

von-Kármán was, at the time, working on aircraft structures, which had a very similar construction to the ship structures that we are considering (plate over longitudinal ribs, in both the fuselage and wings). von-Kármán proposed a very simple (elegant) way to find the edge stress along with the effective width. He suggested that as the plate buckles, the edge stress (over b_e) would be the same as the Euler buckling stress for a plate of b_e width. This idea turned out to be close to the truth. In effect, he proposed to see the plate as two ideal parts: the middle buckled part carrying no load, and a part (the effective part) near the frame carrying all the load. The 'effective' part would be progressively narrowed by the progressive buckling of the middle part.



This is a key point. The idea is that plate buckling is a steady progressive process. When the stress first causes buckling, the plate only just starts to deform. If the stress is held at a level just above the buckling stress, the plate is only slightly deformed (barely noticeable). As the average stress increases, the middle part progressively sheds load to the sides. The whole process is actually steady and stable, right up to the point where the edge stress (and frame stress) reach yield. At least that's the idea, and it is quite close to reality.

The von-Kármán relationships are developed as follows;
Recall that

$$\sigma_{cr} = \frac{3.62}{\beta^2} \tag{3}$$

where

$$\beta = \frac{b}{t}\sqrt{\frac{\sigma_y}{E}} \tag{4}$$

or

$$\sigma_{cr} = 3.62\left(\frac{t}{b}\right)^2 E \tag{5}$$

if we let $b = b_e$, we get ;

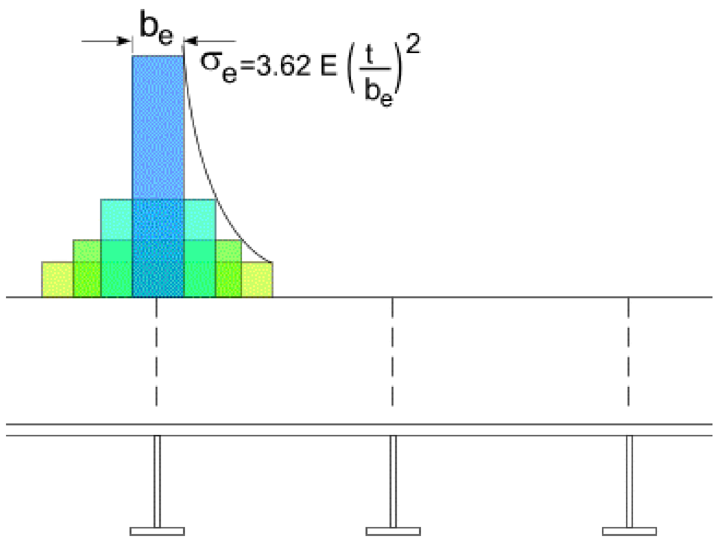
$$\sigma_e = 3.62\left(\frac{t}{b_e}\right)^2 E \tag{6}$$

or

$$b_e = 1.9t\sqrt{\frac{E}{\sigma_e}} \tag{7}$$

These are the equations relating σ_e and be.

The effective stress rises as the effective width decreases. This is plotted below;



As the overall force on the deck increases and the effective width decreases, the von-Kármán model will remain valid until the edge stress reaches yield. At this point the frames will collapse. The minimum effective width can be found by substituting the yield stress into equation (7).

$$b_{em} = 1.9t\sqrt{\frac{E}{\sigma_y}} \tag{8}$$

for E=207000 MPa, σ_y=235 MPa we get;

$$b_{em} = 56t$$

We have said that the plate buckles progressively and that the system can take more and more load as the plate buckles. We can check this by calculating the force on the plate;

$$
\begin{aligned}
F &= b_e \cdot t \cdot \sigma_e \\
&= b_e \cdot t \cdot \frac{3.62 \cdot t^2}{b_e^2} E \\
&= \frac{3.62 t^3 E}{b_e}
\end{aligned} \tag{9}
$$

Quite clearly, the force F will increase as b_e gets smaller.
The limit situation occurs when $b_e = b_{em}$ and $\sigma_e = \sigma_y$. At this point the force is;

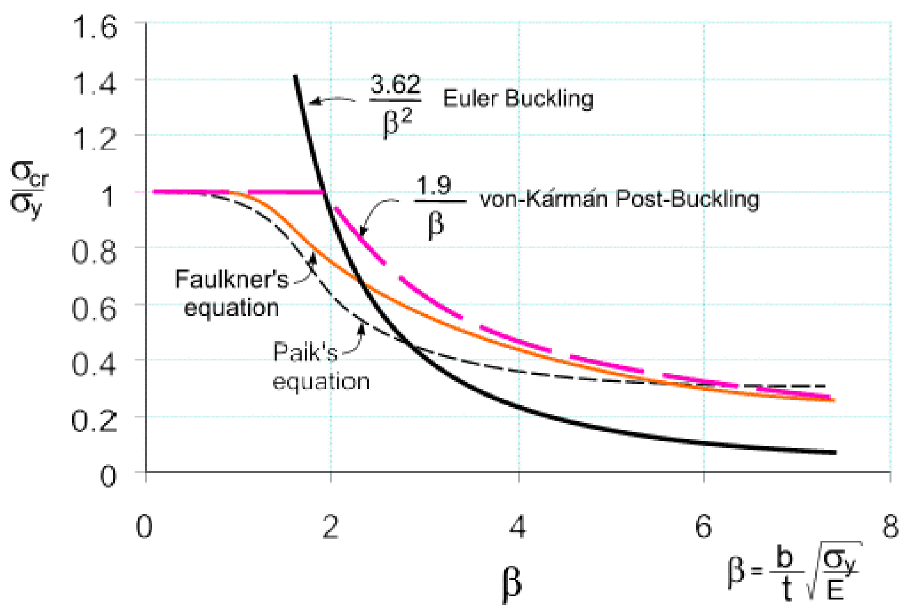
$$F_m = b_{em} \cdot t \cdot \sigma_y \tag{10}$$

The average stress on the plate in the limit condition is;

$$\sigma_m = \frac{F_m}{b \cdot t} = \frac{b_{em} \cdot t \cdot \sigma_y}{b \cdot t} = \frac{1.9t\sqrt{\frac{E}{\sigma_y}}\sigma_y}{b} \tag{11}$$

$$\frac{\sigma_m}{\sigma_y} = 1.9 \frac{t}{b}\sqrt{\frac{E}{\sigma_y}} = \frac{1.9}{\beta} \tag{12}$$

The buckling (3) and post-buckling (12) capacities are shown below;



Faulkner (Prof. at Glasgow), on the basis of experiments proposed a slightly modified equation;

$$\frac{\sigma_m}{\sigma_y} = \frac{2}{\beta} - \frac{1}{\beta^2} \tag{13}$$

Paik (Prof at Pusan), proposed an even more conservative equation, taking imperfections and initial stresses into account, and validated with experimental and numerical data. Paik's equation is;

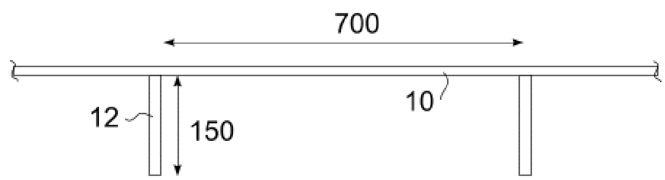
$$\frac{\sigma_m}{\sigma_y} = \begin{cases} -.032\beta^4 + .002\beta^2 + 1 & for\beta \le 1.5 \\ 1.247/\beta & for 1.5 < \beta \le 3.0 \\ 1.248/\beta^2 + .283 & for\beta > 3.0 \end{cases} \tag{14}$$

T.14 – Problems.

1. Find the buckling and post-buckling capacities for the following plates;

	Plate 1	Plate 2	Plate 3
b [mm]	600	800	1200
t [mm]	20	15	10
σ_y [MPa]	235	300	360
E [MPa]	207000	207000	207000

2. For a deck with 10mm plate, longitudinal frames spaced at 700mm, transverse web frames spaced at 5000mm. The frames are 150x12 flats bars. All steel has a yield strength of 360MPa. Determine the buckling and post-buckling strength of the plate. Determine the buckling strength of the frame (assumed pinned). What is the average stress that the deck (plate/frame) can withstand befor failure)? (Hint: how much of the plate do you include in the flange of the frame?)



Topic 15: Local Buckling

Introduction

In this lecture we will

- Discuss a variety of local buckling phenomena

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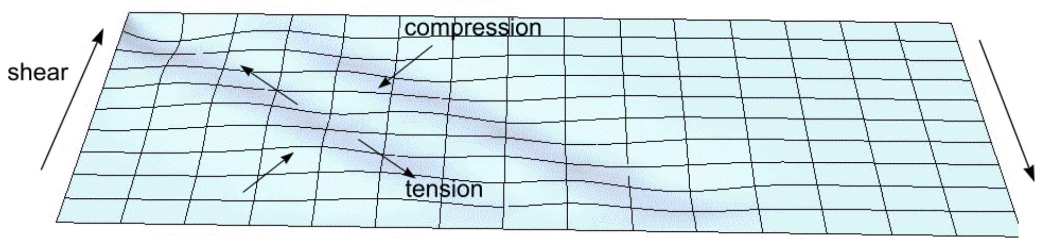
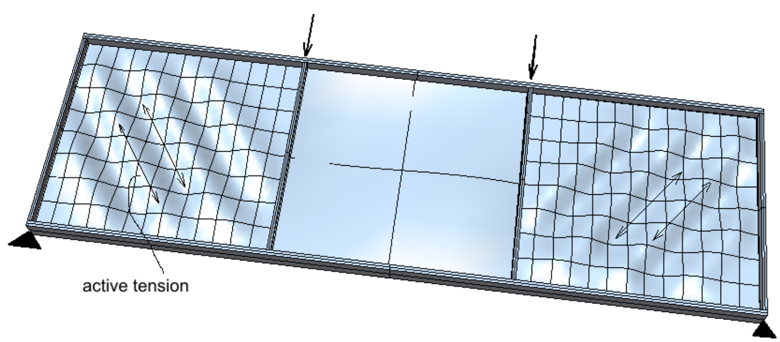
Local buckling refers to a variety of buckling mechanisms involving any part (web, flange, bracket) of a frame. In all cases there is a local compression stress that buckles a section of plate. The compression may be the result of bending, shear or direct pressure.

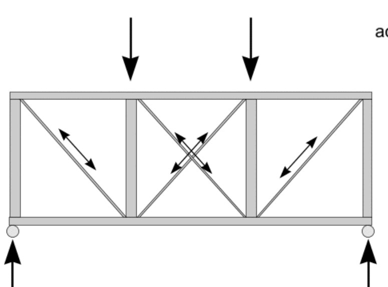
Shear Buckling

When a plate (typically a web) experiences a shear stress, there is a compression field on a 45^o diagonal.



This type of shear buckling is actually common and expected in deep web girders (as might be found in a railway bridge). As long as there are stiffeners on the surface to take the compression, the buckled web will hold the tension in a kind of truss-like structure.





Web and flange buckling
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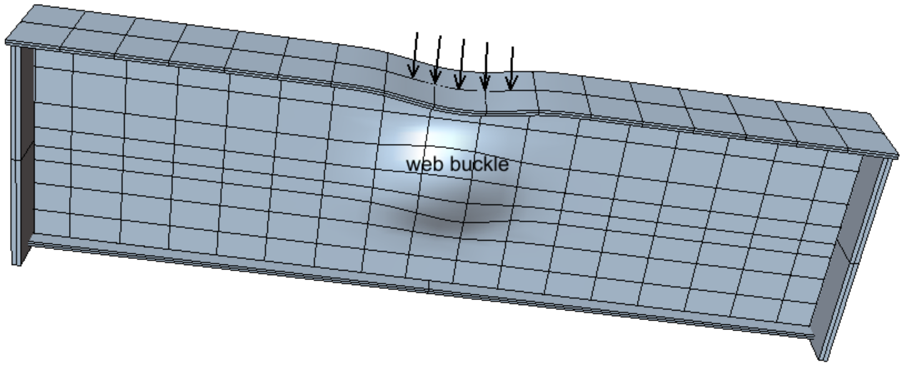
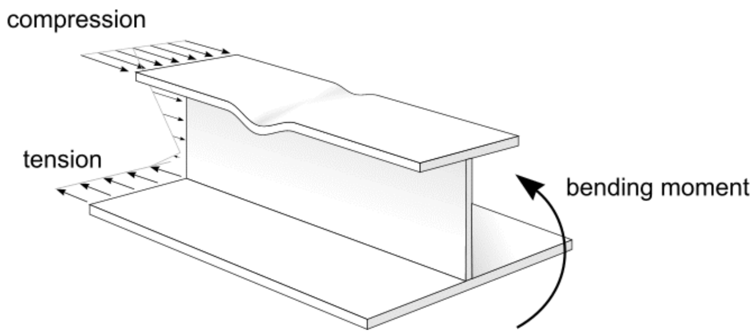
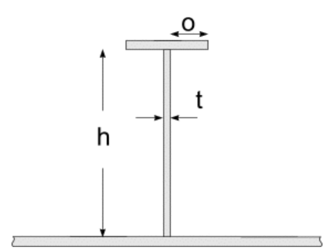
Another common type of local buckling is web buckling, which may be caused by direct compression due to an applied load.



Compression in the flange due to bending can cause the flange to buckle locally.



Local buckling is prevented by limiting the local aspect ratios of parts of the construction. A typical rule for plate with flanges on both boundaries would be;



$$\frac{h}{t} \le \frac{1000}{\sqrt{\sigma_y}}$$ [σ_y in MPa]

or

$$\frac{h}{t} \le 2.2\sqrt{\frac{E}{\sigma_y}}$$ [σ_y and E in same units]

When the plate is connected on one side only (as for a flange) it is called an 'outstand'. The typical local buckling rule is for an 'outstand' is;

$$\frac{o}{t} \le \frac{250}{\sqrt{\sigma_y}}$$ [σ_y in MPa]

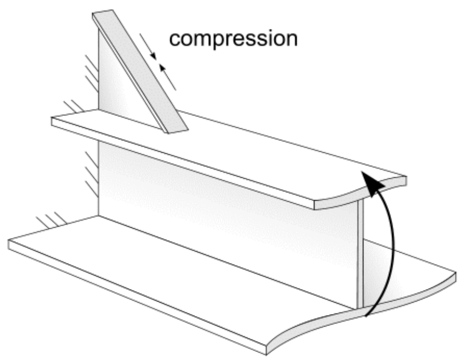
or

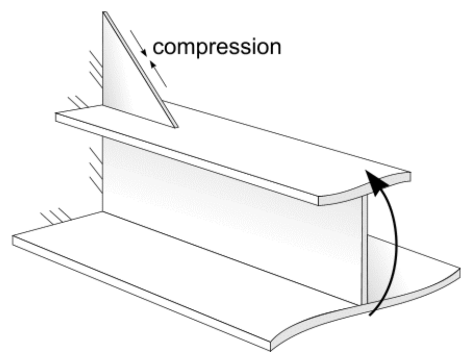
$$\frac{o}{t} \le .55\sqrt{\frac{E}{\sigma_y}}$$ [σ_y and E in same units]

T.15 – Problems.

3. For a 'tee' flange made of 25mm steel with a yield strength of 360MPa, what is the limit width to ensure the flange will remain unbuckled at yield?

4. For the two brackets pictured below, both 400mm (h x w), estimate the minimum thickness needed to prevent buckling. One has a flange on the free edge, while the other does not. Compare your estimate with rule requirements.





Topic 16: Overall Stiffener Buckling

Introduction

In this lecture we will

- Derive the solution for the buckling strength of a frame attached to a plate, assuming that the plate has already buckled.
- Discuss the modified effective width b_e'

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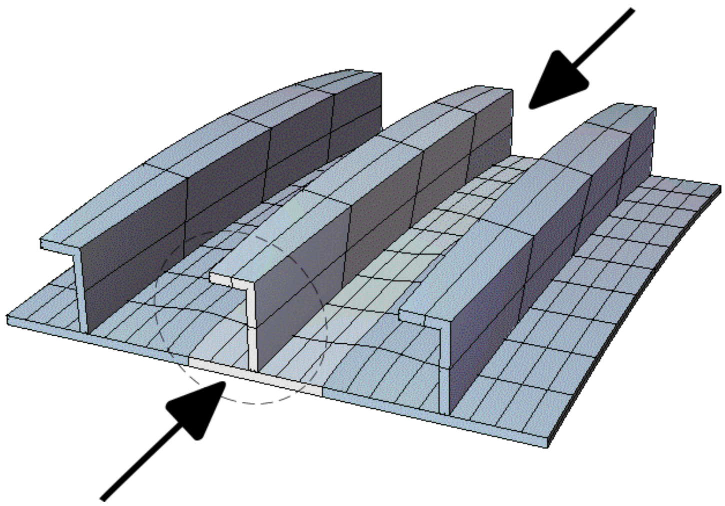
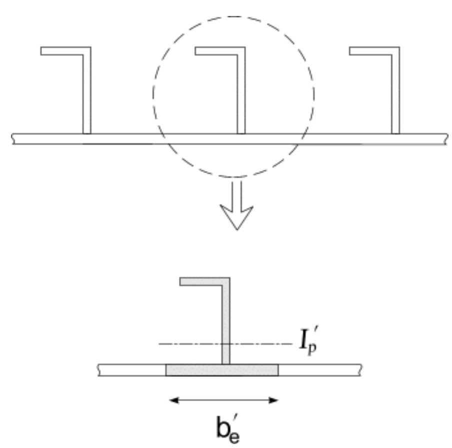
After the shell plate has buckled, the frames act as columns, and may buckle prior to reaching yield (when they will collapse plastically).



We need to find the moment of inertia I_p' of the frame with the part of the plating that is still effective. The effective width b_e is ***not*** what we will use to find I_p'. As the frame buckles, it will compress the plate further and reduce the effective width to something less than b_e. We need to find b_e', which we will do by considering stiffness.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We will start by considering the load carrying capacity of the plate (which is one flange of the frame). Recall from Lecture 19 that the plate in compression carries that load;

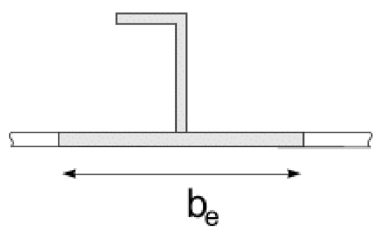
$$P = \sigma_e \cdot b_e \cdot t$$

The flange of any normal beam keeps a fixed width, such that as the beam bends, the flange pushes (or pulls) increasingly hard to resist the bending. We could say that the flange contributes to the bending of the beam by being axially stiff. The axial stiffness of any normal flange would be

$$k = \frac{A \cdot E}{L} = \frac{b \cdot t \cdot E}{L}$$

We can also write that the stiffness is the derivative of the axial load with respect to the shortening. The load (after buckling) is a function of both σ_e and b_e.

$$k = \frac{dP}{d\varepsilon} = \frac{d(\sigma_e \cdot b_e \cdot t)}{d\varepsilon}$$



We can expand this;

$$k = \frac{d(\sigma_e \cdot b_e \cdot t)}{d\varepsilon}$$

$$= t \cdot \left(\frac{d\sigma_e}{d\varepsilon} b_e + \frac{db_e}{d\varepsilon} \sigma_e \right)$$

we can write:

$$\frac{db_e}{d\varepsilon} = \frac{db_e}{d\sigma_e} \cdot \frac{d\sigma_e}{d\varepsilon}$$

which gives;

$$k = t \frac{d\sigma_e}{d\varepsilon} \cdot \left(b_e + \frac{db_e}{d\sigma_e} \sigma_e \right)$$

We define this term as b_e'

so:

$$k = t \frac{d\sigma_e}{d\varepsilon} \cdot b_e'$$

We know that :

$$b_e \cdot \sigma_e = b \cdot \sigma_{avg} \text{ which results in:}$$

$$b_e = b \cdot \frac{\sigma_{avg}}{\sigma_e}$$

We can use this to get a simple expression for b_e':

$$\begin{aligned}
b_e^{'} &= b_e + \frac{db_e}{d\sigma_e}\sigma_e \\
&= b \cdot \frac{\sigma_{avg}}{\sigma_e} + \sigma_e \frac{d}{d\sigma_e}\left(b \cdot \frac{\sigma_{avg}}{\sigma_e}\right) \\
&= b \cdot \frac{\sigma_{avg}}{\sigma_e} + b \cdot \sigma_e \frac{d}{d\sigma_e}\left(\frac{\sigma_{avg}}{\sigma_e}\right) \\
&= b \cdot \frac{\sigma_{avg}}{\sigma_e} + b \cdot \sigma_e \left(\frac{\sigma_e \frac{d\sigma_{avg}}{d\sigma_e} - \sigma_{avg}\frac{\sigma_e}{\sigma_e}}{\sigma_e^2}\right) \\
&= b \cdot \frac{\sigma_{avg}}{\sigma_e} + \frac{b}{\sigma_e}\left(\sigma_e \frac{d\sigma_{avg}}{d\sigma_e} - \sigma_{avg}\right)
\end{aligned}$$

$$so:$$

$$b_e^{'} = b\frac{d\sigma_{avg}}{d\sigma_e}$$

recall that $b_e = b\frac{\sigma_{avg}}{\sigma_e}$. While b_e is related to the ratio of stresses, b_e' is related to the ratio of changes in stress. We need to re-visit von-Karman to find the values. Based on von-Karman:

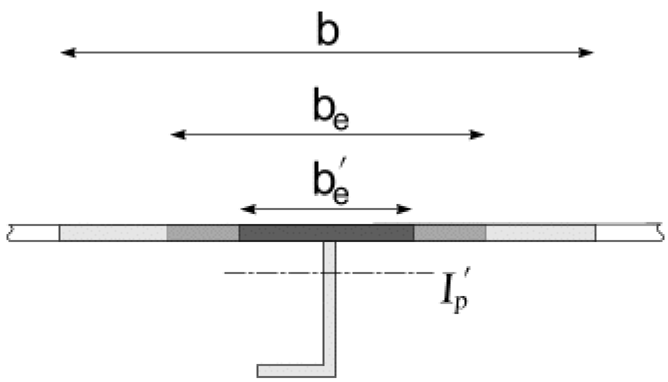
$$b_e = 1.9t\sqrt{\frac{E}{\sigma_e}} = 1.9t\sqrt{E}\sigma_e^{-0.5}$$

$$\begin{aligned}
b_e^{'} &= b_e + \frac{db_e}{d\sigma_e}\sigma_e \\
&= b_e - 0.5 \cdot 1.9t\sqrt{E}\sigma_e^{-1.5}\sigma_e \\
&= \frac{b_e}{2} \Leftarrow \text{ very simple result!}
\end{aligned}$$

As the plate buckles, the minimum b_e is b_{em}. The minimum b_e' is $b_{em}^{'}$

$$b_{em} = 1.9t\sqrt{\frac{E}{\sigma_y}} \quad \text{and} \quad b'_{em} = .95t\sqrt{\frac{E}{\sigma_y}}$$

For stiffener buckling we use b'_{em} as the effective width of plating to calculate I.



$$b_{em} = 1.9t\sqrt{\frac{E}{\sigma_y}} \quad = \ 56 \text{ t for } \sigma_y = 235\text{MPa}$$

and

$$b'_{em} = .95t\sqrt{\frac{E}{\sigma_y}} = 28 \text{ t for } \sigma_y = 235\text{MPa}$$

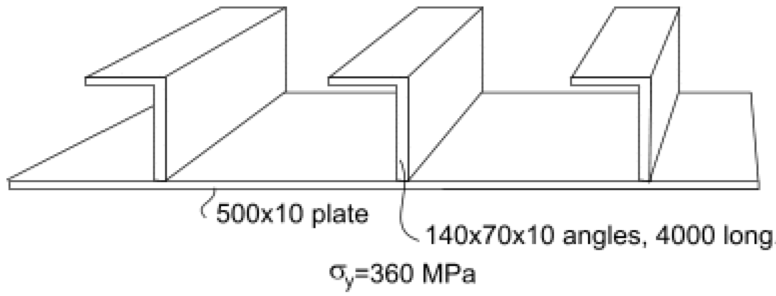
$$F_{\substack{stiffener\\buckling}} = \frac{\pi^2 EI'}{L^2} \quad \text{where I' is found with } b'_{em}$$

$$\sigma_{\substack{stiffener\\buckling}} = \frac{F_{\substack{stiffener\\buckling}}}{A'} \quad \text{where A' uses } b_{em}.$$

If the stiffener buckling stress exceeds yield, then the maximum stiffener force will be F = σ_y A'. The maximum stiffener force (yield or buckling) divided by the total (true) area will give the limit stress in the stiffened panel.

T.16 – Problems.

1. For the plate/frame structure shown, find the average stress that can be applied. What mechanisms determine this limit?



Topic 17: Fatigue/Fracture in Ship Structures

Introduction

In this lecture we will

- Discuss the basic concepts of fracture and relate this to fatigue.

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General

Fatigue is a failure process caused by repeated application of load, even at low stress levels. Hull structure and machinery are both subjected to cyclic stresses and are very prone to fatigue failure. Fatigue damage is common in older ships and represents a significant maintenance challenge. Fatigue damage in machinery will normally result in equipment failure, requiring replacement of the part. Fatigue damage in the hull structure is usually considered 'wear and tear'. Hull structures are very redundant, so that fatigue failure of one element or joint will seldom results in catastrophic failure. There are exceptions, as in the case of the FLARE, as shown below. Fatigue cracks almost certainly were the precursor to the brittle fracture that cut the ship in half.

Figure 1. Forward section of the FLARE [1].

Fatigue starts as the accumulation of micro-structural damage in the material, depending on the amplitude of stress and the number of cycles. This is called the 'crack initiation' stage of fatigue. After small cracks are formed, the state of stress in the material is changed by the presence of the crack, and
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the crack tends to grow stably. This is called the 'crack growth' stage. Once the crack is sufficiently long, it becomes unstable, and a rapid fracture occurs, the 'fracture' stage. In simple structures and many machine parts, small fatigue cracks will grow rapidly and lead to complete failure. In such cases, fatigue can be regarded as a material property. To analyze these situations it is common to employ the S-N curve, Miner's rule and the concept of stress concentrations.

Fracture Mechanics

Fracture, at the material level, is the creation of new surfaces, which necessarily mean the breaking of atomic bonds. Figure 2 shows the three basic ways of creating a crack. While it may seem that tension would create the Type I crack and shear would create the other two, shear results in a tension field at 45° and can also allow Type I cracks. Type II and III cracks can be observed at crystal (grain) boundaries in metals an other polycrystalline materials, but tend to become Type I cracks as they grow to any reasonable size. The cracks turn to become normal to the tension (max. principal stress), and thus tend to align with the direction of the minimum principal stress (which may be compressive).

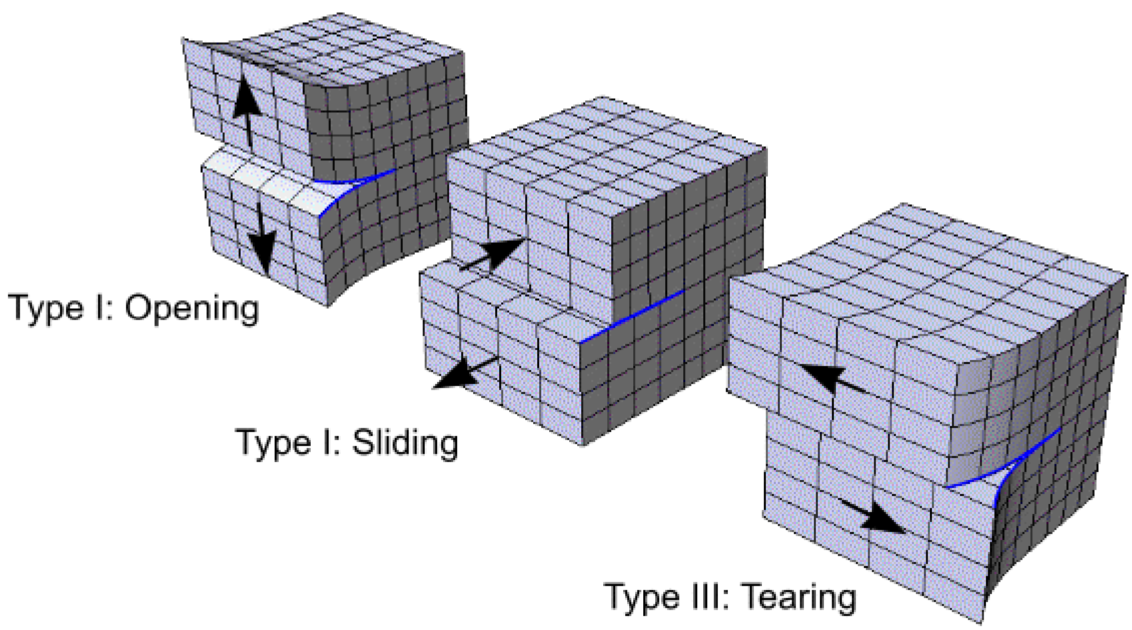

Figure 2. Three types of cracks at the primary material level.

Figure 3 shows the stress field around a sharp crack in a plate subject to constant nominal (far field) stress perpendicular to the crack. The stresses near the crack tip tend to infinity. There is a pattern of stress in the material beside the crack that follows the equation;

$$S_{local} = \frac{K}{\sqrt{2\pi r}} \tag{3}$$

where **r** is the distance from the crack, and;

$$K = S_{nom}\sqrt{\pi a} \tag{4}$$

K is called the "stress intensity" and has units of $MPa\sqrt{m}$. Clearly as **r** goes to zero the stress goes to infinity. One might imagine that all sharp cracks would always grow, because the stress at the crack tip is always infinity. Observations show that sharp crack doesn't always grow. There is always some form of local ductility (plasticity) right around the crack tip. It has been noticed that K is related to the growth of the crack. If *K* exceeds a critical value, called K_C, (or K_{1C}. as we would label the critical value of *K* for a mode I crack) the crack will grow very fast (unstably). If *K* is very low, no crack growth will occur.

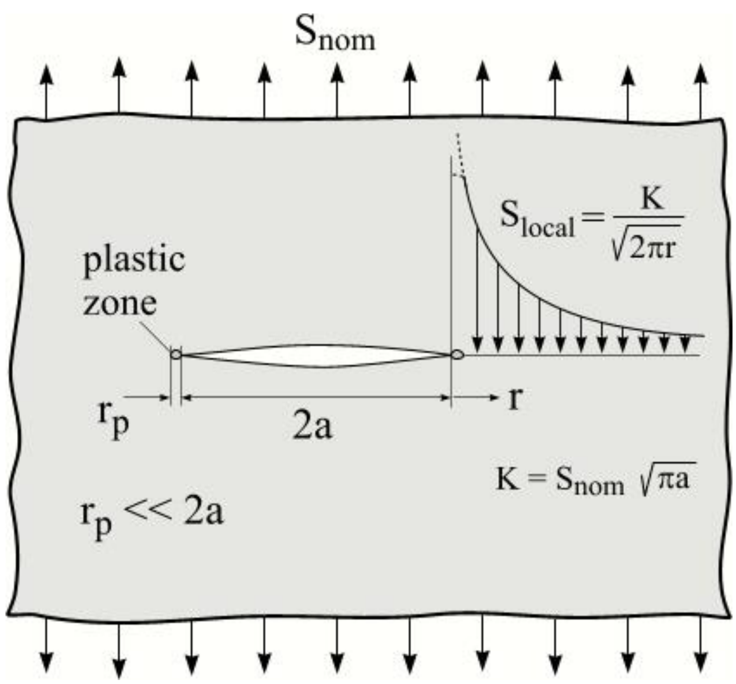

Figure 3. Fracture geometry and stress patterns.

The critical value of K is dependent on plate thickness. This is because plate thickness affects the stress state at the tip of the crack. In thick plates, most of the crack is in a condition of plane strain (i.e. zero strain in the through-plate direction just in front of the crack), while in thinner plates, there is a plane stress condition (i.e. zero stress in the through-plate direction). The plane strain condition is more conducive to fracture, and so the critical value K is lowest (see Figure 4.). To get plain strain, we need a plate thickness of at least ;

$$t_{\min} = 2.5\left(\frac{K_{1c}}{\sigma_y}\right)^2$$

for tough shipbuilding steels, the minimum thickness is extremely thick. For example, for carbon steel of say $\sigma_y = 235$ N/mm², $K_{1c} = 7000$ N/mm$^{3/2}$, we get a t_{min} = 2220mm (over 2 meters thick!). Clearly, cracks in ships are not in a plane strain condition, which probably is part of the reason that they are normally quite resistant to fracture, even when they contain numerous cracks.

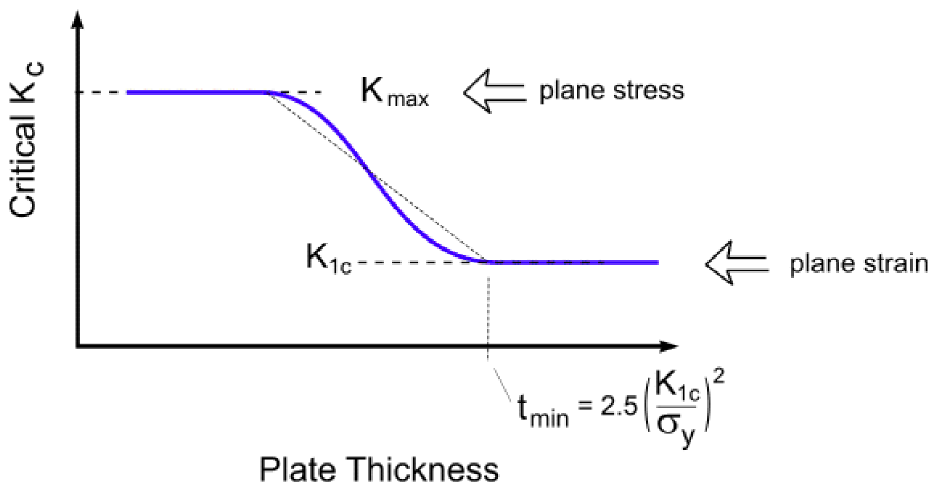

Figure 4. Influence of plate thickness on fracture toughness.

As fracture toughness increases, the type of fracture behaviour changes. Figure 5 illustrates. For very brittle situations, (some or all of: brittle materials, thick materials, large cracks, cold temperatures) failure occurs at stresses well below yield stress. The theory called Linear Elastic Fracture Mechanics (LEFM), is used to describe this type of failure. As the situation becomes more ductile, fractures still occur, but with significant plasticity at the crack tip. In this region we would use Elastic-Plastic Fracture Mechanics

(EPFM). For very ductile materials, substantial plasticity occurs prior to material tearing, and we use Limit Load Analysis (LLA). This discussion refers to the final failure of the structure. In the next part we will discuss fatigue, which is a mechanism by which cracks are created and grown. When fatigue cracks grow large enough, they may become unstable and would then be dealt with by either LEFM or EPFM.

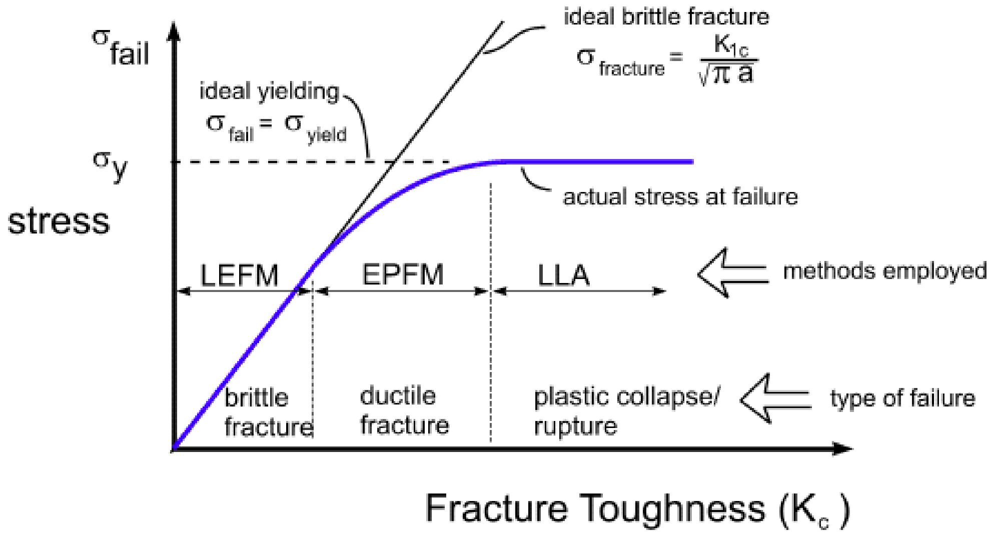

Figure 5. Types of Fracture analysis.

Fatigue as a Material Property

With cyclic stresses, and intermediate values of K, cracks will grow according to how much K changes in each cycle. Paris [2], suggested the 'Paris Law' for crack growth, of the form;

$$\frac{da}{dN} = C \cdot \Delta K^n \tag{5}$$

where $\Delta K = K_{max} - K_{min}$ is the range of K during a cycle of stress. The da/dN term is the increase in the crack size (defined by the half-length a) in each cycle of stress. Figure 6 shows typical experiment data on crack growth. The Paris Law is an idealization of the measured data. And should only be applied for ΔK values in an intermediate range.

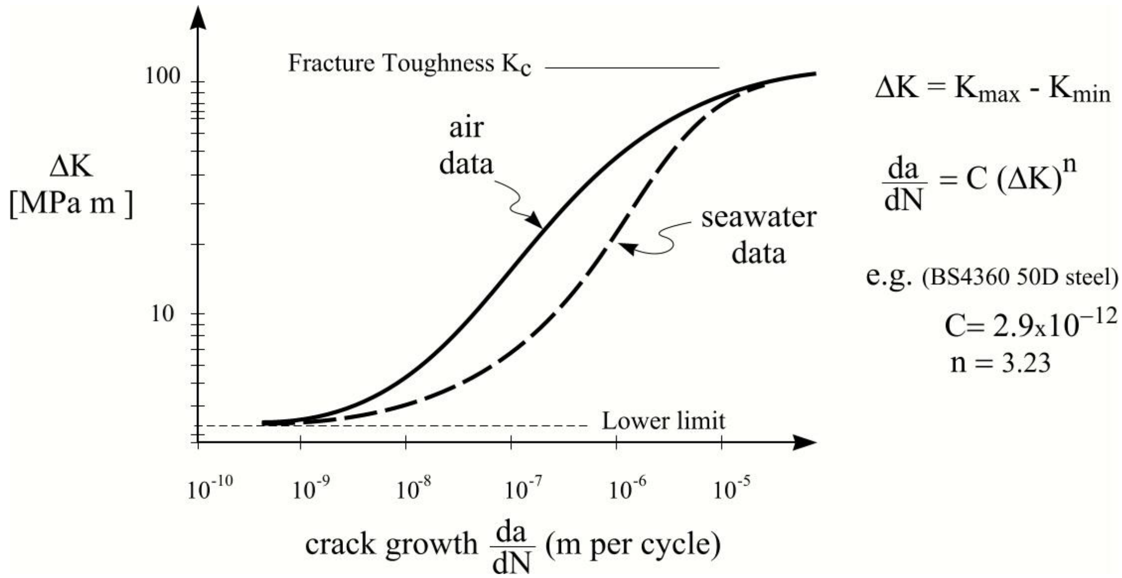

Figure 6. Crack growth diagrams (Paris Law)

Figure 7 shows an experimental arrangement that places a test cylinder in a cyclic stress, which is the same along the test section. There are no notches or other discontinuities in the test. The test section is polished, so that there are not even small surface scratches. In this ideal case the test will continue with no apparent damage until the section snaps. The cyclic stress causes micro-structural damage, either within or between the grains of the metal. Dislocations in the atomic lattice of the crystals are created and pile up, eventually forming micro-cracks. A galaxy of micro-cracks will form, and some of these will eventually coalesce into small cracks. Fatigue tests of this type would take millions to trillions of cycles, and so would be left unattended. Just prior to failure (say a few thousand cycles before) a detectable small crack would exist. Tests of this form are really tests of the formation of small cracks, and are thus “crack initiation” tests. If we plot number of cycles to failure (N) and the cyclic stress (S), we get one point on an “S-N” curve. Repeating the test at varying stress levels allows us to plots the S-N relationship, as shown in Figure 8.

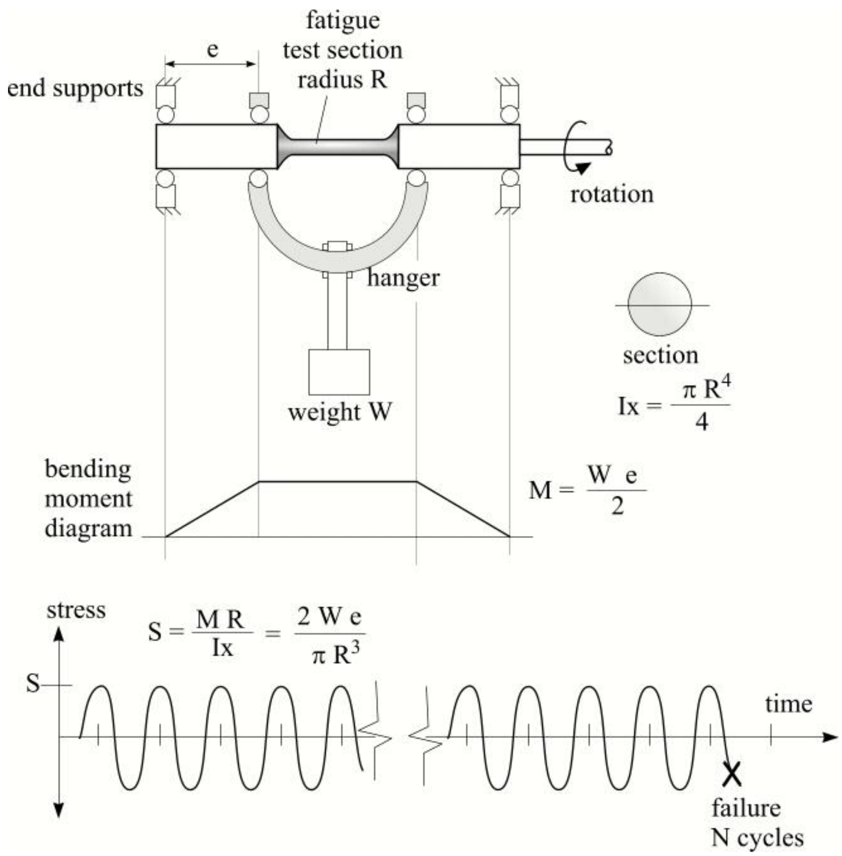

Figure 7. Experimental investigation of material fatigue.
After N cycles at stress S, the section fails.

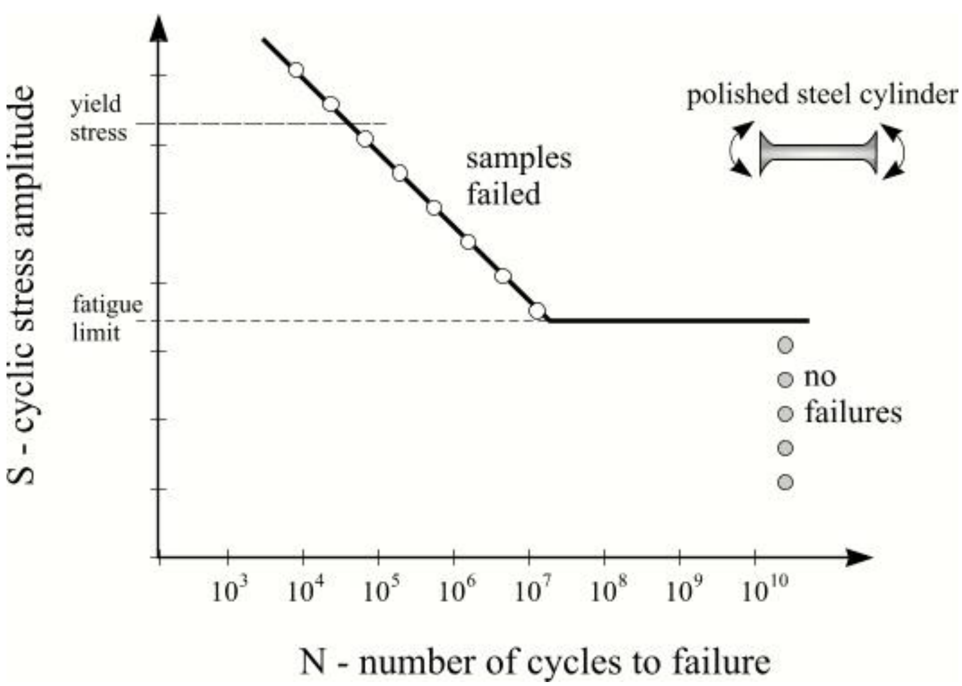

Figure 8. S-N curve for ideal polished cylinder.

Figures 9 and 10 show a typical cross section of a fatigue crack after failure. The location of the small formation crack is labeled the 'initiation point'. The crack then grows steadily and leave beach marks showing the crack front. The marks represent locations where the fracture was temporarily arrested. When the fatigue crack is large enough, the stress intensity at the crack front reaches a critical value and the crack becomes unstable. At this point the bar snaps.

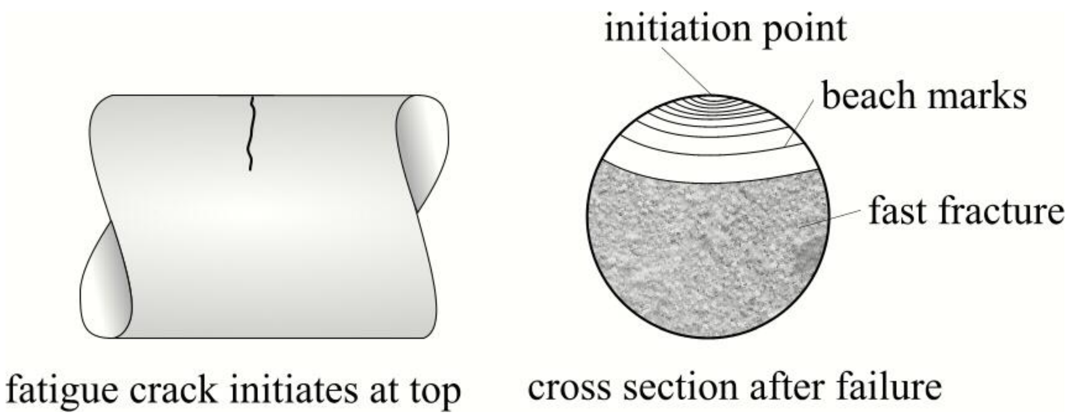

Figure 9. Fatigue failure cross section.

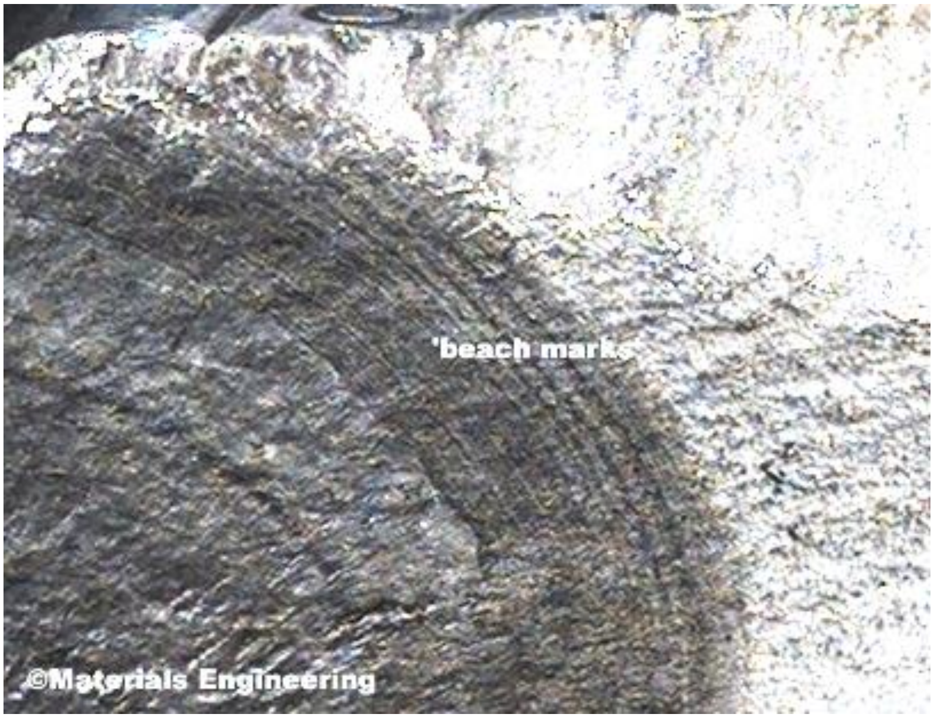

Figure 10. Photo Showing Beach Marks.

References

[1] Marine Investigation Report, "Break-Up and Sinking The Bulk Carrier "FLARE" Cabot Strait 16 January 1998", Transportation Safety Board of Canada, Report Number M98N0001

[2] Paris, P.C., "Fatigue- An interdisciplinary approach", ed. Burke, J.J., Reed, N.L., and Weiss, V., 10th Sagamore Army Mat. Res. Conf. Syracuse University Press, New York, 1964.

[3] Dover, W.D., "Fatigue Behaviour of Offshore Structures" in Offshore Structures, V.2, Ed. by Reddy and Arockiasamy, Kriegar Publishing, Malabar Florida, 1991.

[4] Paik, J.K., Thayamballi, A.K., Ultimate Limit State Design of Steel Plated Structures, Wiley, 2003

T.17 – Problems.

1. What might the physical reason be to explain why there is a low stress fatigue limit in steel?

Topic 18: The Fatigue Process in Ships

Introduction

In this lecture we will

- Discuss Miners rule for dealing with fatigue in a complex loading situation.
- Next we discuss the stages of fatigue and applications to ship structural details.

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In complex structures, fatigue cracks will grow in a slow and stable way, and do not necessarily pose a large risk. In these cases fatigue should be viewed as a sequence of fracture processes, dependant on material properties, stress ranges and geometry. To analyze these situations it is common to employ crack growth models such as the Paris Law, along with fracture mechanics concepts such as "stress intensity", and fatigue life estimation. Alternatively, it is also common to use "detail-specific" S-N curves, which are empirical fatigue obtained from structural tests of complete structural details.

Miner's Rule

In most cases, the stress amplitude is not constant. Most stress time histories are random, and are described by a frequency distribution. With the aid of a Fourier transform, a random time history can be described as the sum of a set of sine waves. Miner [2] proposed a simple method of determining the fatigue strength of a material subjected to variable amplitude cycles. Figure 1 shows the decomposition of a time history into a series of waves. Each wave has an amplitude $\mathbf{S_i}$ and a number of cycles $\mathbf{n_i}$. Miner suggested that each wave is causing damage in the metal. Stress $\mathbf{S_1}$ would result in fatigue failure after $\mathbf{N_1}$ cycles. After $\mathbf{n_1}$ cycles, $\mathbf{S_1}$ has caused a damage ratio $\mathbf{D_1}$, where;

$$D_1 = n_1/N_1 \qquad (1)$$

The total damage **D** is the sum of the contributions from all the waves;

$$\mathbf{D = \Sigma_{\text{all i}} \quad n_i/N_i} \qquad \mathbf{(2)}$$

When the damage ratio **D** becomes 1, failure will occur. Miner's rule actually applies only to crack initiation. For cases where this is most of the fatigue life, it is a good approximation to the fatigue life.
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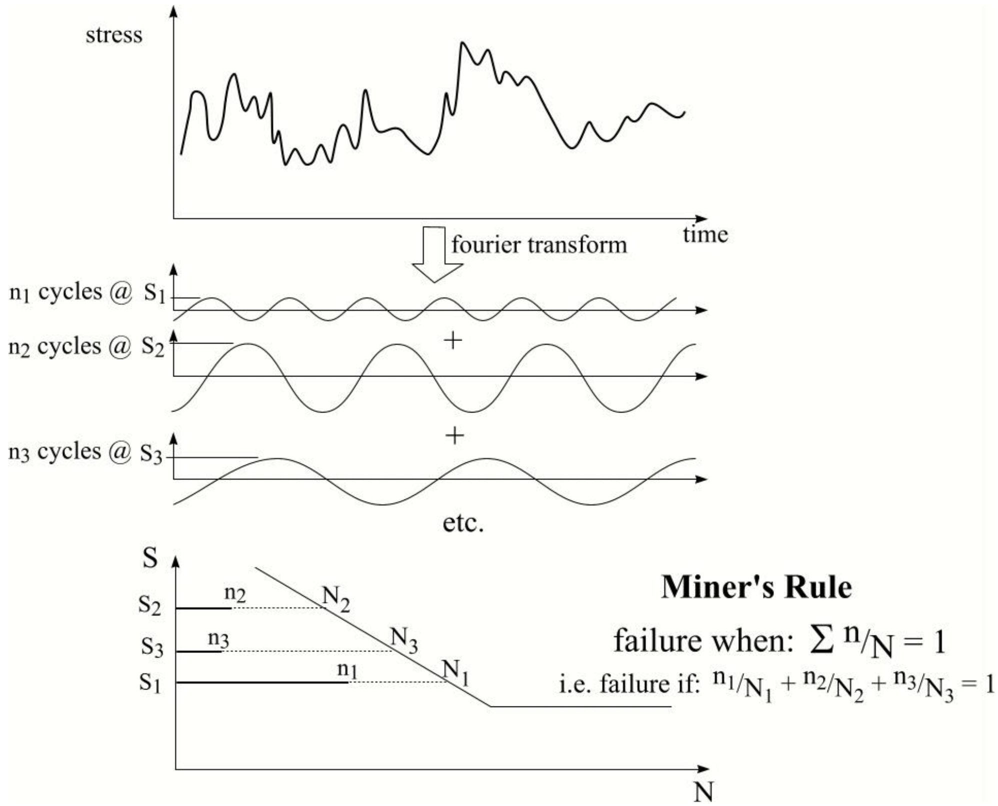

Figure 1. Miner’s Rule for variable stress.

Fatigue as a Process

The above discussion relates to a perfect geometry, with no initial material flaws, and no sharp corners or other “stress–risers”. In real structures, the stress is not uniform. There are sharp corners and flaws that result in localized stress peaks, called ‘stress risers’. Figure 7 shows a typical detail. The end of a bracket, although tapered and sniped, causes a local stress peak called a ‘hot spot’. The stresses are highest at the toe of the weld where a corner exists. The stress in this region is a multiple of the nominal stresses. So, as the nominal stresses oscillate, the ‘hot-spot’ stresses will oscillate with greater magnitude. Obviously, the material at the toe of the weld will fatigue fastest, and cracks will form here first. The relative complexity of the geometry causes the stress peaks. Figure 8 shows how fatigue data for welded joints tends to differ from ideal fatigue tests. The fatigue strength, for the same material, tends to be lower, and the lower cutoff is much less well defined.

Figure 9 shows that fatigue is composed of three phases; initiation, propagation, and final failure. In simple tests the initiation phase is the most important. Once the crack has formed it grows very rapidly and the part fails.

In more complex structures, the stresses that caused the crack to initiate are quite localized. As the crack grows it tends to relieve the stresses. The crack can grow slowly or even stop. In such cases the crack propagation phase dominates the failure process. In real structures, there are often many cracks left over from the fabrication process. These can be weld flaws or just geometric discontinuities. In such cases the cracks are already initiated, and fatigue life depends on the crack growth process.

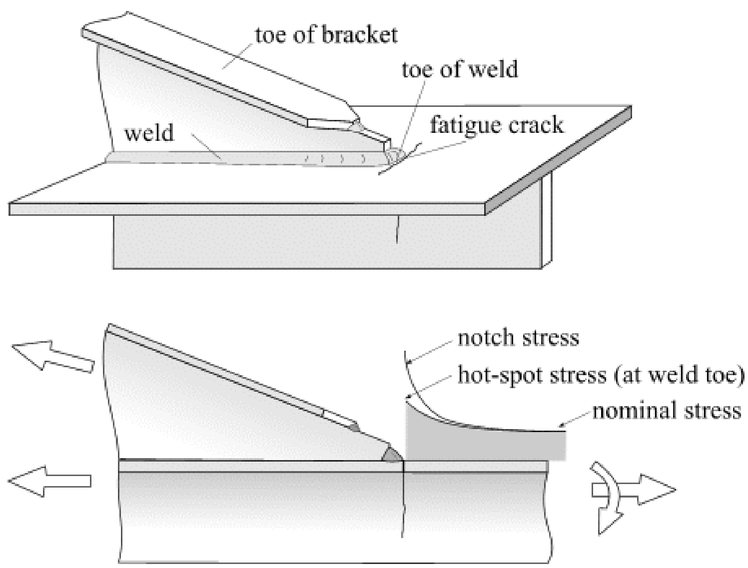

Figure 2. Fatigue cracks in structural details. Localized geometric discontinuities cause local stress peaks.

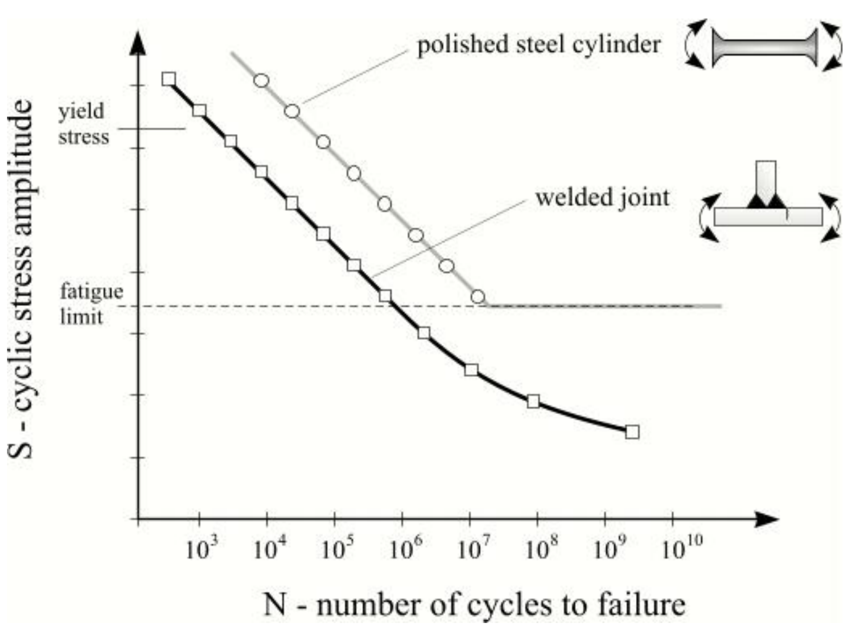

Figure 3. Typical S-N curves for welded connections compared with ideal specimens.

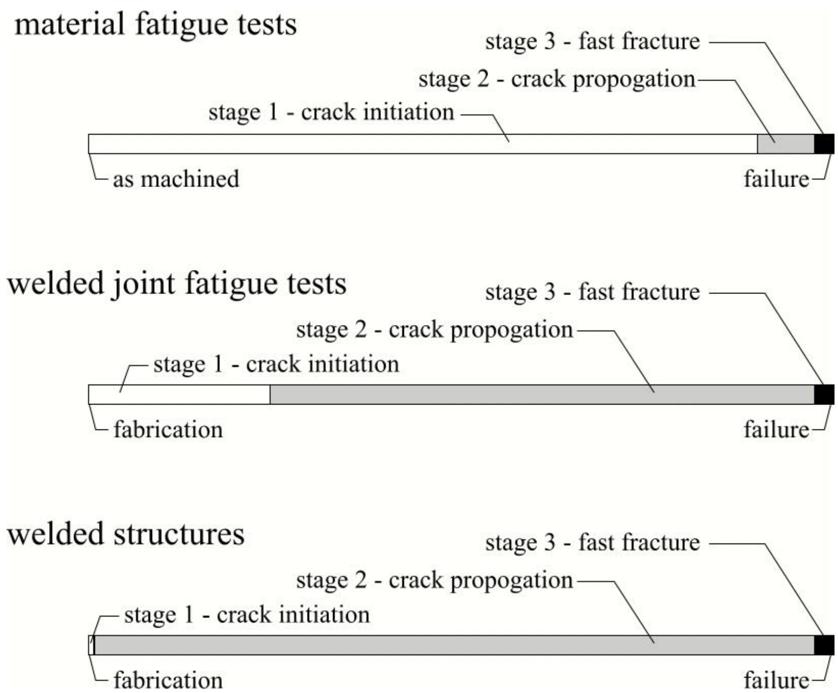

Figure 4. The 3 stages of fatigue.

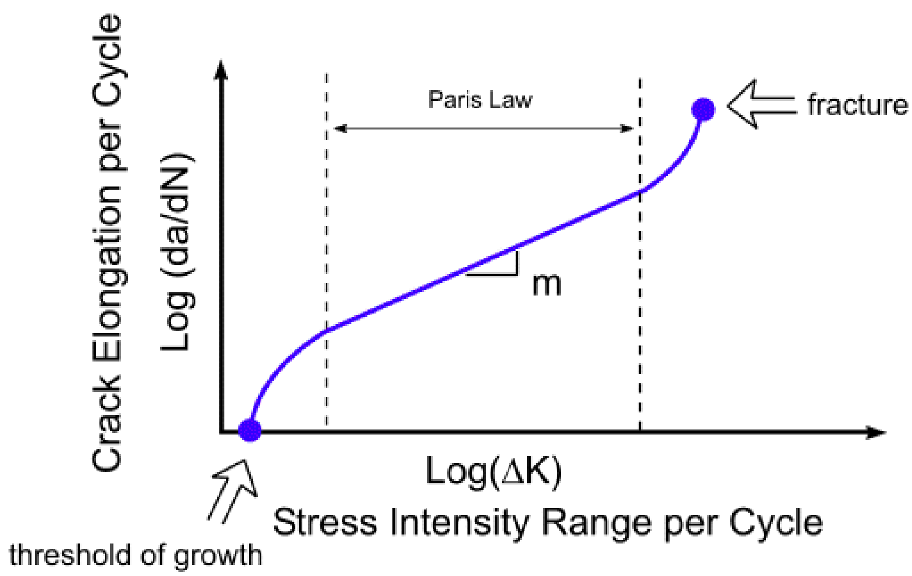

Figure 5 The 3 regions of crack growth rate.

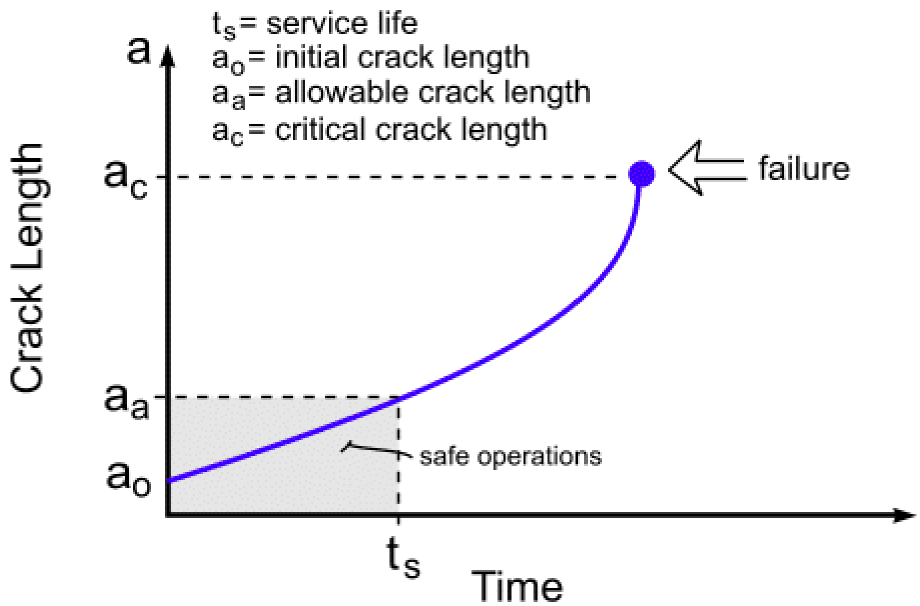

Figure 6 Crack growth during 'service' life.

References

[1] Dover, W.D., "Fatigue Behaviour of Offshore Structures" in Offshore Structures, V.2, Ed. by Reddy and Arockiasamy, Kriegar Publishing, Malabar Florida, 1991.

[2] Miner, M.A., Jnl. Appl. Mech., ASME, Vol. 12. Sept 1945

T.18 – Problems.

1. For the following stress history and SN curve, determine if the fatigue failure is likely to have occurred or not.
   Stress History: 20,000 cycles of amplitude 300MPa
   100,000 cycles of amplitude 200MPa
   880,000 cycles of amplitude 100MPa

   The SN curve: $\log(N)=12.164 - 3 \times \log(\Delta\sigma)$

2. Discuss why crack propogation is the most significant part of the fatigue like of ships, even though initiation is the dominant part of the fatugue life in lab tests.

Topic 19: Discussion of Hot Spot Fatigue Analysis

Introduction

In this lecture we will

- Discuss the DnV Recommended Practice for Fatigue Analysis of Offshore Structures (RP-C203, 2001), especially the idea of hot spot analysis

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Intro:

Read RP-C203, especially the parts related to plated structures (not the tubular joints or pipelines). I've included some parts below.

Concept

On the basis of experiments, S-N curves for various structural details were found. Table 2.3.3 is for structures in a seawater environment with cathodic protection. Figure 2.3-2 shows the same S-N data in graphical form. Take for instance, S-N curve 'D'. The table shows that $\log(\bar{a}) = 11.764$.

The equation for the S-N curve is;

$$\log N = \log(\bar{a}) - m \log\left(\Delta\sigma\left(\frac{t}{t_{ref}}\right)^k\right)$$

where

N : number of cycles of $\Delta\sigma$ to failure.

m: the 'negative inverse' slope, (=3 for stresses above the fatigue limit)

For example, at $\Delta\sigma$ = 200MPa;

N=10^(11.764-3 log (200)) = 72,595 cycles

The figure shows the 'D' curve crossing the 200MPa line just above 70,000 cycles, as we would expect.

The letter 'D' and all other letters refer to the detail category. Table 5 below shows some category 'C' and 'D' details. The more prone the detail is to fatigue, the higher the letter and the lower the S-N curve.

Category 'D' can be used for a more general purpose. Note that in table 2.3.3, on the right hand side, the stress-concentration factor as derived by the hot spot method is 1.0. This means that we can use the 'D' line with stresses calculated by the hot-spot method for any structural detail. The hot spot method is a way to do fatigue analysis on a detail that is not in the table.
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2.3.3 S-N curves in seawater with cathodic protection

S-N curves for seawater environment with cathodic protection are given in Table 2.3-2 and Figure 2.3-2. The T curve is shown in Figure 2.3-3.

Table 2.3-2 S-N curves in seawater with cathodic protection

S-N curve	$\log \bar{a}_1$ $N \le 10^6$ cycles $m_1 = 3.0$	$\log \bar{a}_2$ $N > 10^6$ cycles $m_2 = 5.0$	*Fatigue limit at 10^7 cycles *)*	*Thickness exponent k*	*Stress concentration in the S-N detail as derived by the hot spot method*
B1	12.513	16.856	93.57	0	
B2	12.339	16.566	81.87	0	
C	12.192	16.320	73.10	0.15	
C1	12.049	16.081	65.50	0.15	
C2	11.901	15.835	58.48	0.15	
D	11.764	15.606	52.63	0.20	1.00
E	11.610	15.350	46.78	0.20	1.13
F	11.455	15.091	41.52	0.25	1.27
F1	11.299	14.832	36.84	0.25	1.43
F3	11.146	14.576	32.75	0.25	1.61
G	10.998	14.330	29.24	0.25	1.80
W1	10.861	14.101	26.32	0.25	2.00
W2	10.707	13.845	23.39	0.25	2.25
W3	10.570	13.617	21.05	0.25	2.50
T	11.764	15.606	52.63	0.25 for SCF ≤ 10.0 0.30 for SCF >10.0	1.00

*) see also 1.4

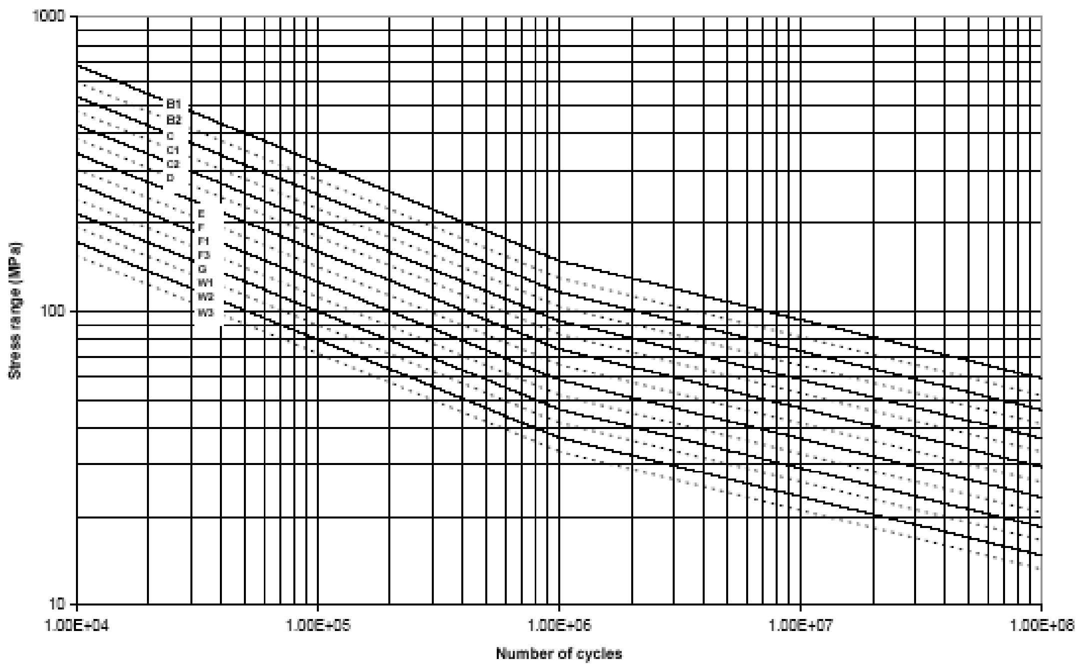


Figure 2.3-2 S-N curves in seawater with cathodic protection

Table 5 Transverse butt welds, welded from both sides

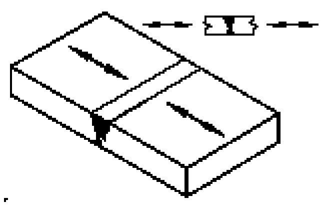
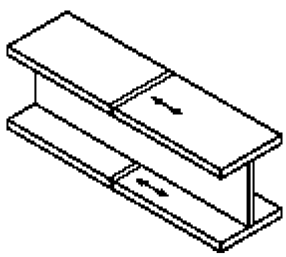
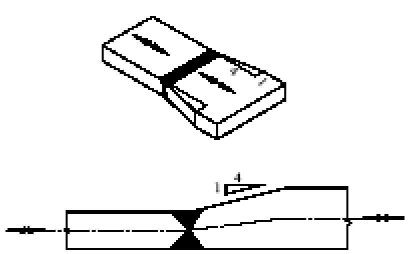
Notes on potential modes of failure

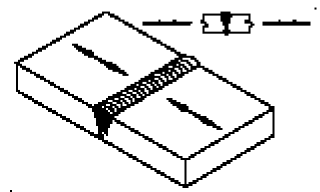
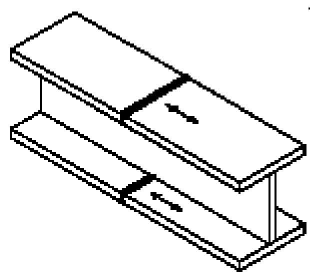
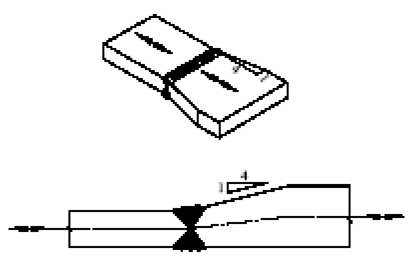
With the weld ends machined flush with the plate edges, fatigue cracks in the as-welded condition normally initiate at the weld toe, so that the fatigue strength depends largely upon the shape of the weld overfill. If the overfill is dressed flush, the stress concentration caused by it is removed, and failure is then associated with weld defects.

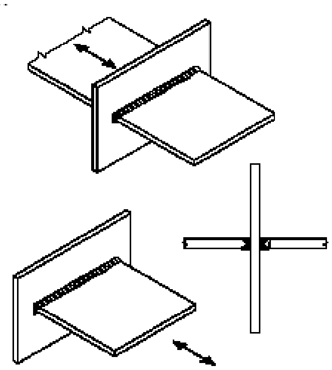
Design stresses

In the design of butt welds that are not symmetric about the root and are not aligned, the stresses must include the effect of any eccentricity (see section 2.5 to 2.9).

With connections that are supported laterally, e.g. flanges of a beam that are supported by the web, eccentricity may be neglected.

Detail category	Constructional details	Description	Requirement
C1	1.  2.  3. 	1. Transverse splices in plates flats and rolled sections 2. Flange splices in plate girders. 3. Transverse splices in plates or flats tapered in width or in thickness where the slope is not greater than 1:4.	1. and 2.: - Details 1. and 2. may be increased to Category C when high quality welding is achieved and the weld is proved free from significant defects by non-destructive examination (it is assumed that this is fulfilled by inspection category I). 1., 2. and 3.: - All welds ground flush to plate surface parallel to direction of the arrow. - Weld run-off pieces to be used and subsequently removed, plate edges to be ground flush in direction of stress. - All welds welded in horizontal position in shop.

Detail category	Constructional details	Description	Requirement
D	4.  5.  6. 	4. Transverse splices in plates and flats. 5. Transverse splices in rolled sections or welded plate girders 6. Transverse splices in plates or flats tapered in width or in thickness where the slope is not greater than 1:4.	4., 5. and 6.: - The height of the weld convexity to be not greater than 10% of the weld width, with smooth transitions to the plate surface. - Welds made in flat position in shop. - Weld run-off pieces to be used and subsequently removed, plate edges to be ground flush in direction of stress.

Detail category	Constructional details	Description	Requirement
F	1. 	1. Full penetration butt welded cruciform joint	1.: - Inspected and found free from significant defects. The detail category is given for: - Edge distance ≥ 10mm - For edge distance < 10mm the detail category shall be downgraded with one SN-curve

The Hot Spot Method

As you can see in the table above, category 'D' details are quite simple, while higher categories (e.g. 'F') are more complex and have more geometric discontinuities which lead to stress concentrations. Note that for category 'D', the only significant aspect is the presence of a weld in an otherwise straight section.

The hot-spot method treats fatigue by separating two types of stress concentration effects. The stress concentration due to the overall structural geometry is separated from the stress concentration due to the very local geometry of the weld. The reason for this is that it is very difficult to properly model the detailed shape of a weld. On the other hand, it is reasonably practical to model the general stress concentration in a detail. We use the following definitions;

Hot spot stress : the stress in the region of the toe of the critical weld, accounting for the general detail geometry, but without the very sharp stress concentration due to the weld toe geometry.

Notch stress: This is the stress at the weld toe, including the stress concentration due to the weld toe shape. (very hard to model)

When making a finite element model, we often have very large (though very localized) stresses at discontinuities. For this reason, the hot spot stress is not taken directly from the FE model right at the discontinuity (the toe of the weld). Rather, the hot spot stress is projected to the hot spot, from values nearby. This is a practical way to avoid the spurious (false) results that an FE model gives at sharp discontinuities. Figure 2.13-1 shows how the stresses are projected from nearby. Once we get the (projected) hot-spot stress, we can use category 'D' to estimate the fatigue life of the hot-spot (and the detail).

2.13.3 Welded connections other than tubular joints

The stress range at the hot spot of welded connections should be combined with S-N curve D. The C-curve may be used if machining of the weld surface to the base material is performed. Then the machining has to be performed such that the local stress concentration due to the weld is removed.

The aim of the finite element analysis is not normally to calculate directly the notch stress at a detail, but to calculate the geometric stress distribution in the region at the hot spot such that these stresses can be used as a basis for derivation of stress concentration factors. Reference is made to Figure 2.13-1 as an example showing the stress distribution in front of an attachment (A-B) welded to a plate with thickness t. The notch stress is due to the presence of the attachment and the weld. The aim of the finite element analysis is to calculate the stress at the weld toe (hot spot) due to the presence of the attachment, denoted geometric stress, $\sigma_{\text{hot spot}}$. The stress concentration factor due to this geometry effect is defined as,

$$SCF = \frac{\sigma_{\text{hot spot}}}{\sigma_{\text{nominal}}} \qquad \textbf{(2.13.1)}$$

Thus the main emphasis of the finite element analysis is to make a model that will give stresses with sufficient accuracy at a region outside that effected by the weld. The model should have a fine mesh for extrapolation of stresses back to the weld toe in order to ensure a sufficiently accurate calculation of SCF.

FEM stress concentration models are generally very sensitive to element type and mesh size. By decreasing the element size the FEM stresses at discontinuities will approach infinity. It is therefore necessary to set a lower bound for element size and use an extrapolation procedure to the hot spot to have a uniform basis for comparison of results from different computer programs and users. On the other hand, in order to pick up the geometric stress, σ_g, increase properly, it is important that the stress reference points in t/2 and 3t/2 (see Figure 2.13-1) are not inside the same element. This implies that element sizes of the order of the plate thickness are to be used for the modelling. If solid modelling is used, the element size in way of the hot spot may have to be reduced to half the plate thickness in case the overall geometry of the weld is included in the model representation.

Element stresses are normally derived at the gaussian integration points. Depending on element type it may be necessary to perform several extrapolations in order to determine the stress at the location representing the weld toe. In order to preserve the information of the direction of principal stresses at the hot spot, component stresses are to be used for the extrapolation. When shell elements are used for the modelling and the overall geometry of the weld is not included in the model, the extrapolation shall be performed to the element intersection lines. If the (overall) weld geometry is included in the model (3D model), the extrapolation is related to the weld toe as shown in Figure 2.13-1. If 8 node shell elements are used the hot spot is considered to be at the element intersection line.

Two different definitions for hot spot stresses are used:

1. The stress is derived by extrapolating the stress to the weld toe (intersection line).
2. The stress at 0.5t from the considered hot spot

Method 1: The stresses are first extrapolated from the gaussian integration points to the plate surface. A further extrapolation to the line A - B is then conducted. The final extrapolation of component stresses is carried out as a linear extrapolation of surface stresses along line A - B at a distance t/2 and 3t/2 from either the weld toe, or alternatively the element intersection line (where t denotes the plate thickness). Having determined the extrapolated stress components at the hot spot, the principal stresses are to be calculated and used for the fatigue evaluation.

Some comments on element size are given in Appendix 4, Commentary.

It is recommended to perform a verification of the procedure on a detail that is S-N classified and that is similar in geometry and loading to that being analysed. If the verification analysis comes out with a different SCF (SCF Verification) than that inherent in the S-N detail, ref. e.g. Table 2.3-1, a resulting stress concentration factor can be calculated as

$$SCF = SCF_{\text{Analysis}} \cdot \frac{SCF_{\text{S-N Table 2.3-1}}}{SCF_{\text{Verification}}}$$

where

$SCF_{\text{S-N Table 2.3-1}}$ = Stress concentration in the S-N detail as derived by the hot spot method, see Table 2.3-1.

SCF_{Analysis} = Stress concentration factor for the analysed detail.

It should be noted that the hot spot concept can not be used for fatigue checks of cracks starting from the weld root of fillet/partial penetration welds. The weld should be checked separately considering the stresses in the weld itself, ref. section 2.2.3.

Method 2: The hot spot stress is derived directly from the finite elements at a distance 0.5t from the weld toe using 20 node solid elements or 0.5t from the intersection line using 8 node shell elements.

It is also here recommended to perform a verification of the procedure on a detail that is S-N classified and that is similar in geometry and loading to that being analysed. If the verification analysis comes out with a different SCF (SCF Verification) than that inherent in the S-N detail, ref. e.g. Table 2.3-1, a resulting stress concentration factor can be calculated as shown in the example above.

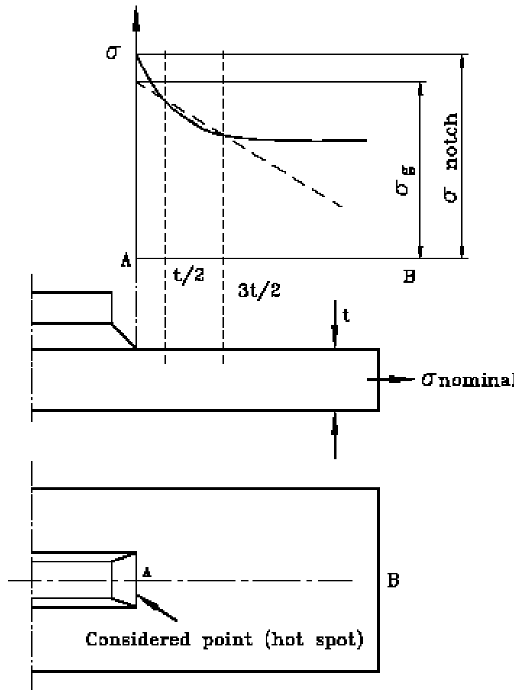


Figure 2.13-1 Stress distribution at an attachment and extrapolation of stresses

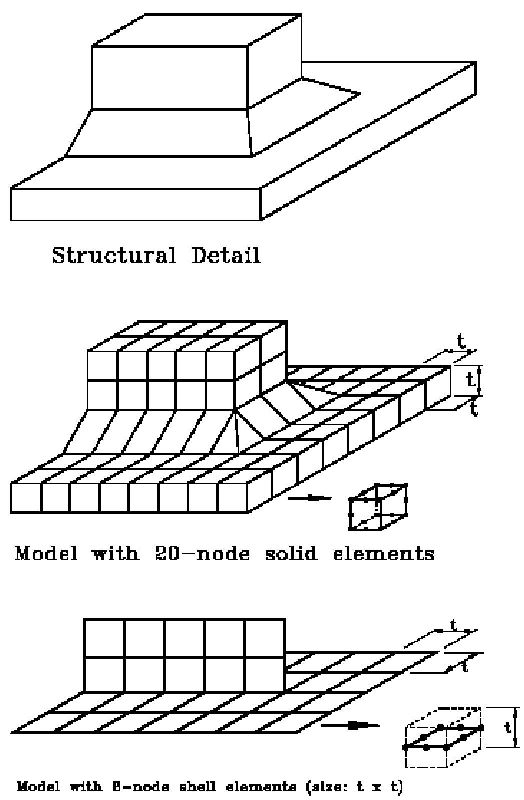


Figure 2.13-2 Examples of modelling

T.19 – Problems.

1. Explain how you would verify that you are doing the hot spot analysis correctly for your detail of interest.

Topic 20: Section Modulus – a measure of moment capacity

Introduction

In this lecture we will

- Review elastic section modulus
- Describe plastic section modulus

~~~~~~

Elastic section modulus

A section modulus is a geometric measure of the ability of a cross-section to carry a moment. An elastic section modulus measures the ability to carry an elastci moment, while a plastic section modulus indicates the ability to carry a plastic moment. We denote section modulus with S (or sometimes Z).

Recall that the moment of inertia of an area is;

$$Ix = \int y^2 da$$

where y is measured from some baseline. Ix will be a minimum if the baseline runs through the centroid. In this case we call the Ix : Ina (through the neutral axis).

For a rectangle w wide and h high, we have Ina = 1/12 w h^3. In general we can find I by integration or by the parallel axis theorm;

$$I_{base} = \sum\left(ay^2 + i\right)$$

where y, in this case, is measured from a baseline

and

$$I_{na} = I_{base} - A \cdot h_{na}^2$$

where

$$A = \sum a \quad \text{and} \quad h_{na} = \frac{\sum ay}{\sum a}$$

For elastic stresses, the stress formula is;
~~~~~~

$$\sigma = \frac{M \cdot y}{I}$$

where in this case y is measured from the neutral axis.

at the elastic limit, the moment is M_y and the stress is the yield stress (σ_y at y=c, c being the largest distance from the neutral azis on the section), so we have;

$$\sigma_y = \frac{M_y \cdot c}{I}$$

We can combine I and c to give the elastic section modulus;

$$\sigma_y = \frac{M_y}{S_e} \text{ where } S_e = \frac{I}{c}$$

Note that we can write;

$$S_e = \frac{M_y}{\sigma_y},$$

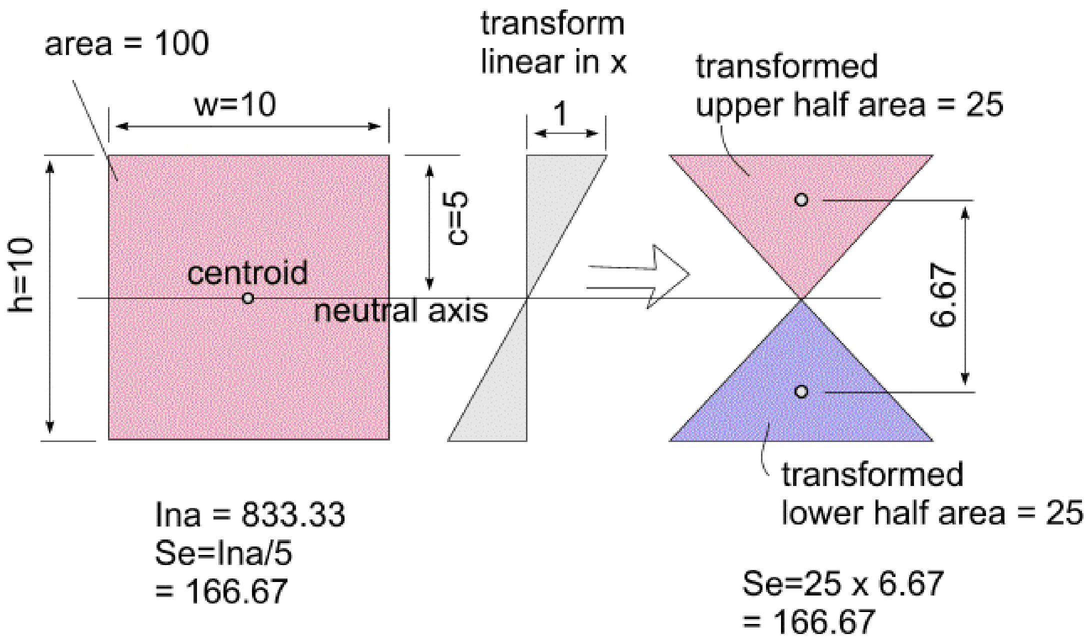
and more importantly;

$$M_y = S_e \cdot \sigma_y$$

We can think of the section modulus as the geometric part of the moment capacity, while σ_y is the material part.
From this perspective, S_e can be thought of as the moment per unit strength (i.e. the moment for σ_y = 1.0)

We can find S_e from I/c.

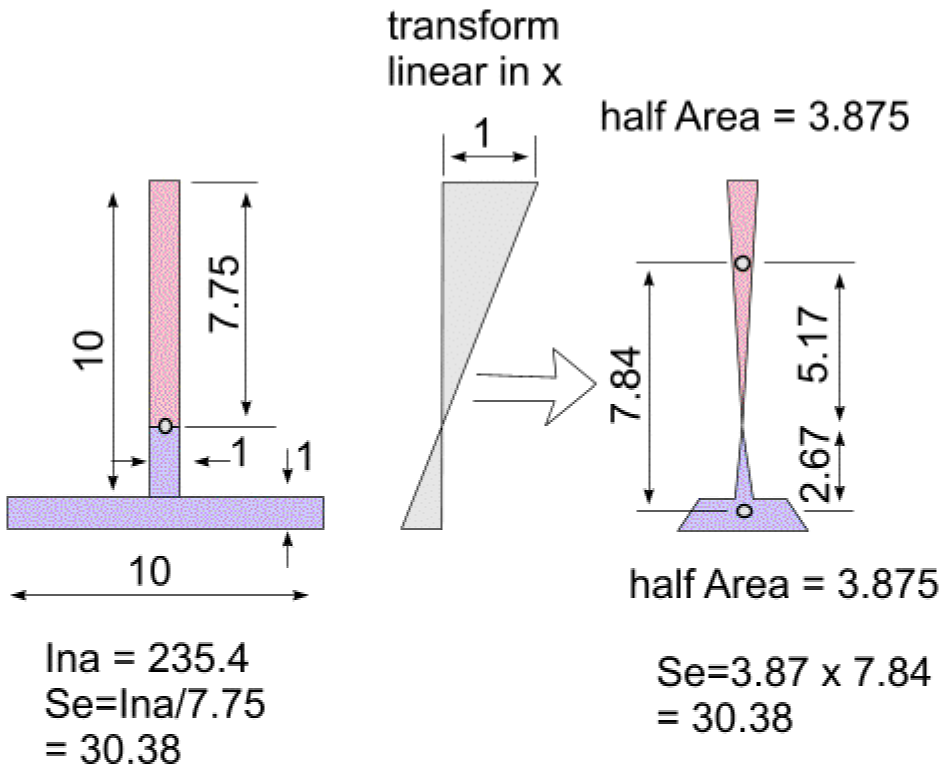
Alternatively we can find S_e from moment considerations (graphically). The figure below shows how to find S_e in this way. If we imagine that the stress at c is 1.0, for elastic bending the stress is linear and zero at the centroid. The force in any horizontal strip (b wide, dh high) will be equal to b dh y/c. We can transform the section by shrinking all widths according to y/c. This is a linear 'squeezing' of the area, with the new width at the centroid going to zero, the width at c remaining unchanged, and all other widths adjusted accordingly. We now have a modified section, with two parts (above and below the centroid).



The elastic section modulus can be found as the moment of the modified area (i.e. the half-area times the distance between the centroids of the upper and lower area).

Why should this be?

In any pure bending situation the axial force is zero. This means that the compressive and tensile forces balance. By transforming the section, we create a pattern of force, reflecting the real force, and equivalent to the force that would occur if there were a uniform pressure over the transformed area. The center of action of the uniform pressure would be at the centroid of the area. For a unit pressure, the forces equal the area, and the moment equals the product of the half-area times the distance between the centroids. It may seem obvious that in the case shown that the two areas (upper and lower) are equal, but this is generally true even for unsymmetrical sections as shown below.


transform
linear in x
half Area = 3.875
1
10
7.75
1
1
10
7.84
5.17
2.67
half Area = 3.875
Ina = 235.4
Se=Ina/7.75
= 30.38
Se=3.87 x 7.84
= 30.38

Plastic section modulus

Again, the plastic section modulus is a geometric measure of the ability of a cross-section to carry moment. In the plastic case we assume that the stress is uniformly at yield everywhere. As such there is no need to transform the area.

As with elastic moments, there is no net axial force. This means that the compressive and tensile forces must balance. As the stresses are all at yield (in both tension and compression) it means that exactly half the area is in compression and exactly half is in tension. This means that the plastic neutral axis is at the half-area. For symmetrical sections this is the same as the centroid, but different for unsymmetrical sections.

Whether symmetrical or not;

$$S_p = \frac{M_p}{\sigma_y},$$

and;

$$M_p = S_p \cdot \sigma_y$$

S_p, like S_e has units of length^3, (e.g. mm^3). The plastic section modulus, S_p, can be found as the moment of the half-area (i.e. the half-area times the distance between the centroids of the upper and lower half-areas).

This gives a physical (graphical) meaning to Sp. For typical ships frames, with plating, web and flange, it is convenient to have a generic formula for the plastic section modulus. Its rather long;

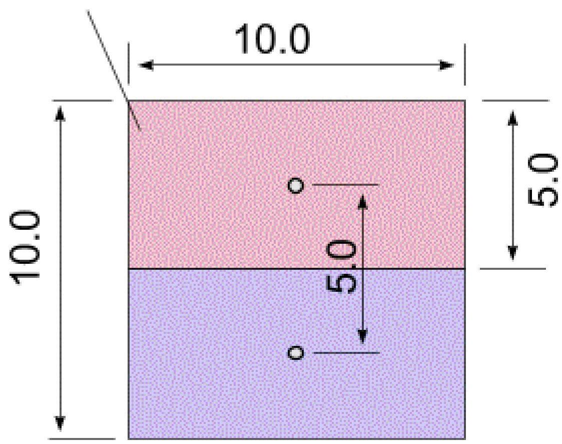
$$S_p = \begin{cases} C_1 + \dfrac{C_2}{bp} + C_3 & ap > aw + af \\ C_1 + \dfrac{C_2}{tw} + C_3 & ap < aw + af \end{cases}$$

$$C_1 = \frac{1}{2}\left(bp \cdot tp^2 + tw \cdot hw^2 + bf \cdot tf^2\right)$$

$$C_2 = -\frac{1}{4}\left(bp \cdot tp - tw \cdot hw - bf \cdot tf\right)^2$$

$$C_3 = hw \cdot bf \cdot tf$$

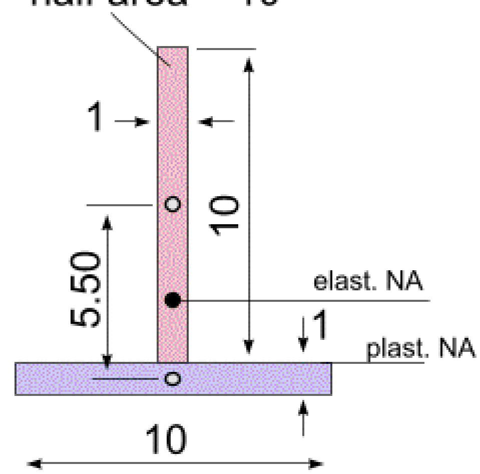
Ina = 833.33
Se=Ina/5
= 166.67
half area = 50



Sp=50 x 5
=250

Sp/Se = 250/166.67
=1.5

Ina = 235.4
Se=Ina/7.75
= 30.38
half area = 10

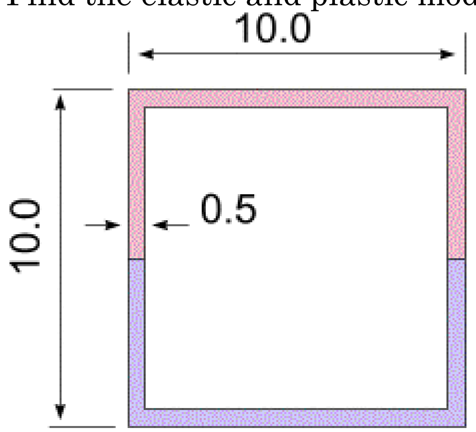


Sp=10 x 5.5
= 55

Sp/Se = 55/30.38
=1.81

T.20 – Problems.

1. Find the elastic and plastic modulii for the hollow section;



Topic 21: Strain Rate Effects

Acknowledgement: This chapter prepared by B.W.T. Quinton

Introduction

In this lecture we will:

- Introduce the concept of material strain rate effects.
- Use the Cowper-Symonds expression to resolve a dynamic yield stress.

So far during our analysis of ship structures we have implicitly assumed that all loads are applied *quasi-statically*; that is, applied sufficiently slowly such that *inertial* and *material strain rate effects* are insignificant, and therefore ignorable. What if we are dealing with a material whose *constitutive equations* are sensitive to *strain rate* (like steel is)?
In this lecture we will learn how to account for *material strain rate effects* in the context of our assumption in previous lectures that steel behaves in an *elastic-perfectly plastic* manner.

Dynamic Yield Stress

Steel is a material whose behaviour is different for different *strain rates*; i.e. it exhibits *material strain rate effects*. As the term implies, *strain rate* is the speed at which the *strain*, ε, in a material changes with respect to *time*.

Strain is (as we have seen before) given by the *change* in length of a geometry divided by the original length:

$$\varepsilon = \frac{l - l_0}{l_0}$$

Strain rate is the derivative of ε w.r.t. time:

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{d(l - l_0)}{dt \cdot l_0}$$

In the limit of time from $0 \to t$, we have:

$$\dot{\varepsilon} dt = d\varepsilon$$

$$\dot{\varepsilon}(t - t_0) = \varepsilon - \varepsilon_0$$

If we assume that $t_0 = 0$ and $\varepsilon_0 = 0$ than we have:

$$\dot{\varepsilon} = \frac{\varepsilon}{t}$$

Specifically, *material strain rate effects* affect the *constitutive relationships* of a material. In this course we have assumed an *elastic-perfectly plastic constitutive model* for steel.

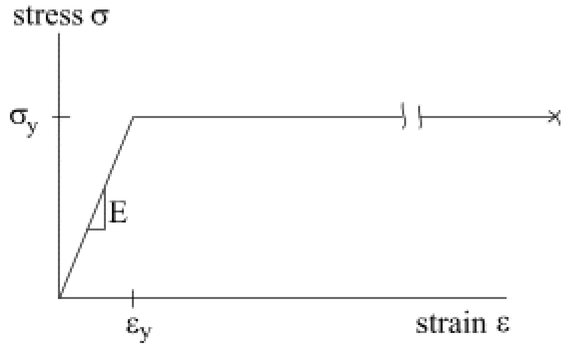


FIGURE 1: *ELASTIC-PERFECTLY PLASTIC CONSTITUTIVE MODEL.*

In this context, the *strain rate* affects only the *yield stress*, σ_y; *Young's Modulus*, E, remains unchanged.
Experiments (particularly by Cowper and Symonds[3,4]) have shown that the *yield stress*, σ_y, of steel increases as *strain rate*, $\dot{\varepsilon}$, increases. We may call this new value for *yield stress*, the *dynamic yield stress*, σ_y', where:

$$\sigma_y' > \sigma_y$$

This phenomenon implies that structures loaded at high *strain rates* exhibit *linear elastic behaviour* at stress levels above the *yield stress*, σ_y. This in turn implies that *plastic collapse* will occur at σ_y' instead of σ_y.

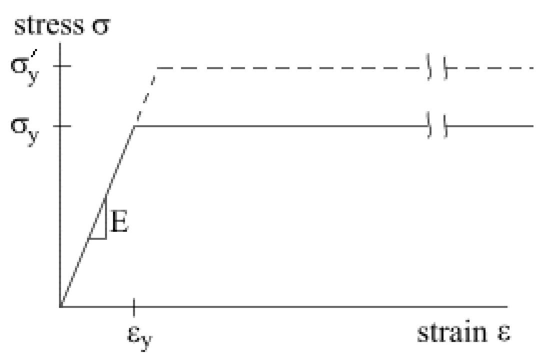


FIGURE 2: *ELASTIC-PERFECTLY PLASTIC CONSTITUTIVE* MODEL ALTERED BY *MATERIAL STRAIN RATE EFFECTS.*

Therefore, for rapidly loaded structures, we may replace the *static yield stress*, σ_y, with the *dynamic yield stress*, σ_y'. So how do we calculate the *dynamic yield stress*?...

---

[3] Symonds, P.S., *Survey of Methods of Analysis for Plastic Deformation of Structures under Dynamic Loadings*, Brown University, Report BU/NSSRDC/1-67 (1967)
[4] Symonds, P.S., Viscoplastic behaviour in response of structures to dynamic loading, in *Behaviour of Materials Under Dynamic* Loading, p. 106, ed. N. J. Huffington, ASME (1965)

The Cowper-Symonds Relationship

Based on experimental data, Cowper and Symonds[5] used the following regression equation to predict the *dynamic yield stress*, σ_y':

$$\sigma_y' = \sigma_y \left[1 + \left(\frac{\dot{\varepsilon}}{C}\right)^{1/p}\right]$$

where: C and p are constants called the *Cowper-Symonds Parameters*

> Note: C has the same units as *strain rate*, i.e. $[1/s]$ (or equivalently $[(mm/mm)/s]$ as sometimes strain is expressed in units of $[mm/mm]$) and p is *dimensionless*.

This relationship has attained almost universal acceptance in the engineering community because analytical and numerical predictions agree remarkably well with actual experimental data (Jones, 1983).
For hot-rolled mild steel, Cowper and Symonds determined that:

$$C = 40.4 \quad \text{and} \quad p = 5$$

Values for other materials are given in the following table[6]:

Material	C [-/s]	p
Stainless Steel 304	100	10
Alpha-Titanium (Ti-50A)	120	9
Aluminum	6500	4

If we compare hot-rolled mild steel to other *material strain rate* sensitive materials, we find that it is particularly sensitive. The figure below shows the *dynamic scale factor*, γ, for these various materials plotted verses *strain rate*. If we notice that the *static yield stress*, σ_y, is simply scaled by the term in brackets then we may say:

$$\frac{\sigma_y'}{\sigma_y} = \gamma$$

where: γ is the *dynamic scale factor* and

$$\gamma = \left[1 + \left(\frac{\dot{\varepsilon}}{C}\right)^{1/p}\right]$$

[5] See the above Symonds references.
[6] Taken from:
Jones, N., Structural crashworthiness, p. 321, ed. N. Jones and T. Wierzbicki, Butterworths (1983)

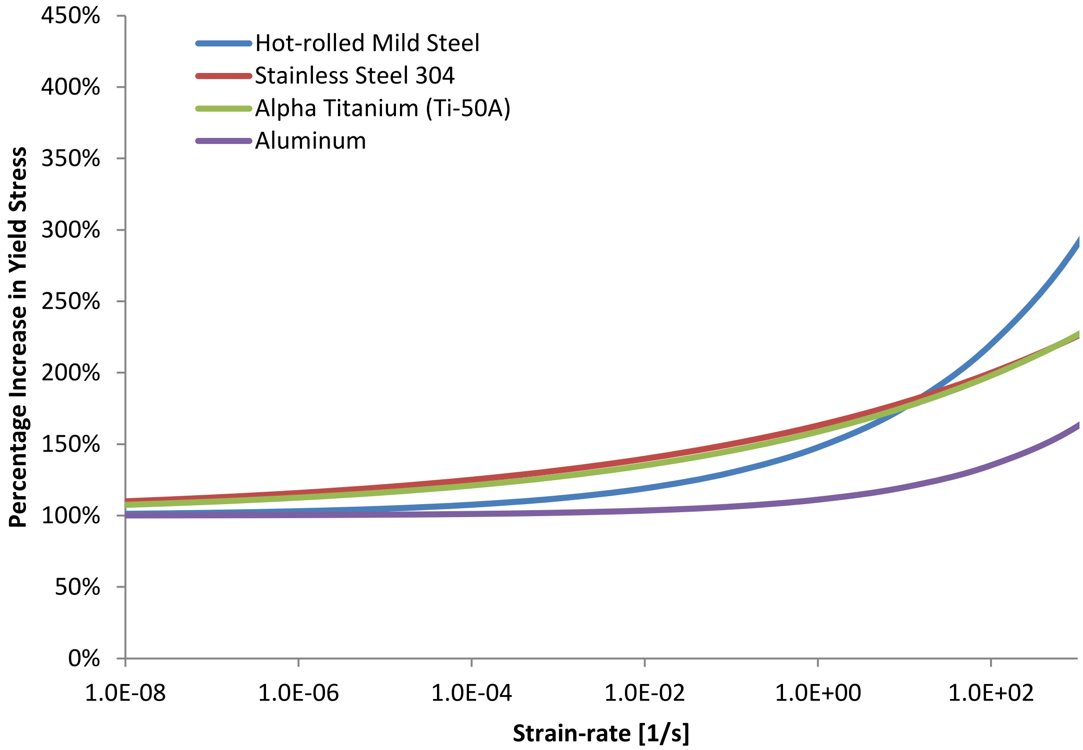


FIGURE 3: COWPER-SYMONDS *DYNAMIC SCALE FACTOR* FOR THE *YIELD STRESS* OF VARIOUS MATERIALS.

Figure 3 shows us that hot-rolled mild steel is very sensitive to *material strain rate effects*, while aluminum is virtually unchanged.

Limits on the Cowper-Symonds Relationship

It should be noted that the above given values for C and p are valid only up to a few percent strain (e.g. $0 < \varepsilon < 0.05$ for hot-rolled mild steel). This is because the *dynamic stress*, σ', is a local maximum at the *dynamic yield stress*, σ_y', and drops off significantly at higher strains for most normal *strain rates*.
Since our *elastic-perfectly plastic constitutive model* assumes that the stress is constant once *yield* is achieved, than for strains higher than say, 5%, we cannot use the above C and p values because they will overestimate the resulting constant stress. We require new values of C and p for strains above, about 5%.
Figure 11.5 below illustrates this concept. Notice for the STATIC curve that all stresses after *yield* are larger than the *perfectly plastic* assumption. The curve above the STATIC curve is for a *strain rate* of 0.02 [-/s] and shows that the stress after the *dynamic yield stress*, σ_y', is significantly less than the *perfectly plastic* assumption.

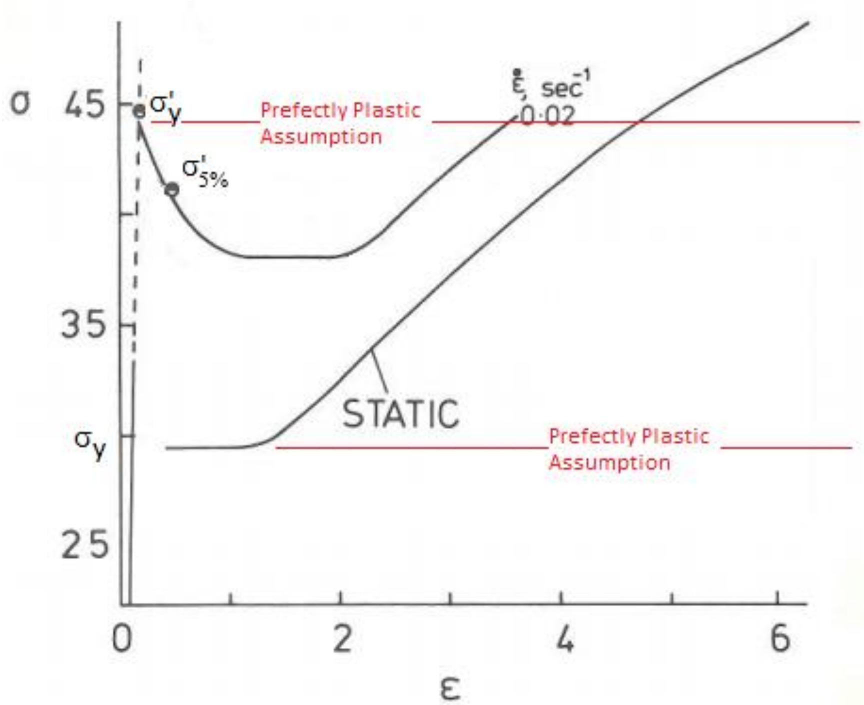


Figure 11.5 Stress–strain curves for mild steel at various strain rates according to Marsh and Campbell[75] (1 unit of ordinate is 10^3 lb in^{-2})

Estimating Strain Rate

Do to non-linear geometry effects, *strain rate* is difficult to determine for a ship's structure; however we may estimate the *average strain rate* by estimating the *dynamic loading time*.

We get the *dynamic loading time* from either experiments or by consulting published data. For example, Giannotti and Associates, Inc.[7] estimate that the collapse of structural members of a container ship occurs within 0.18 seconds. Thus if we assume that the local maximum strain is 0.36, then the *average strain rate* in [-/s] is:

$$\dot{\varepsilon} = \frac{\varepsilon}{t} = \frac{0.36}{0.18} = 2$$

Thus for hot-rolled mild steel:

$$\gamma = \left[1 + \left(\frac{\dot{\varepsilon}}{C}\right)^{1/p}\right] = \left[1 + \left(\frac{2}{40.4}\right)^{1/5}\right] = 1.55$$

and:

$$\sigma_y' = 1.55\sigma_y$$

[7] Giannotti and Associates, Inc., *Determination of Strain Rates in Ship Hull Structures. A Feasibility Study*, Report 81-114 for Ship Structure Committee (1982)

Concluding Remarks

We have seen that we may use the *Cowper-Symonds relationship* to determine the *dynamic yield stress*, σ_y' for structural analyses that involve high *strain rates*. Since the *dynamic yield stress* is greater than the *static yield stress* it is advantageous to use this relationship when applicable, as significant weight savings in structural steel may be possible during ship design.

Further to the above lecture, the following two items should be noted:

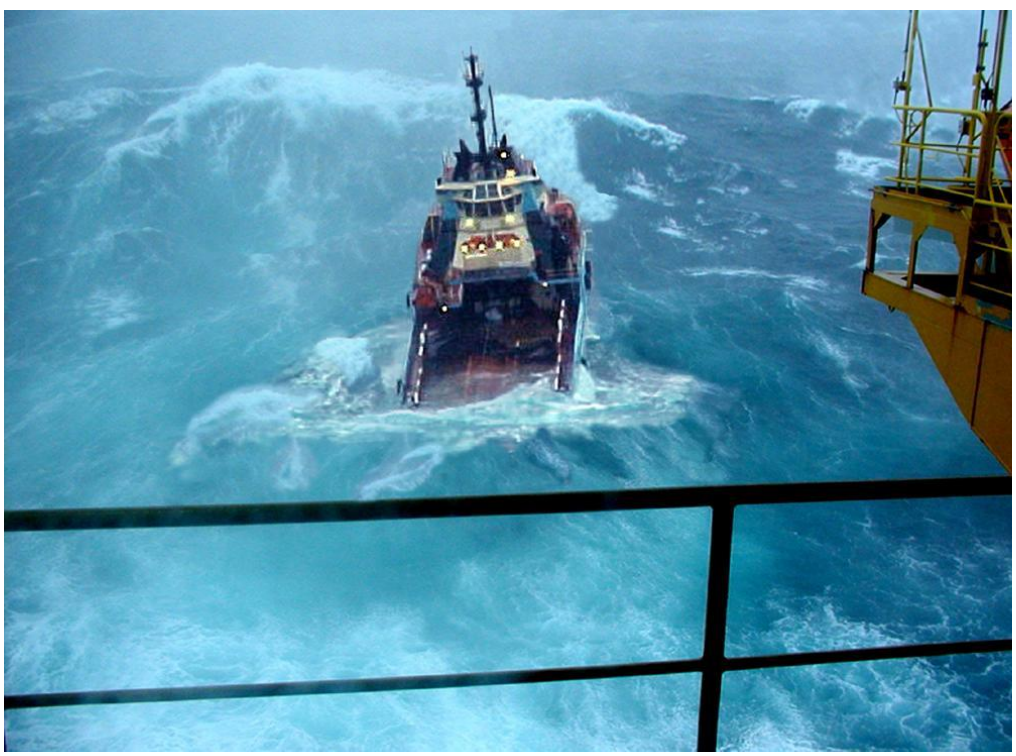
1. The *Cowper-Symonds relationship* may be significant for *quasi-static* loads as the $\left(\frac{\dot{\varepsilon}}{C}\right)$ may be sufficiently large in cases where *inertial effects* can still be ignored.
2. The *Cowper-Symonds relationship* is based on uniaxial data, but its application has been extended to multi-dimensional stress states in practice; with little to no experimental data to support its applicability.

Topic 22 : Dealing with Uncertainty in structural design

Introduction

We live with uncertainty. We build structures to limit that uncertainty. The earliest structures were built to cope with the uncertainties of the weather. Ships must cope with the uncertain sea.

The theory of probability and statistics gives us a way of understanding and coping as best we can, with uncertainty. There are several theories and methods developed to specifically deal with engineering design problems involving uncertainty. These methods have become popular in recent years and are being implemented in many codes and standards, as well as in engineering design and analysis. Not all the developments are sensible. Few professionals can really claim to understand the issues involved, yet many believe it to be the correct approach. Much further development of these methods will likely occur. An understanding of the methods and issues is a crucial element of a professional education in structures.



Somewhere out there

Probability Concepts

ray-traced image by Jeremy Ginsberg and Matt Ginzton June 1998

Uncertainty vs Certainty.

When you roll a single die:

- You are certain to have one of the following: 1,2,3,4,5 or 6
- You are certain not to have 7 or 3.2
- You have an equal chance of getting any of the 6 numbers
- What if the die is flawed? What changes?

Deterministic vs Random.

- Deterministic implies determination by exact natural laws. The outcome is conditionally pre-determined. e.g. drop an object and it accelerates at 9.81 m/s^2
- Random means that the outcome is determined by a fully or partially unknown cause. e.g roll a die.

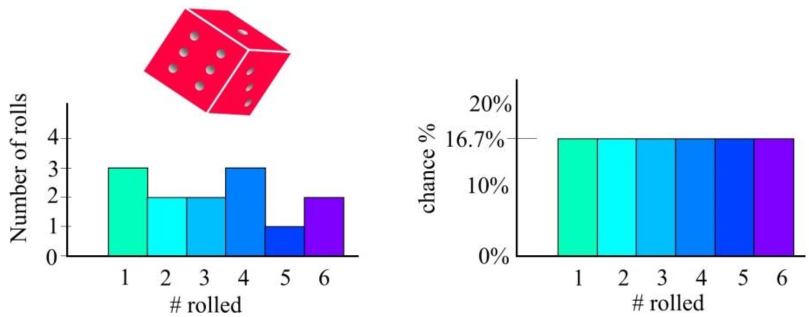
Dynamic vs Static

- Dynamic means varying in time.
- Static means constant in time.
- Probability concepts can be applied to both static (is there oil 6000m below me) and dynamic (what will be the largest wave to strike the Hibernia platform tomorrow?)

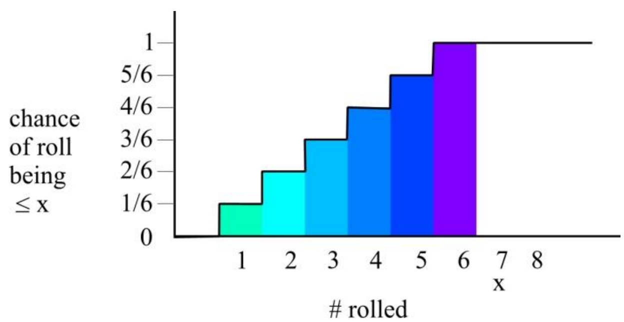
A "Random Variable" means an uncertain dynamic variable, such as the sea surface.

Histogram: plot of variable occurrences

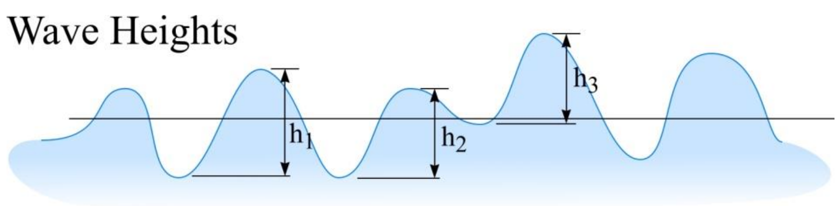
- May be expressed as observation counts.
- May be expressed as % (observed or predicted chance).

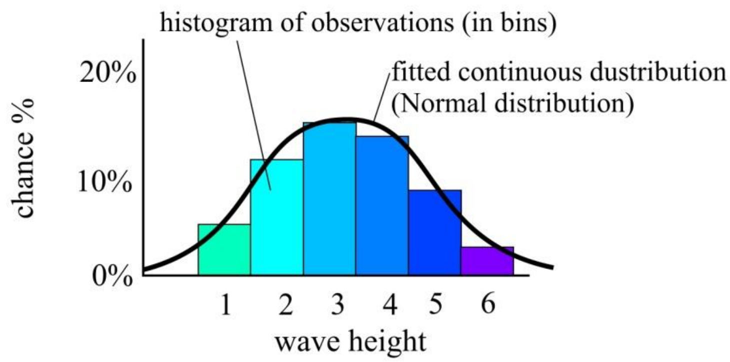


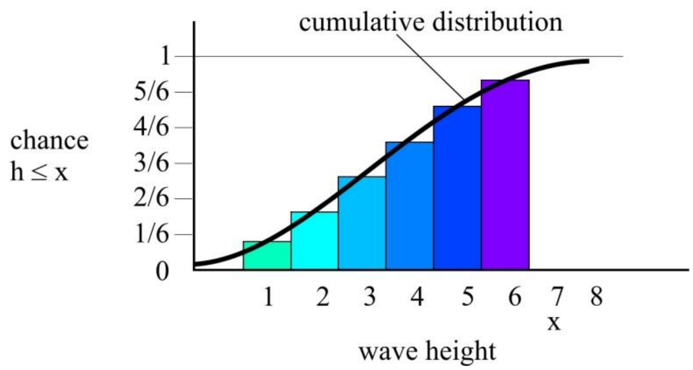
- Cumulative probability for 1 roll of a 'fair' die:



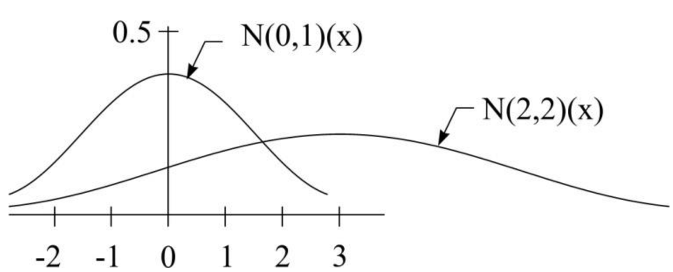
While dice and cards have discrete values (integers), waves have continuous values (real numbers), and can be modeled with smooth distributions:







The most common distribution is the Normal distribution:



- The normal distribution is denoted X=N(μ,σ)(x),

$$N(\mu,\sigma)(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[-1/2\left(\frac{x-\mu}{\sigma}\right)^2\right]}$$

where
μ = the mean = (Σ x_i) / n
σ = standard deviation = ((Σ (x_i - μ)^2 / (n-1))$^{0.5}$
(in excel this is STDEV(n1,n2,n3...))
X is the random variable
x is one specific value (measurement) of X

The "Standard" Normal distribution is N(0,1). All other Normal distributions are just stretched/shifted versions of N(0,1).

We usually want to find the probability that X is in a certain range. (e.g. pr(X<10) or pr(3<X<5). For this the more useful function is the cumulative Normal distribution : Φ(x)

Φ(x) : the probability that the value of X will be less than x.

$$\Phi(\mu,\sigma)(x) = \frac{1}{\sigma\sqrt{2\pi}} \int e^{\left[-1/2\left(\frac{x-\mu}{\sigma}\right)^2\right]} dx$$

There is no closed form solution for this integral. Instead we use tables or built-in functions.
(in excel Φ(μ,σ)(x) is "=NORMDIST(x, μ,σ,1)"

See the handout for the Normal Distribution.

e.g. for Φ (0,1) pr(x<1.52) = .9357
for Φ (2,3) pr(x<0) = 1- Φ (0,1)(2/3) = 1-.748 = 0.252

Discussion : Design for safety:

You are asked to design a ferry in which the chances of dying are "acceptable".

Define "acceptable" in a rigorous, defensible way.

What are your chances of dying?
- today?
- this year?
- if you go on a ferry?
- What about specific risks – e.g. collision with another ship?

Are current ferries safe enough?
Who should say what is safe enough?

The *Herald of Free Enterprise*

Extra Reading:

See pdf extract from **Target Risk** by Gerald J.S. Wilde
See pdf : Personal observations on the **reliability of the Shuttle** by R. P. Feynman

L.2 – Problems.

1. What are the chances of drawing a Queen from a deck of cards?
2. What have you assumed in the previous question?
3. What if you thought that someone had lost some of the cards in a standard deck of playing cards. Then what would you estimate the chances of drawing a Queen be?
4. A rocket booster is expected to crash in the ocean. The potential crash zone is 170 x 80 km. Estimate the chance of the rocket hitting a 200,000 tonne ship, moored in the crash zone?
5. Find: $\Phi(7,4)(10)$, $\Phi(7,4)(4)$, $\Phi(7,4)(-1)$

Standard Normal Probabilities, (mean = 0, SD = 1)

(probability that a value, randomly chosen from X, is less than x)

Pr(X<x)

x	0	1	2	3	4	5	6	7	8	9
0	0.5	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.55567	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.57926	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.60642	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.6293	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705402	0.70884	0.71226	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.77035	0.773373	0.776373	0.77935	0.782305	0.785236
0.8	0.788145	0.79103	0.793892	0.796731	0.799546	0.802338	0.805106	0.80785	0.81057	0.813267
0.9	0.81594	0.818589	0.821214	0.823814	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1	0.841345	0.843752	0.846136	0.848495	0.85083	0.853141	0.855428	0.85769	0.859929	0.862143
1.1	0.864334	0.8665	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881	0.882977
1.2	0.88493	0.88686	0.888767	0.890651	0.892512	0.89435	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914656	0.916207	0.917736
1.4	0.919243	0.92073	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.93822	0.939429	0.94062	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.95254	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.96407	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971284	0.971933	0.972571	0.973197	0.97381	0.974412	0.975002	0.975581	0.976148	0.976705
2	0.97725	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.98983	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.99224	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.99379	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.99506	0.995201
2.6	0.995339	0.995473	0.995603	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.99702	0.99711	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.99825	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3	0.99865	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.99893	0.998965	0.998999
3.1	0.999032	0.999064	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.99955	0.999566	0.999581	0.999596	0.99961	0.999624	0.999638	0.99965
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.99972	0.99973	0.99974	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.9998	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.9999	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.99993	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.99995
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967
4	0.999968	0.99997	0.999971	0.999972	0.999973	0.999974	0.999975	0.999976	0.999977	0.999978

Topic 23: R-Q calculation of the probability of structural failure

Introduction

We must design structures for random loads. In such circumstances there can be no absolute guarantee of safety. The best we can do is to create a structure with a known and acceptable probability of failure.

We define failure as any circumstance in which load>strength.
This would be true for any internal (load effect) as well as any external load.

Both load and strength are random quantities (random variables).

The probability of failure is thus the probability that load exceeds strength. We have a system of two random variables, and failure will occur for every cases in which load>strength.

Typically we use open-tail probability density functions (eg Normal) so that values theoretically go to $\pm\infty$.

The total probability of failure must cover all possible values of load and strength. Failure can occur for any strength level, because it is always possible that the load is even higher.

Somewhere out there

A *Simple* approach to safety

We will derive a mathematical expression for the probability of failure, assuming that we have of a model of the probabilities of both load and strength.

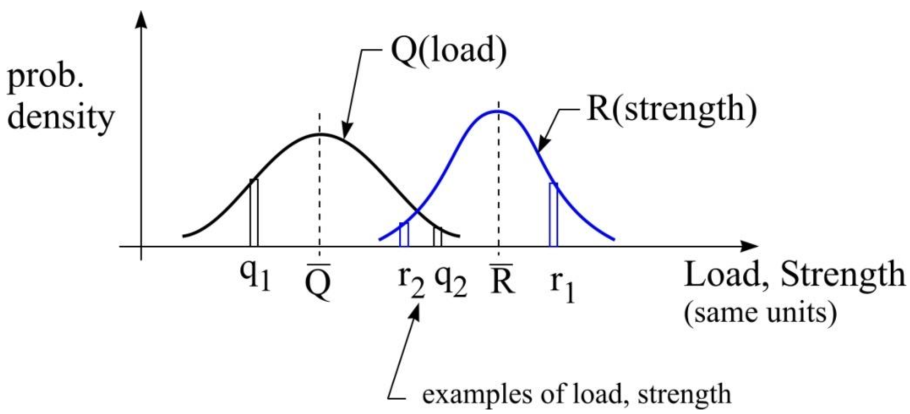
Define:

Q ≡ load (action, demand)
R ≡ strength (re-action, capacity)

q ≡ a specific load (a particular occurrence or example of..)
r ≡ a specific strength

we say that: $q \in Q$, q is a member of Q
and : $r \in R$

we have failure when q>r (we can't say Q>R)



the mean load is $\overline{Q}$

the mean strength is $\overline{R}$

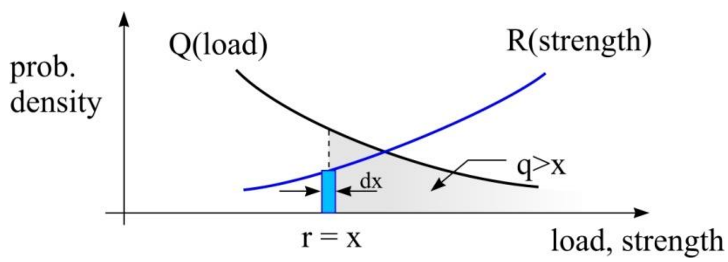
$\overline{Q} < \overline{R}$ so 'on average' we are safe,

but if we choose examples:
$q_1 < r_1$ safe
$q_2 > r_2$ failure !

note : the factor of safety *on the mean*, FOS_m, is $\dfrac{\overline{R}}{\overline{Q}}$

unfortunately the FOS_m has little relation to the probability of failure.

To determine to probability of failure, we need to consider all possible circumstances. We start by picking either load or strength (lets pick strength – but it doesn't matter which). For each possible value (of strength), we calculate the chance of failure. Once we've done this for all possible values (of strength), we add up the partial chances to get the total probability of failure. It would give the same answer if we picked load. The calculation can be visualized with the following plot of Q, R, q, r:



all possible values of load and strength are values along the x axis. The chances of either of these values are given by the Q(x) or R(x) probability density curves. The chances of r falling in the small range 'dx' at x is:

Prob(r=x) = R(x) dx (remember R is probability density)

for r=x, the (conditional) probability of failure is the same as the chance that q>x. We can express this as:

Prob(q>x) = area under Q curve to the right of x

$= \int_x^\infty Q(x)dx$ (this is the shaded area)

Note: for two independent things, A,B, to happen together, the chance is Prob(A and B) = Prob(A) x Prob(B)

Therefore the chance of failure, when r=x is:

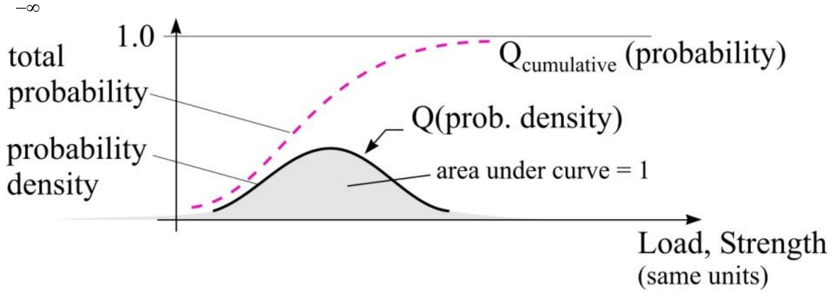
$\text{Prob(fail)}|_{r=x} = R(x)\, dx \int_x^\infty Q(x)dx$

The total probability of failure must include all possible values of r. We need to sum the above probabilities for all r. This is written as:

$$\text{Prob(fail)} = \int_{-\infty}^{\infty} \left[R(x)\, dx \int_x^\infty Q(x)dx \right]$$

This integral of an integral is called a convolution integral. However, we can simplify this to a simple integral.

$$Q_{cumulative}(x) = \int_{-\infty}^{x} Q(x)dx$$



therefore

$$\int_{x}^{\infty} Q(x)dx = 1 - \int_{-\infty}^{x} Q(x)dx$$
$$= 1 - Q_{cumulative}(x)$$

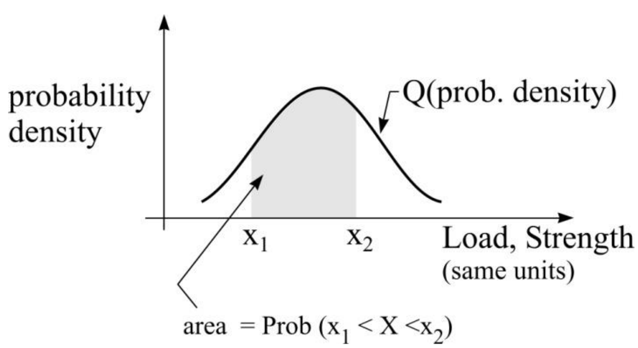
The total probability of failure is:

$$\text{Prob(fail)} = \int_{all\ x} R(x) \cdot (1 - Q_{cumulative}(x))dx\,]$$

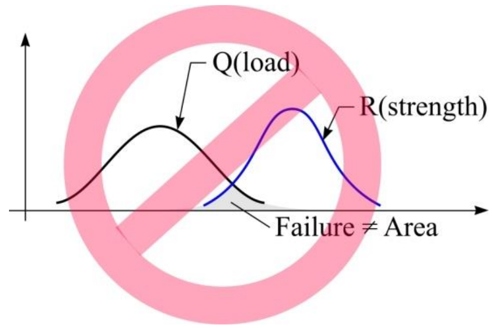
A word about geometry ----------------

It is always nice to see an idea geometrically. In this case there are a couple of geometric analogs to the probabilities.

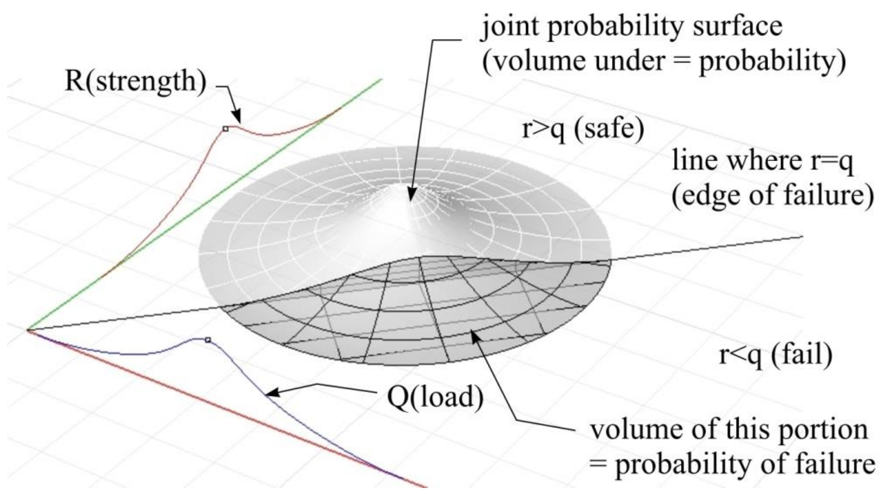
In the case of a probability density function, the probability of having a value between any two points (on the x axis) is equal to the area between those points.



In the case of the failure probability, the value is determined by combining all possible probabilities for R and Q. A common misunderstanding is that this is equal to the area of the overlap between the two curves. This is NOT true.



However, there is a geometric analog to the total probability of failure. We plot R on one horizontal axis, Q on another, and probability density on the vertical axis. In this space, the probabilities of all combinations of R and Q form a surface (like a Mexican hat). The volume under the whole surface is 1. A line R=Q forms a diagonal line on the horizontal plane. One side of this line represents Q>R and includes all the combinations that result in failure. The failure probability is the volume under the surface in this region.



Summary

We have calculated the exact probability of failure. To do so we used the probability density functions for both load and strength. For simple cases of one type of load and one failure mechanism, we need good data for both, so that we could fit good probability distributions. For cases of multiple types of load or multiple failure mechanisms, we would need to combine these factors in some way. This gets complicated in practice.

Exercise 1: do on your own.

Recall that when we derived the failure equation, we said that we could start with either R or Q, but we started with R and we got:

Prob(fail) = $\int_{all\ x} R(x) \cdot (1 - Q_{cumulative}(x))dx\,]$

If we had started with Q, which of the following formulas would we have derived?

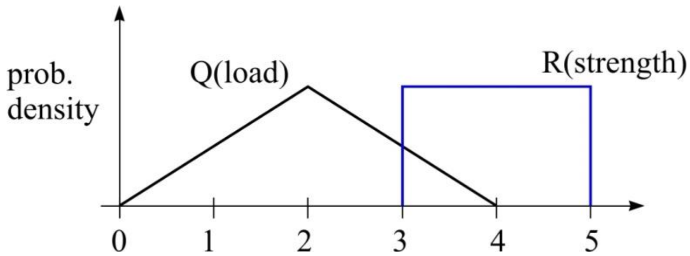
a) Prob(fail) = $\int_{all\ x} Q(x) \cdot (1 - R_{cumulative}(x))dx\,]$

b) Prob(fail) = $\int_{all\ x} Q(x) \cdot R_{cumulative}(x)dx\,]$

c) Prob(fail) = $\int_{all\ x} R_{cumulative}(x) \cdot (1 - Q(x))dx\,]$

Example: Calculation of Failure Probability

For these load and strength distributions, calculate the exact probability of failure;



If a failure takes place, it must occur with load and strength in the 3→4 range. We will use the equation

$$\text{Prob(fail)} = \int_{all\ x} R(x)\cdot(1-Q_{cumulative}(x))dx$$

But before we do, lets estimate (guess) what the answer should be.
1/2 of all R values are in the 3→4 range
1/8 of all Q values are in the 3→4 range.

Failure happens when both Q and R are in the 3→4 range. The chance of this is:
P_fail = P(Q ∈ 3→4 AND R ∈ 3→4) = P (Q ∈ 3→4) x P (R ∈ 3→4)
= 1/2 x 1/8 = 1/16

But even in the 3→4 range, r could be greater than q. A quick estimate is that 1/2 of the time we have r>q, so

P_fail = 1/2 x 1/16 = 1/32

But most of the Q values are in the lower part of the range, so lets guess that in the 3→4 range P(R<Q) = 1/3. This gives us an estimate of:

P_fail = 1/3 x 1/16 = 1/48 ⇐ our best guess

Now lets calculate the value exactly.

First we need to find the height of the triangle and rectangle. Both have area of 1, so, by inspection, both have a height of 0.5.
In the range x ∈ 3→4
R(x) = 0.5 (constant)
Q(x)= 1-.25x (check @x=4 Q=0, and @x=2, Q=0.5 ⇐ OK)

We need to get a mathematical expression for $Q_{cumulative}$. Note that there are two branches for this. We must know the area up to x=2, as this will be the constant in the integral equation for the x ∈ 2→4. It is obvious that the area up to x=2 is 0.5. Therefore:

$$Q_{cumulative} = 0.5 + \int_2^x Q(x)dx = 0.5 + \int_2^x (1-.25x)dx = -1 + x - x^2/8$$

(check, $Q_{cumulative}$@ 4 = -1+4-16/8 = 1 ⇐ OK)

Prob(fail) = $\int_{all\ x} R(x)\cdot(1-Q_{cumulative}(x))dx = \int_3^4 0.5\cdot(2-x+x^2/8)dx$

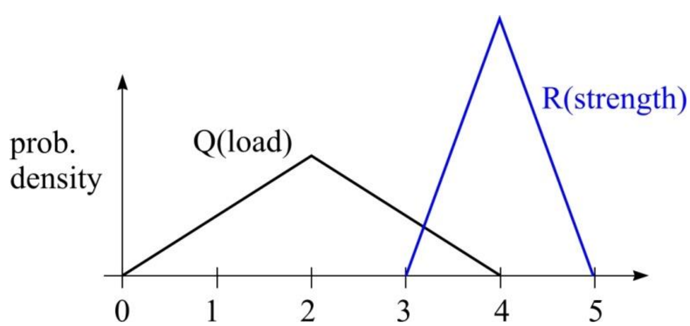
= 0.02083 (=1/48)

So our 2nd guess was correct!

Note that the area of overlap was 1/8, so obviously the area of overlap is not the failure probability.

L.3 – Problems.

1. For these load and strength distributions, calculate the exact probability of failure;



Topic 24 : Engineering Safety Metrics

Introduction

To implement the concepts of probability into engineering design, without making the process too complex, we need to extract simple metrics from the statistical data. In this lecture we will define the **Margin of Safety**, and contrast this with the **Factor of Safety**. Next we will define what we mean by **Characteristic Values** and show how the use of these is related to safety.

Margin of Safety

The margin is the reserve (in units of load) between the load and the strength:

Margin = Strength - Load

M=R-Q

It is easy to see that failure will occur if the margin is negative.

M<0 = Failure

Contrast this with Factor of Safety FOS, which is the % reserve:

FOS = Strength/Load = R/Q
FOS<1 = Failure

The Margin of Safety is more useful when trying to determine the probability of failure. The reason has to do with one of the properties of the Normal distribution. When variables with Normal distributions are added (or subtracted), the result is also normal. For example:

C = A + B, with A, B Normal

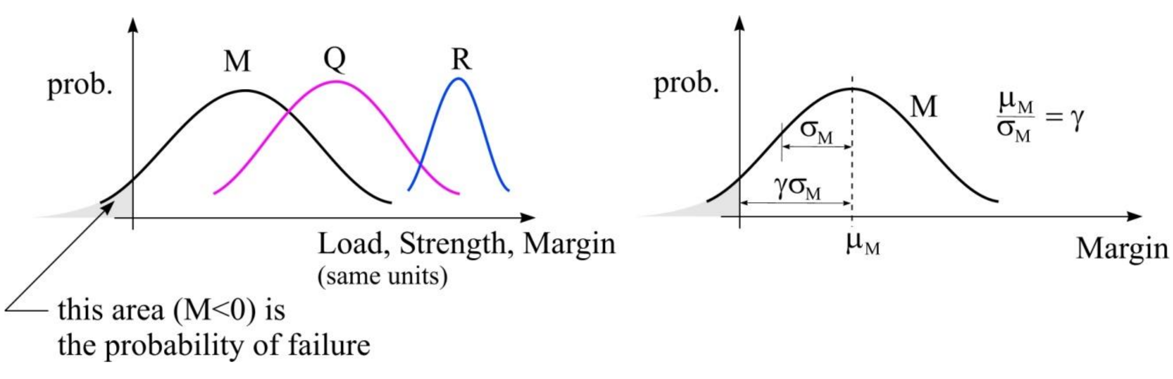
then C is Normal with:
mean(C) = mean(A) + mean(B)
$(\text{stdDev}(C))^2 = (\text{stdDev}(A))^2+(\text{stdDev}(B))^2$

For Strength R and load Q, Normally distributed, with :
Mean strength = μ_R Mean load = μ_Q
std. dev of strength = σ_R std. dev of load = σ_Q

M=R-Q
M will also be Normally distributed with:
$\mu_M = \mu_R - \mu_Q$

$$\sigma_M = \sqrt{\sigma_R^2 + \sigma_Q^2}$$

The probability of failure is the probability that M<0. As M is Normal, it is easy to find the Prob(M<0).



We want to find the probability that M<0. We know μ_M and σ_M
We can define γ as:

$\gamma = \mu_M / \sigma_M$

This term gamma γ, is the number of standard deviations that the mean of the margin is above zero. This sounds awkward, but it is quite simple and useful. The standard Normal table lets us find the probability of a value being below *n* standard deviations above the mean. We can use this to find the faiulre probability as follows:

Probability of Failure = 1- NC(γ)
where NC is the standard Normal Cumulative function (see Table)

e.g. if γ = 1.5, Prob. Failure = 1-0.933193 = 0.066807 = 6.7%

There is always a unique relationship between g and the probability of failure. This is not true for the Factor of Safety (why?).

We call γ the **Safety Index**.

Note : the term 'coefficient of variation' or COV for a random variable A is defined as: COV(A) = σ_A / μ_A

We can see that the safety index γ = 1/COV(Margin)

Characteristic Values

We often describe certain things with values that are actually not averages, but special or rare values. Examples include, say, water depth in a lake, or heights of a mountain range. We might describe a lake as being 'deep' because there are spots with over 100ft depth. The actual average depth may be only 20 feet, with the single deepest spot may be 120 ft. Yet we might say the lake is 100feet deep. Similarly we might say that the Rocky Mountains are over 10,000 ft, when the actual average elevation is 3000ft (within the overall territory called the Rocky Mountains). What we are doing focusing on a typical rare (high) value, but not the most rare (highest) value.

This approach is very common in engineering. When we say that steel has a yield strength of 300MPa, we don't mean that the average strength is 300MPa, we mean that no more than 5% of the tested values fall below 300MPa. This is because if more than 5% are below, the steel would fail to be rated at 300MPa. There are cases where none of the samples would be below 300, possibly with a minimum test at 320MPa, but yet the steel is rated as 300MPa.

We define values such as the 5% value as Characteristic Values. We might be more specific and call this the 5% characteristic value. In naval architecture we use the term 'significant wave height' which we might call Hs. Hs, is approximately equal to the average of the highest one-third of the waves. This is a 'characteristic value', at about the 10% level (est.).

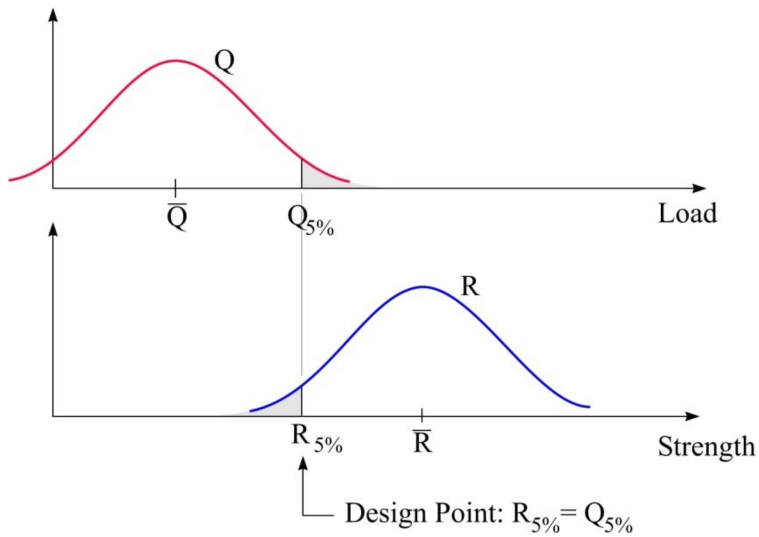
Note: Hs is calculated precisely using:

$$Hs = 4.0 * \mathrm{sqrt}(m0)$$

where m0 is the variance of the wave displacement time series acquired during the wave acquisition period.

The "Design Point'

Typically we define load and strength with characteristic values. For strength we use a rare low value, and for load a rare high value. In the sketch below we have used the 5% value of each. Further we have defined a design point at which the two characteristic values are equal.



You might wonder how we can set the 5% strength to equal the 5% load. Usually the load is defined by nature, though it may be set as a design decision (e.g. allow 500 lb/ft2 of equipment to be placed on the floor of a room). Strength is an outcome of the design process, so that once we know the 5% load (the design point) we can adjust the size (and material) of the structure to give a strength such that the 5% characteristic value is at the design point.

Example: for the long plate sketched below, the value p_c is calculated using average values of t and L and the 5% value for σ_y . Consequently, Pc is the 5% characteristic value of strength (i.e. $R_{5\%}$)

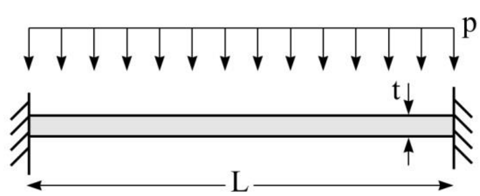


plate capacity at 3 hinge : $p_c = 4.5\,\sigma_y\left(\frac{t}{L}\right)^2$

σ_y = 5% characteristic value

so $R_{5\%} = p_c = 4.5\,\sigma_y\left(\frac{t}{L}\right)^2$

When we use characteristic values, the probability of failure can be found. For example, in the case where $Q_{5\%} = R_{5\%}$ we express the characteristic values as the mean plus 1.645 standard deviations (see the N(0,1) table):

$$Q_{5\%} = \mu_Q + 1.645 \cdot \sigma_Q$$
$$R_{5\%} = \mu_R - 1.645 \cdot \sigma_R$$

which lets us write:

$$1.645 \cdot \left(\sigma_Q + \sigma_R\right) = \mu_R - \mu_Q = \mu_M$$

We need to find σ_M , but to do so we need to combine σ_Q and σ_R .

Typically we have more variation in Q than in R, so lets assume that $\sigma_Q = 2 \cdot \sigma_R$. This is used to give;

$$\mu_M = 1.645 \cdot \left(3 \cdot \sigma_R\right)$$
and
$$\sigma_M = \sqrt{4\sigma_R^{\ 2} + \sigma_R^{\ 2}} = \sqrt{5} \cdot \sigma_R$$

consequently;

$$\gamma = \frac{\mu_M}{\sigma_M} = \frac{1.645 \cdot 3 \cdot \sigma_R}{\sqrt{5} \cdot \sigma_R} = 2.21$$

from which we get;

Probability of failure = 1-N(0,1)(2.21) = 0.0136 = 1.36%

The number would be a bit different if we had chosen a different ratio for σ_Q/σ_R. In a 'rule' development exercise, the rule developers can look at the typical ratios for σ_Q/σ_R and specify the appropriate characteristic values to achieve the desired target probability of failure. Use of characteristic values allows for the consideration of variability in a way that is easy to formulate as a rule, and will look similar to current rule formats. They are one aspect of a formal Limit States Design approach.

Exercise: Find Prob(Failure) for $R_{2\%}$, $Q_{1\%}$ with $\sigma_Q/\sigma_R = 3$.

Topic 25 : Partial Safety Factors

or – dealing with the unexpected!

Introduction

If all the randomness in the world could be measured and quantified, we wouldn't need anything other than a good model of the distribution of Q&R (load & strength). Unfortunately, a lot of uncertainty can not be readily measured. Some of it hasn't even been thought of. As a result of this, we define two broad categories of uncertainty. One is called **statistical uncertainty**. This is the type of uncertainty that we can measure and quantify, including things like steel strength, wave heights and the size of welding distortions.

The other type of uncertainty is called **approximational uncertainty**. This is the type of uncertainty that we haven't measured. Quite often we can't practically measure this type. One example is called **model uncertainty**. This is the uncertainty in our theoretical models (i.e. what if our understanding of the world is wrong?). Another type of approximation uncertainty covers unmeasured variables. There is an almost endless list of factors that might lead to an accident, but haven't been formally studied and measured (i.e what % of people are able to execute a successful iceberg collision avoidance maneuver, on the first actual attempt?). We have a name for all the things we don't know, and we do have a standard method of dealing with this **approximational uncertainty.** We use partial safety factors.

Why do we call these "partial" safety factors. The reason is that we have several of them, to cover different concerns, and the partial factors combine to give a total factor of safety. Lets call these partial safety factors:

γ_{S1} , γ_{S2} , γ_Q , γ_R

and the total factor of safety:

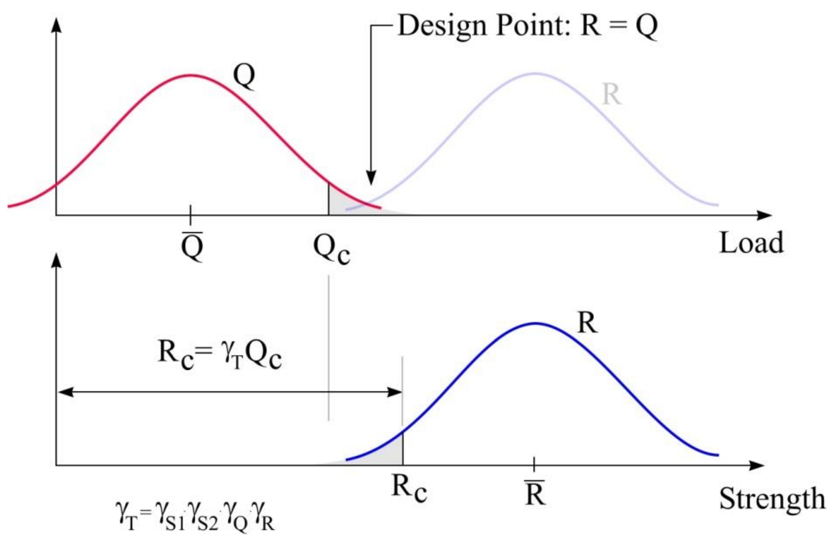
γ_T

where:

$$\gamma_T = \gamma_{S1}\, \gamma_{S2}\, \gamma_Q\, \gamma_R$$

Once we have γ_T we use it to increase the strength so that Rc (the characteristic value of R) is above the design point. Without partial safety factors, we would set Rc=Qc at the design point. With partial safety factors we set:

$Rc = \gamma_T\, Qc$



Now for important point #1: These partial safety factors do help our situation, and do help safety, but we can not say by how much. In part we add them to account for things that we can't quantify, so we can only modify our calculated probability of failure (as we found when we set Rc=Qc), for some of the partial safety factors.

Typical Partial Safety Factors

Different codes use different specific partial safety factors. The four specified below are typical of the kinds of issues covered. Each structural component may have a different set of partial safety factors. The four factors are:

γ_{S1} – this factor reflects the degree of seriousness of the items failure in terms of safety (of people or environment, e.g. hull rupture, fire.)

γ_{S2} – this factor reflects the degree of seriousness of the items failure in terms of serviceability (operability, usefulness, e.g. crane damage, loss of processing equipment)

γ_{Q} – this factor reflects our confidence in the loads (e.g. weights are relatively certain, wave loads are less certain, ice loads are not certain.)

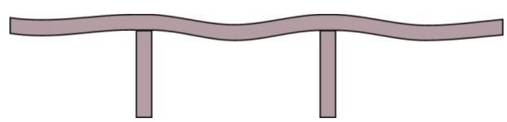
γ_{R} – this factor reflects our confidence in the strength (easy to predict plastic hinge, harder to predict bucking, very had to predict structural aging effects)

Examples for component failures:

Hull plate plastic set

safety: no safety concerns $\gamma_{S1}=1.0$

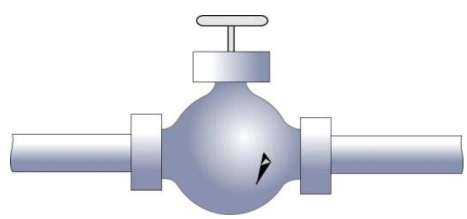
service: minor speed loss $\gamma_{S2}=1.02$



Oil Valve Rupture

safety: possible major fire $\gamma_{S1}=1.4$

service: delay to clean mess $\gamma_{S2}=1.02$



In codes that use partial safety factors, various components and structures would be classed as being more or less important, or more or less uncertain. The example below is from a Danish paper describing the partial safety factors in the Danish Structural Codes.

> The partial safety factor γ_m is determined by $\gamma_m = \gamma_0\, \gamma_1\, \gamma_2\, \gamma_3\, \gamma_4\, \gamma_5$ where γ_0 takes into account the consequences of failure (safety class: low: 0.9, normal: 1.0 and high: 1.1), γ_1 takes into account the type of failure (ductile with reserve: 0.9, without reserve: 1.0 and brittle: 1.1), γ_2 takes into account the possibility of unfavorable differences from the characteristic value of the material parameter (uncertainty), γ_3 takes into account the uncertainty in the computational model (good: 0.95, normal: 1.0 and bad: 1.1), γ_4 takes into account the uncertainty in connection with determination of the material parameter in the structure on the basis of the controlled material parameter (large: 0.95, average: 1.0 and small: 1.1) and γ_5 takes into account the amount of control (extended: 0.95, normal: 1.0 and reduced: 1.1).
>
> *from* : "Calibration of Partial Safety Factors and Target Reliability Level in Danish Structural Codes" by J. D. SØRENSEN, S. O. HANSEN, T. A. NIELSEN

Summary

Partial safety factors provide a way to raise the structural strength for those cases in which a rational person would have increased concerns (i.e. cases where the designer is more worried about either the design parameters, or the consequences of failure).

Pros and Cons

There are many good reasons for taking a partial safety factor approach to structural design. The most obvious benefit comes when we compare two different designs built to the same code. **Partial safety factors let us express our reasonable concerns about uncertainty and consequences**. We will, hopefully, achieve a consistent level of safety, for both critical structures and for relatively unimportant components.

Nevertheless, there are some serious flaws with the idea of partial safety factors. Designers should be sensitive to the limitations of the approach, as they are always ultimately responsible for the design.

An obvious flaw with partials safety factors is **the conceit that we can quantify what we don't know**. This suggests that knowledge is incremental, such that our current knowledge is a large % of the truth, with some small % of uncertainty. Unfortunately, the truth may be a long way from our current understandings. The pretense that we can use statistics to quantify and, in effect, sanitize this type of 'approximational uncertainty' is quite irrational. Only by studying the issue and removing our ignorance can we properly deal this approximational uncertainty. When we don't know something we should just state our assumptions and admit the potential for error.

Another problem has to do with the way probability models commonly work. Most are essentially linear, in that they assume that different effects are independent and add. The Normal distribution was derived on the basis of addition of small independent errors. Reality can often be quite different. We still have little idea of how large systems behave. It is certain that most systems are non-linear and are full of interacting (non-independent) components. **We use the models we have, because we have nothing better, not because they are correct**.

The most obvious problem with application of a partial safety factor approach is the assumption that more (specified) strength will result in greater safety. **Most major failures involve some sort of gross error**, often some human error. Gross errors, such as not following the approved welding standard, or leaving out a crucial structural member, or driving a vessel onto the rocks, **can never be accounted for with small factors**. Safety is a multi-dimensional thing. The people involved at all stages must be trained, alert and cautious. The design must have layers of independent backup systems. If the designers are using a code that claims to be capable of producing an extremely safe design (e.g. 10^{-6} failure per year per structure), there may be a tendency for the designers to become overly confident that all will be well. Such target levels of safety are common in probability-based codes, and are overly optimistic. This is particularly true with offshore structures. The operational experience with many designs is quite limited, and the actual failure rate for offshore structures in general is very high.

The following is taken from the abstract of a Norwegian paper, calling for caution in the area of offshore structures:

> The purpose of this paper is to review the worldwide historical structural failure data in the 1990s on offshore structures, and compare this with the present risk analyses of Norwegian offshore structures.
>
> The paper describes an overview of registered accidents to offshore structures based on the databases WOAD and CODAM. The accident data is given for fixed platforms, jack-ups and for floating platforms. Estimates of risk level in annual frequencies and PLL values are given for each platform type.
>
> The paper concludes that:
>
> - The risk connected to marine operations and structures give a significant contribution to the total risk.
> - The historical risk to marine operations and structures is significant higher than the results from risk analyses.

- Neither component nor system based reliability analyses of structures give adequate descriptions of the real risk connected to structures.
- Human errors are probably the dominating cause of accidents connected to structural failure.

from “On the Risk of Structural failure on Norwegian Offshore Installations” By A. Kvitrud, G. Ersdal and R.L. Leonhardsen, ISOPE 2001

Topic 26 : Risk Analysis - trying to think of everything!

Introduction

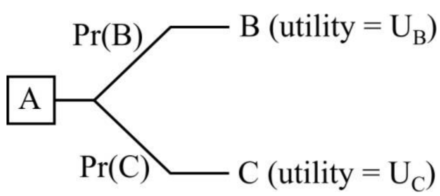
Until now we have talked about the relatively simple question of whether load exceeds strength. Accidents most often occur as a complex sequence of events. Further, there are many possible unwanted events.
We can say that risk is the combination of probability and consequences. More accurately we could say that risk is the combination of probability, consequences and context. The addition of context will be discussed after we discuss the probability/consequences.

We would like to use risk analysis to make decisions. We are trying to make intelligent forecasts, so that we can make the **best decisions**. It may seem odd, but we must start with a good definition of the word 'best'. In a complex and multi-dimensional world, how do we compare apples and oranges? We need a way to compare outcomes, so that we can say which one is 'best'.

Utility and Expected Value

The usual way to compare outcomes is to place a value, either in terms of 'utility' or money, on all possible outcomes. Utility is an economist's term meaning anything of value, whether actual or perceived, such that at least some human being will want it. It includes things like 'satisfaction' and 'reputation', as well as things with a market value.

Once we know the value of each outcome, we can calculate the 'expected value' of the utility, as follows;
Given a choice A, we determine that there is a Pr(B) chance of outcome B, with a utility of U_B . There is also a Pr(C) chance of outcome C, with a utility of U_C.



The **expected value** of utility of choice A is therefore;

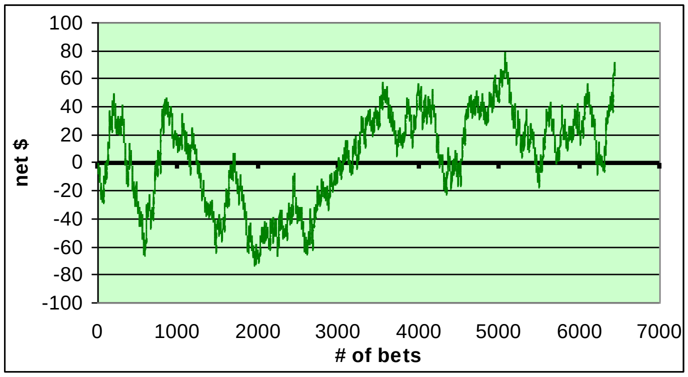
$EV(U_A) = Pr(B) \times U_B + Pr(C) \times U_C$

The actual utility if we take choice A will be U_B or U_C, depending on whether B and or C actually occur. It is important to understand the idea of expected value. To further illustrate, let's take an example from gambling.

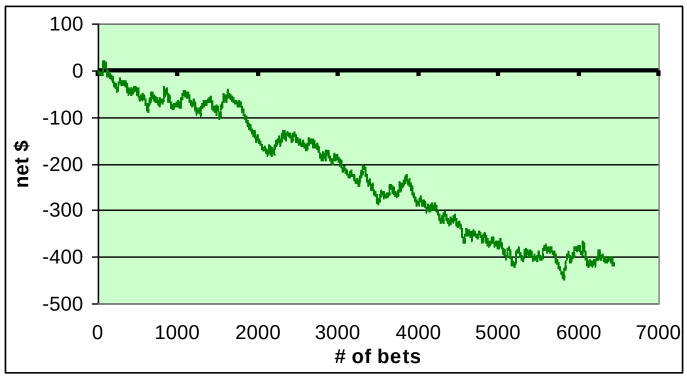
Case 1: you and 5 friends each place $1 on the table. Each or you are sitting in a numbered seat (1 to 6). Someone rolls a single die. The person sitting at the seat with the # on the die gets the $6. If the odds are fair, the expected value of the payout for any individual is 1/6x$6 = $1. But each person paid $1, so the expected net winnings is 1/6x$6 - $100 = 0$.

Universal truth #1: your expected net value when gambling with honest friends is $0. All the money stays with the group! That does not mean that after a long night of gambling you will go home with all the money you came with. See the plots below.

These plots show the cumulative winnings (or loses) that might occur for one of the friends.



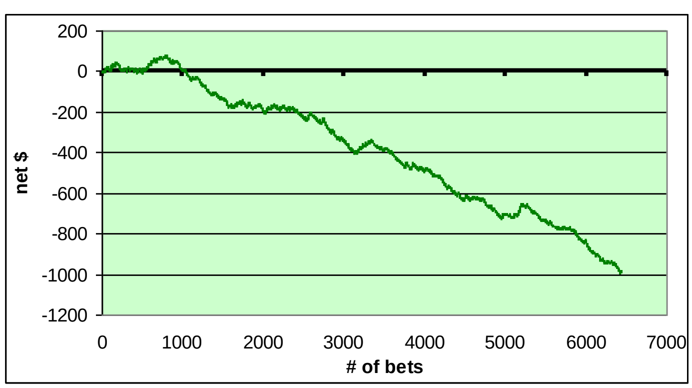
things might average out, or…



they might not!

Case 2: you place a $1 bet in a typical game in Las Vagas. You may win $6 if you win, but the odds of winning are 1/7. The expected value of the winnings is 1/7 x $6 = $0.86. But your expected net winnings is 1/7x$6 - $1 = -$0.14

A 2nd universal truth: your expected net value when gambling in Las Vegas is negative. If you gamble long enough, you will surely lose your money! Below, the plot shows the outcome of a series of 1$ bets, where each bet had a 1/7 chance of winning $6. The gambler was ahead for a while, but after about 1000 bets, all was downhill. One would expect that after 6500 bets, our gambler would be at a net value of 6500 x –$0.14 = -$910, quite close to what happened!



A word about Context

When calculating expected value, we are implicitly assuming that 'a buck is a buck', meaning that winning a dollar would produce a positive effect that would just balance the negative effect of loosing a dollar. If such were true, we would be happy to risk loosing one year the same as we might win the next. This is only true for people betting small amounts (small relative to their wealth). For most of us, the loss of $10,000 hurts far more than the positive effect of winning $10,000. The loss of $10,000 threatens our well being by far more than $10,000 would add to our well being. This winning and loosing is not symmetrical.

E.g. We will not be happy betting $1Million even if we would win $2.2 Million on a coin toss. (expected value is: EV=.5 x 2.2 – 1 = + $0.1M). We'd be foolish not to take such odds if the bet was $1.00, because we could afford to loose. We could not afford to loose the $1M.

The size of the outcome is part of the context that we need to consider. There are many other issues where two things may seem to be equal, but are not, or are asymmetrical. The death of an employee should have the same consequences regardless of context, but this is not the case. Context is very important in determining the utility of various outcomes.

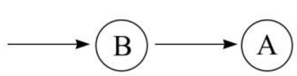
Conditional Probability

Before we formulate a risk model we need one more concept. We need a way to combine various events. We need the idea of '**conditional probability**'. The condition is the circumstance for which we want the probability. The probability of winning the season is not the same as the probability of winning the finals. We need to know the probability of getting through the semi-finals and the probability of winning the final game. we say:

Prob(A) = Prob(B) x Prob(A | B)

read Prob(A | B) as probability of A given B.

Event Probabilities:



for sequential events:

Prob(A) = Prob(B) x Prob(A|B)

eg:
Prob(Win Season) =
Prob(Win Semis) x Prob(Win Final | Enter)

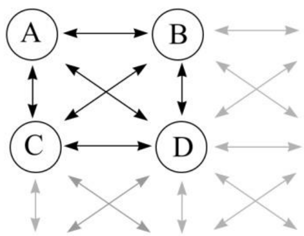
Events can have one or several precursors:

in general:

$$\text{Prob(A)} = \sum_{all\ \ i} \text{Prob(i) x Prob(A|i)}$$

We need this for a variety of probability models. In a “State” based probability model, can (may) move from any state to any other state.

State Probabilities:



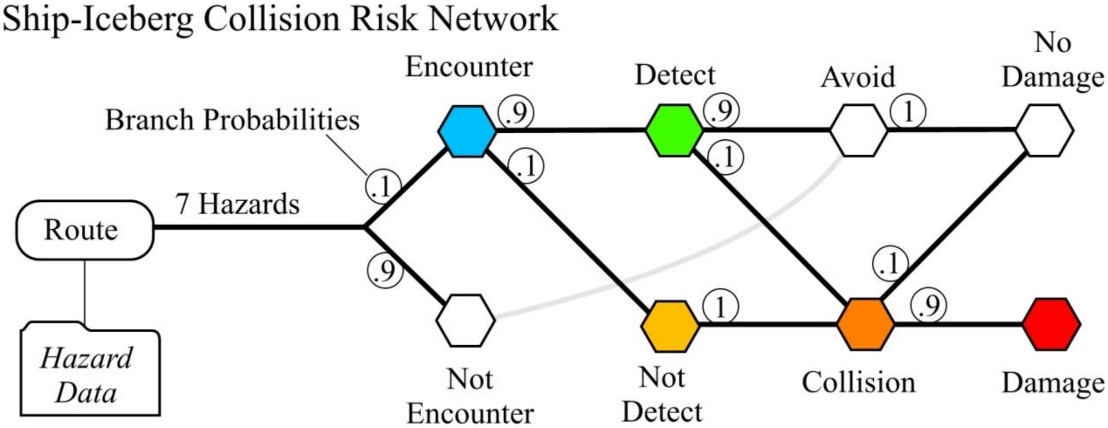
for state space:

Prob(A) = Prob(B) x Prob(A|B)
+ Prob(C) x Prob(A|C)
+ Prob(D) x Prob(A|D)

eg:
State A: Abandon Ship
State B: Bottom Damage, sinking on even keel
State C: Loss of Roll Stability, cargo shifted
State D: Loss of Steering Gear

Now back to risk models. All risk models aim at calculating the expected value of utility for the choices and random events considered. Lets look at two examples. In the first example we want to estimate the chance of structural damage to a ship traveling along a certain route. The **event tree** is :

Ship-Iceberg Collision Risk Network

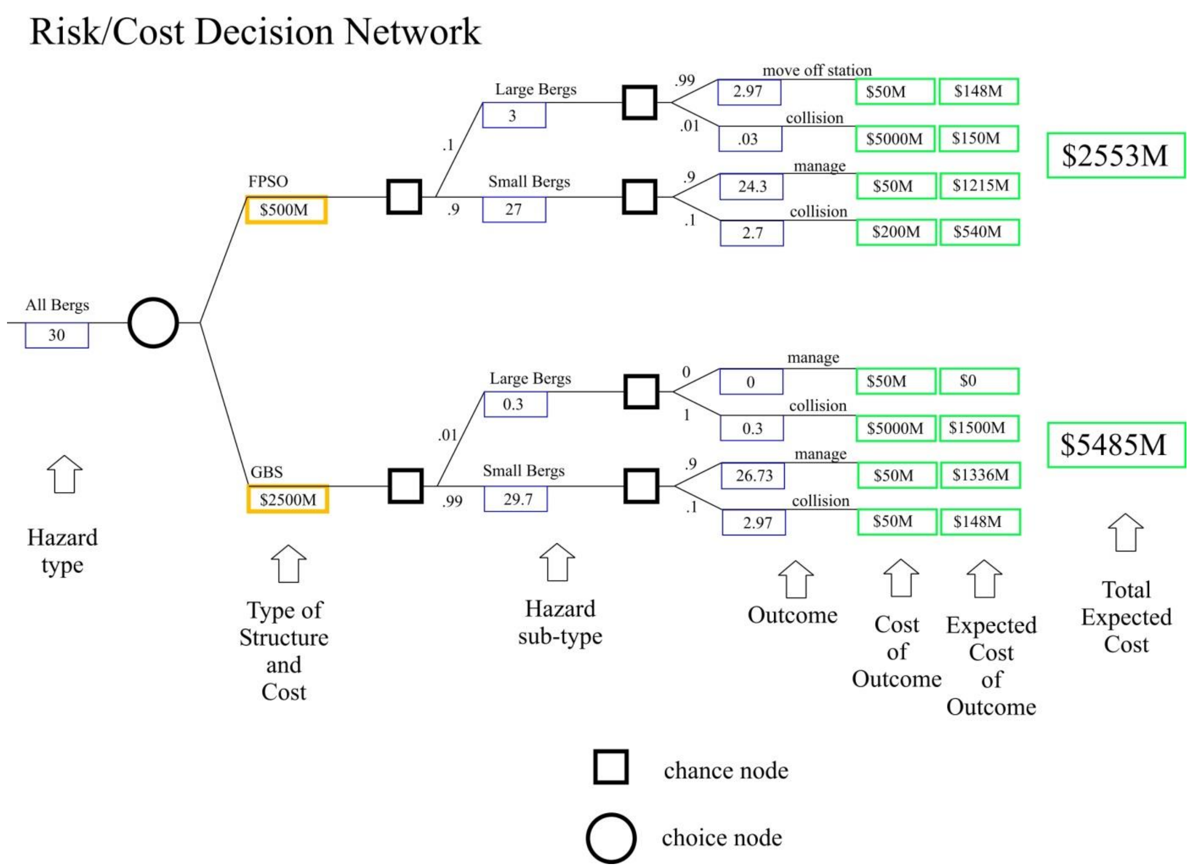


Prob(Damage) = 7 x 0.1 x (0.1 + 0.9 x 0.1) x 0.9
= 0.1197 (= 12%)

To get a collision we either do not detect the berg in front, or we detect but fail to avoid. Then we recognize that some of the collisions will cause damage. The numbers (conditional probabilities) given here are taken from thin air!

Next we will examine a slightly different type of model. This is called a decision network. In it there are decision nodes (rather than all random branches). We examine both sides of the decision branch (or branches), and pick the branch with the lowest cost (greatest utility).

In this case we are trying to compare a GBS with an FPSO for use on the grand Banks. To do this calculation we need both the conditional probabilities and the costs of the various outcomes.



Again, these numbers are hypothetical.

In general, risk analyses only focus on a specific set of risks. One will normally go through a process of hazard identification (HAZID) to find the set of risky events that you want to model. As mentioned in Lect. 5, this set is often smaller than the actual set of risks that the design is subject to. The attached extract from a paper on nuclear waste re-cycling gives a description of the formulation and use of a risk model.