# Unit 3

# Linear Wire Antennas and Antenna Arrays

While the poor efficiency of the "small" antennas discussed in the last unit limits their practicality, the ideas encountered in analyzing them are very useful in establishing the radiation characteristics of larger, more efficient, structures. The main restriction in what follows is the imposition of the requirement that the linear wires involved have a diameter ($d$) which is small compared to a wavelength ($\lambda$) – i.e., they are *thin* antennas. If $d \ll \lambda$, the current along the arbitrarily long dipole is essentially sinsoidal with nulls at each end (more on this shortly) and the analysis is fairly straightforward. The case of $d \not\ll \lambda$ is not considered in this course.

A linear antenna may be characterized as an arbitrary current on a thin straight conductor lying along the $z$-axis and centred at $(0, 0, 0)$. This structure is fed at its midpoint by a transmission line – i.e., this is a dipole antenna with length, $\ell$, unrestricted. The two "arms" of the dipole are the regions where curents can flow and charges accumulate.

## 3.1 Thin, Linear Wire Dipole Antennas

Consider a thin dipole antenna of arbitrary length being fed at its centre as shown below. The antenna is chosen for the purpose of this analysis to lie along the $z$-axis and to be centred at $(0, 0, 0)$. We shall examine the far field of this structure.

Facts: (1.) The dipole is thin and the current is essential sinusoidal – we shall use the form $I(z')$ as the phasor current.

(2.) The current goes to zero at the ends of the dipole – i.e., current nulls exist at the ends.

(3.) The current is given by

$$I(z') = I_0 \sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right] \tag{3.1}$$

where $k = 2\pi/\lambda$ as before. It may be verified experimentally – as well as from a variety of theoretical methods – that equation (3.1) is an extremely good representation of the current on thin linear dipoles. The expression $\sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right]$ is referred to as the *form factor* for the current on the antenna. Here, then, equation (1) is taken as a "given".

### 3.1.1 The Vector Potential and Resulting Fields of the Thin Linear Dipole

The usual approach of determining the potential, $\vec{A}(\vec{r})$, in order to find the $\vec{E}$ and $\vec{H}$ fields will again be followed. As previously, we have for the vector potential, in general for a line current,

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\hat{z}\int_{-\ell/2}^{\ell/2} \frac{I(z')e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dz' \tag{3.2}$$

Because the dipole is "thin" and lies along the $z$-axis, $x' = y' = 0$ and

$$|\vec{r}-\vec{r}'| = \sqrt{x^2+y^2+(z-z')^2}\ . \tag{3.3}$$

For the far-field, we have shown in Section 2.2.2 that, since $r \gg z'$, equation (3.3) reduces to

$$|\vec{r}-\vec{r}'| \approx r - z'\cos\theta\ . \tag{3.4}$$

Using (3.4) in the phase term and $|\vec{r}-\vec{r}'| \approx r$ in the amplitude term of (3.2), we write

(3.5)

Putting the expression for the dipole current of equation (3.1) into (3.5) yields

(3.6)

Doing the integration, it is easily verified that

$$\vec{A}(\vec{r}) = \hat{z}\frac{\mu I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right)-\cos\left(\frac{k\ell}{2}\right)}{k\sin^2\theta}\right]\ . \tag{3.7}$$

Beginning with $\vec{H} = \dfrac{1}{\mu}\vec{\nabla}\times\vec{A}$, it is straightforward to determine that in the far-field (where we retain terms $\propto \frac{1}{r}$ and drop higher orders of $\frac{1}{r}$)

$$\vec{E}(\vec{r}) = E_\theta\hat{\theta} = \frac{j\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right)-\cos\left(\frac{k\ell}{2}\right)}{\sin\theta}\right]\hat{\theta} \tag{3.8}$$

and

$$\vec{H}(\vec{r}) = H_\phi\hat{\phi} = \frac{jI_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right)-\cos\left(\frac{k\ell}{2}\right)}{\sin\theta}\right]\hat{\phi}\ . \tag{3.9}$$

The magnitudes of the expressions in the square brackets are referred to as the $E$- and $H$-field pattern, $f(\theta)$, respectively (or, in general, $f(\theta,\phi)$).

### 3.1.2 Power Density and Other Parameters

The other far-field characteristics, such as the Poynting vector, radiation intensity, radiation resistance and so on may now be determined from equations (3.8) and (3.9).

Time-Averaged Poynting Vector, $\vec{\mathcal{P}}_a$:

The time-averaged Poynting vector, $\vec{\mathcal{P}}_a$ may be determined as usual according to

$$\vec{\mathcal{P}}_a =$$

$$=$$

$$\Rightarrow \vec{\mathcal{P}}_a = \frac{\eta |I_0|^2}{8\pi^2 r^2} \left[ \frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta} \right]^2 \hat{r} \ \ \mathrm{W/m^2} \ . \tag{3.10}$$

Radiation Intensity, $U \equiv U(\theta, \phi)$:

The radiation intensity, $U$ may be determined as usual according to

Therefore,

$$U = \frac{\eta |I_0|^2}{8\pi^2} \left[ \frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta} \right]^2 \ \ \mathrm{W/sr} \tag{3.11}$$

Average Radiated Power, $P_r$, and Radiation Resistance, $R_r$:

Again, considering the power density over a sphere surrounding the dipole we get as before (see equation (2.33))

$$P_r = \frac{\eta |I_0|^2}{4\pi} \int_0^{\pi} \frac{\left[\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)\right]^2}{\sin\theta} d\theta \ \ \mathrm{W} \ . \tag{3.12}$$

This is the general form of the *radiated* power from a dipole of arbitrary length. This final integral is not "simple" and it doesn't reduce to a closed-form result. After four or five pages of tedious manipulations, it transpires that

$$\begin{aligned} P_r &= \frac{\eta |I_0|^2}{4\pi} \left\{ C + \ln(k\ell) - C_i(k\ell) + \frac{1}{2}\sin(k\ell)\left[S_i(2k\ell) - 2S_i(k\ell)\right] \right. \\ &\left. + \frac{1}{2}\cos(k\ell)\left[C + \ln\left(\frac{k\ell}{2}\right) + C_i(2k\ell) - 2C_i(k\ell)\right]\right\} W \end{aligned} \tag{3.13}$$

where $C = 0.5772$ is called Euler's constant and the Cosine and Sine integrals are defined as follows:

$$C_i(x) = -\int_x^\infty \frac{\cos y}{y} dy = \int_\infty^x \frac{\cos y}{y} dy \quad \text{(Cosine Integral)}$$

$$S_i(x) = \int_0^x \frac{\sin y}{y} dy \quad \text{(Sine Integral)}$$

These integrals do not exist in closed form, but they may be "performed" as series expansions. Of course, with numerical integration (eg., Matlab, Mathematica, Maple, etc.) numerical answers may usually be obtained very efficiently.

From equation (2.28) and (3.13), the radiation resistance is given by

$$\begin{aligned} R_r = \frac{2P_r}{|I_0|^2} &= \frac{\eta}{2\pi} \left\{ C + \ln(k\ell) - Ci(k\ell) + \frac{1}{2}\sin(k\ell)\left[Si(2k\ell) - 2Si(k\ell)\right] \right. \\ &\left. + \frac{1}{2}\cos(k\ell)\left[C + \ln\left(\frac{k\ell}{2}\right) + Ci(2k\ell) - 2Ci(k\ell)\right]\right\} \Omega \end{aligned} \tag{3.14}$$

### 3.1.3 Special Cases of the Thin Dipole

Having the general forms for the important parameters of the thin dipole, we will now consider one special case in detail and briefly refer to some others.

The Half-Wave Dipole

Current Distribution

For the half-wave dipole, by definition, $\ell = \lambda/2$. Substituting this into equation (3.1) yields for the current distribution

$$I(z') \;=$$

$$=$$

$$\Rightarrow I(z') \;=\; I_0 \cos(k|z'|) \tag{3.15}$$

<u>$\vec{E}$ and $\vec{H}$ Fields:</u>

Noting in (3.8) and (3.9) that $k\ell/2 = (2\pi\lambda)/(\lambda \cdot 2 \cdot 2) = \pi/2$,

(3.16)

(3.17)

Clearly, the field patterns are given by

$$\left| f(\theta)|_{\ell=\lambda/2} \right| = \left| \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right| \tag{3.18}$$

The field pattern may be easily established numerically (or may be checked analytically for maxima and minima):

(3.19)

Setting $f'(\theta) = 0$ to seek maxima and minima eventually leads to

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta = \cos\theta\cos\left(\frac{\pi}{2}\cos\theta\right)$$

provided we put the restriction $\sin\theta \neq 0$ (hold on for this case). A quick check (not a proof!) reveals that $\theta = \pm\frac{\pi}{2}$ creates equality in the last expression. From (3.19)

Now, to get the minima in the pattern, we can't (without checking) simply set the numerator to zero in (3.19) because the "solution" is then $\theta = 0, \pm\pi, \ldots$ and this makes the denominator 0 also – i.e. $f(\theta) \longrightarrow \frac{0}{0}$ which is an indeterminate form. However, using L'Hopital's rule on (3.19) gives

Therefore, there are indeed nulls at $\theta = 0, \pm\pi$. These results should not be surprising in view of the current pattern on the dipole as given above.

Principal Plane Patterns:

Time-Averaged Poynting Vector, $\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}}$:

Using $\ell = \lambda/2$ and noting that $k\ell/2 = \pi/2$, equation (3.10) readily gives for the half-wave dipole

$$\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}} = \frac{\eta|I_0|^2}{8\pi^2 r^2} \frac{\cos^2\left(\frac{k\ell}{2}\cos\theta\right)}{\sin^2\theta}\hat{r} \ \ \text{W/m}^2 \ . \tag{3.20}$$

Radiation Intensity, $U_{\frac{\lambda}{2}}$:

Using $\ell = \lambda/2$ and noting that $k\ell/2 = \pi/2$, equation (3.11) readily gives for the half-wave dipole

$$U_{\frac{\lambda}{2}} = r^2 \left|\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}}\right| = \frac{\eta|I_0|^2}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \ \ \text{W/sr} \ . \tag{3.21}$$

Note that the power pattern shape is given by

$$F(\theta, \phi) = |f(\theta, \phi)|^2 = \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Radiated Power, $P_{r_{\frac{\lambda}{2}}}$:

From equations (2.33) and (3.21), it is easily seen that the radiated power is given by

This must be done numerically, in which case it can be shown that the final integral has a value $\int_0^\pi = 1.218$. Then, using the free-space value of $\eta_0 = 120\pi$,

$$P_{r_{\frac{\lambda}{2}}} = 36.54|I_0|^2 \ \text{W} \tag{3.22}$$

Note that this result could have been readily obtained by programming equation (3.13).

Radiation Resistance, $R_{r_{\frac{\lambda}{2}}}$:

As usual, set $P_{r_{\frac{\lambda}{2}}} = \frac{1}{2}|I_0|^2 R_{r_{\frac{\lambda}{2}}}$ and from (3.22)

$$R_{r_{\frac{\lambda}{2}}} = 2 \times 36.54 = 73.1\Omega \,.$$

This is the radiation resistance of the thin half-wave dipole in free space.

Current Distributions on Other Dipoles and Relative $\vec{E}$-Field Patterns

Recall $I(z') = I_0 \sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right]$.

$\ell < \frac{\lambda}{2}$:

$\frac{\lambda}{2} < \ell < \lambda$:

Up to a limit, the beam becomes more directive, with maximum gain in the horizontal direction, as the antenna length increases – at $\ell > \frac{5\lambda}{4}$ other lobes appear in the pattern; that is, the beam splits up.

$\lambda < \ell < \frac{3\lambda}{2}$:

## 3.2 Images and Monopoles

The concept of "image theory" in electromagnetics states that "any given charge configuration above an infinite, perfectly conducting plane is electrically equivalent to the combination of the the given charge configuration and its image configuration with the conducting plane removed". For this to "work"

1. the *image* charge configuration must be placed in the region (half-space) containing the conductor

   and

2. the image charge configuration must be placed so that the potential at the conductor surface is a constant (generally zero – at infinity the potential is also zero).

Illustration:

Repeating our previous theory for the second situation leads to the same $\vec{E}$, $\vec{H}$, and $\vec{\mathcal{P}}_a$ expressions as before – but only for the $z > 0$ region! The results DO NOT APPLY in the $z < 0$ region. The second illustration above is that of a *monopole* antenna. It could be realized, for example, by extending the centre conductor of a coaxial cable through a ground plane while the outer conductor is attached to the plane.

In the present context of thin wire antennas, consider a quarter-wave ($\lambda/4$) monopole:

While the fields and Poynting vector using the antenna and the image have the same expressions as for the $\lambda/2$ dipole in free space, the radiated power is only half that of the dipole. This is easy to believe since in the $z > 0$ region (region of analysis validity), $0 \leq \theta \leq \frac{\pi}{2}$ while for the dipole in free space, $0 \leq \theta \leq \pi$. Therefore, the following hold:

Note: Here we have considered the antenna to be at the level of the ground plane (not raised above it).

Similar analysis could be carried out for any antenna configuration. For example:

Note, additionally, that if the ground is not perfectly conducting – i.e. the case for all real-life situations – the field patterns will not be exactly as discussed. In fact, significant modifications may exist depending on the electrical characteristics, in particular $\sigma$ and $\epsilon$, of the ground. Other "real" problems which may need consideration arise from the fact that the ground may not be planar in the the antenna's field of view. There are theoretical analyses as well as practical computer programs which may be used to address such issues. We will not consider them further here.

## 3.3 Antenna Arrays

We have seen that enlarging the dimensions electrically – which may be done by enlarging the physical dimensions or by electrical means (for example, recall the "capacitive hats" discussed in the context of the small element in Section 2.2.1) – makes an antenna more directive (higher gain). Another way of increasing the directive characteristics is to use multiple elements in special electrical and geometrical configurations. Such a combination of antenna elements is called an *antenna array.* In this section, we shall consider arrays of dipole elements. The nature of the resulting antenna pattern will depend on several controlling factors:

1. the geometrical layout of the array – i.e., the shape of the combination of elements, whether it be linear, circular, rectangular, etc..

2. the relative displacement between the elements – usually referred to in terms of wavelength.

3. the excitation amplitude of the individual elements.

4. the excitation phase of the individual elements.

5. the relative pattern (power or field pattern) of the individual elements.

A very important class of antenna elements is the *linear array* formed by placing identical, equally spaced elements along a line. As well, more complicated analytically, are *planar arrays* and *circular arrays* (a special case of planar arrays) arranged as shown.

We shall begin our analysis with the 2-element array.

### 3.3.1 Two-Element (Dipole) Array – Free Space Analysis

To simplify the analysis, we consider the following:

1. both elements have the same physical and electrical properties (i.e., length, material, and thus radiation resistance, etc. are the same).

2. there is no coupling between the two elements.

Property (2) is not likely to be satisfied in "real life", but as a reasonable approximation it allows us to get some idea of the resulting pattern without undue mathematical complications. Remember, too, that in actual implementations neither free space nor perfect ground are exactly realizable. For particular implementatioms of an antenna array, the patterns may be measured directly after the initial design and installation. Several numerical (computer) techniques provide an excellent means of giving a good idea of antenna performance before the actual fabrication and installation occurs. With these facts in mind, we proceed with the case of the ideal two-element array in free space and seek the $E$-field patterns.

First, recall the following examples:

1. Elementary (i.e. Infinitesimal) Dipole Source: To emphasize that, in general, $\vec{E}$ may be a function of $r$, $\theta$, and $\phi$ and noting $\eta = 120\pi$ for free space while $k = 2\pi/\lambda$, we write equation (2.16) as

   which implies

   $$\vec{E}(r, \theta, \phi) = \frac{CI_0}{r} e^{-jkr} \vec{E}_a(\theta, \phi) \tag{3.23}$$

   where $C = \dfrac{j60\pi\ell}{\lambda}$ is a complex constant and

   $$\vec{E}_a(\theta, \phi) = \sin\theta\hat{\theta}$$

   is the direction-dependent factor. Recall that the far-field, principal plane patterns are:

2. Half-Wave Dipole: From equation (3.16) (note that we could use (3.8) and discuss the general dipole)

   $$\vec{E}(r, \theta, \phi) = \frac{CI_0}{r} e^{-jkr} \vec{E}_a(\theta, \phi) \tag{3.24}$$

   where, now, $C = j60$ and

   $$\vec{E}_a(\theta, \phi) = \left[ \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \hat{\theta} \ .$$

   The principal plane patterns are

The idea that we get from (3.23) and (3.24) is that the far-field $\vec{E}$ of a vertical dipole antenna may be written in general form as

$$\vec{E}(r,\theta,\phi) = [\text{ Complex Constant }] \times \left[\frac{I_0}{r} e^{-jkr}\right] \times [\text{ Direction Dependent Factor }] \tag{3.25}$$

With this information in place, let us now consider an "array" of two identical vertical dipole antennas. Furthermore, we shall consider only the far field (i.e. $r \gg d$ where $d$ is the distance between the elements). For reference purposes, let's put the elements along the $x$-axis as shown (this is totally arbitrary – we could use any axis). Because $r \gg d$, the $\theta$ and $\phi$ coordinates are approximately the same with respect to both elements. We shall label the first antenna 0 and the second, 1.

Element 0 carries current $I_0$ and element 1 carries current $I_1$ – this means that $I_1$ replaces $I_0$ in the field equation for the element involved. We note the following:

1. From the point of view of magnitude, in the far field, $r_0 \approx r_1$ and, of course,

$$\frac{1}{r_1} \approx \frac{1}{r_0} \qquad \text{FACT 1}$$

2. However, as discussed earlier, we can't make this approximation in the phase. Rather, from the above diagram,

$$r_1 = r_0 - d\cos\alpha \qquad \text{FACT 2}$$

We'll use $\vec{E}_0(r_0,\theta,\phi)$ and $\vec{E}_1(r_1,\theta,\phi)$ for the far fields of elements 0 and 1 respectively, with corresponding direction-dependent parts of $\vec{E}_{0a}$ and $\vec{E}_{1a}$ as in equations

(3.23) or (3.24) or (3.25). Then, for the first element

$$\vec{E}_0(r_0, \theta, \phi) = \frac{CI_0}{r_0} e^{-jkr_0} \vec{E}_{0a}(\theta, \phi) \tag{3.26}$$

and for the second element

$$\vec{E}_1(r_1, \theta, \phi) = \frac{CI_1}{r_1} e^{-jkr_1} \vec{E}_{1a}(\theta, \phi)$$

or, because the direction characteristics of identical elements are identical,

$$\vec{E}_1(r_1, \theta, \phi) = \frac{CI_1}{r_1} e^{-jkr_1} \vec{E}_{0a}(\theta, \phi) \tag{3.27}$$

Now, the total $\vec{E}$-field, $\vec{E}_T$, may be written as

(3.28)

on using (3.26) and (3.27) in (3.28) along with FACTS 1 and 2,

$$\vec{E}_T(r_1, \theta, \phi) = C \left[ \frac{I_0 e^{-jkr_0}}{r_0} + \frac{I_1 e^{-jk(r_0 - d\cos\alpha)}}{r_0} \right] \vec{E}_{0a}(\theta, \phi) \tag{3.29}$$

Let's further assume that $I_1$ is related to $I_0$ via

where $A = \frac{|I_1|}{|I_0|}$ and $\beta$ is the phase difference between the two currents – i.e. we are allowing for scalar multiplication in amplitude and a phase shift. Clearly, (3.29) becomes

$$\vec{E}_T = \left[ \frac{CI_0 e^{-jkr_0}}{r_0} \vec{E}_{0a}(\theta, \phi) \right] \left[ 1 + Ae^{j(kd\cos\alpha + \beta)} \right]$$

or

$$\vec{E}_T = \vec{E}_0(r_0, \theta, \phi) \left[ 1 + Ae^{jkd\cos\alpha + j\beta} \right] \tag{3.30}$$

<u>Important note:</u>

Pattern Multiplication

Let's write equation (3.30) in the form

$$\vec{E}_T = \vec{E}_0(r_0, \theta, \phi) E'(\alpha, \beta) \tag{3.31}$$

where $E'(\alpha, \beta) = (1 + Ae^{(jkd\cos\alpha + j\beta)})$ is called the *array factor* and $\vec{E}_0(r_0, \theta, \phi)$ is the field pattern for an individual element. We have seen that the general shape of the $\vec{E}_0(r_0, \theta, \phi)$ pattern in both the horizontal and vertical planes for vertical dipoles on several different occasions already.

Now, the pattern based on the array factor is called the *group pattern* and is found by plotting the magnitude $|E'(\alpha, \beta)|$, while bearing in mind the earlier definition of $\alpha$. Thus, based on (3.30) and (3.31), we see that

$$\boxed{\text{Resultant Pattern} = \text{Unit Pattern} \times \text{Group Pattern}} \tag{3.32}$$

and this may be readily deduced for either the horizontal or vertical planes – i.e., azimuth and elevation.

**Special Case** $|I_1| = |I_0|$

In this case, $A = 1$ in equation (3.30). Then,

$$|E'(\alpha, \beta)| = \left|1 + e^{(jkd\cos\alpha + j\beta)}\right| \ ,$$

which is easily shown to be (do this)

$$|E'(\alpha, \beta)| = 2\left|\cos\left(\frac{kd\cos\alpha + \beta}{2}\right)\right| \ , \tag{3.33}$$

It will thus be observed that varying the phase, $\beta$, is a technique which will allow the direction of the array factor to be altered – i.e., $\beta$ may be altered to "steer" the beam of the array.

Special Case 1. $\beta = 0°$ for the Two-Element Array

Let us suppose that the equal magnitude currents that led to (3.33) are also in phase. That is, $\beta = 0°$. Furthermore let us assume that the array spacing is $d = \lambda/2$ so that $kd = \pi$.

Horizontal Plane ($\alpha = \phi$):

We note from (3.33) that

$$|E'(\alpha,\beta)|_{\substack{\alpha=\phi \\ \beta=0°}} = 2\left|\cos\left(\frac{kd\cos\alpha+\beta}{2}\right)\right|_{\substack{\alpha=\phi \\ \beta=0°}} = 2\left|\cos\left(\frac{\pi\cos\phi}{2}\right)\right|$$

Note:

Note that the maxima are in the direction perpendicular to the array line (which was taken along the $x$-axis from the outset of this analysis). Thus, with $\beta = 0$, that is with the currents in phase, the array is said to operate in *broadside* mode.

Vertical Plane ($\alpha = 90° - \theta$):

Special Case 2. $\beta = 180°$ for the Two-Element Array

Let us suppose that the equal magnitude currents that led to (3.33) are out of phase by 180° – that is, $\beta = 180°$. Furthermore let us assume that the array spacing is still $d = \lambda/2$ so that $kd = \pi$.

Horizontal Plane ($\alpha = \phi$):

We note from (3.33) that the group pattern is given by

$$|E'(\alpha, \beta)|_{\substack{\alpha = \phi \\ \beta = 180°}} = 2\left|\cos\left(\frac{kd\cos\alpha + \beta}{2}\right)\right|_{\substack{\alpha = \phi \\ \beta = 180°}} = 2\left|\cos\left(\frac{\pi\cos\phi + \pi}{2}\right)\right|$$

Comparing with the $\beta = 0°$ case, we note that the maxima are in the direction parallel to the array line (which was taken along the $x$-axis from the outset of this analysis). Thus, with $\beta = 180°$, that is with the currents out of phase by 180°, the array is said to operate in *endfire* mode.

Vertical Plane ($\alpha = 90° - \theta$):

Now the group pattern becomes

$$|E'(\alpha, \beta)|_{\substack{\alpha = 90° - \theta \\ \beta = 180°}} = 2\left|\cos\left(\frac{\pi\sin\theta + \pi}{2}\right)\right|$$

It is possible to determine that when $d > \lambda/2$, additional lobes appear. (Try this with a small Matlab program).

### 3.3.2 The Linear Array

Consider, next, the more general case of $M$ equally-spaced identical elements lying along a line as indicated. Again, we are interested in the far field. As usual, $d$ is the element spacing, $r_0$ is the distance from the $0^{\text{th}}$ element and so on up to $r_{M-1}$.

Assumption: The array length $(M-1)d \ll r_n$ for $0 \leq n \leq M-1$.

Consequence: The $\theta$ and $\phi$ coordinates of the field point are the same with respect to each element.

Thus, the total field at the observation point due to all of the elements may be written as

$$\vec{E}_T = \sum_{n=0}^{M-1} \vec{E}_n(r_n, \theta, \phi) \tag{3.34}$$

Proceeding as for the two-element case (see equation (3.26))

(3.35)

and, in general, for the $n^{\text{th}}$

$$\begin{aligned} \vec{E}_n(r_n, \theta, \phi) &= \\ &= \end{aligned} \tag{3.36}$$

since the direction-dependent factor is the same for each element. As before, in the amplitude sense, for the far field,

$$\frac{1}{r_n} \approx \frac{1}{r_0}\,, \tag{3.37}$$

but in the phase term we must specify

$$r_n = r_0 - nd\cos\alpha \ . \tag{3.38}$$

Using (3.36), (3.37), and (3.38) in (3.34),

$$\begin{aligned}\vec{E}_T &= \sum_{n=0}^{M-1} \frac{CI_n}{r_0} e^{-jkr_0 + jknd\cos\alpha} \vec{E}_{0a}(\theta,\phi) \\ &= \left[\frac{CI_0 e^{-jkr_0}}{r_0} \vec{E}_{0a}(\theta,\phi)\right] \left[\sum_{n=0}^{M-1} \frac{I_n}{I_0} e^{jknd\cos\alpha}\right]\end{aligned}$$

Therefore,

$$\vec{E}_T = \vec{E}_0(r_0,\theta,\phi) \left[\sum_{n=0}^{M-1} \frac{I_n}{I_0} e^{jknd\cos\alpha}\right] \tag{3.39}$$

The term containing the summation is commonly referred to as the *array factor* (AF). It may be clearly observed from equation (3.39) that the following statement may be made about the "resultant pattern" of a linear array:

| resultant pattern = unit pattern × group pattern. |
|---|

Now, some special cases of linear arrays may be considered.

Special Case 1: The Uniform Linear Array

In a *uniform* linear array, the current magnitudes are equal, but there is a uniform progression in the phase of the current from one element to the next. This phase shift, $\beta$, may take positive or negative values. For this case, the currents may be written as

$$I_n = I_0 e^{jn\beta} \ , \tag{3.40}$$

Then, equation (3.39) may be written as

$$\vec{E}_T = \vec{E}_0(r_0,\theta,\phi) E'(\alpha,\beta)$$

where

$$E'(\alpha,\beta) = \sum_{n=0}^{M-1} e^{jn(\beta + kd\cos\alpha)}$$

or

$$E'(\alpha,\beta) = \sum_{n=0}^{M-1} e^{jn\Psi} \tag{3.41}$$

where

$$\Psi = \beta + kd\cos\alpha\ . \tag{3.42}$$

The RHS of (3.41) is a $M$-term geometric progression whose first term is 1 and whose constant ratio is $e^{j\Psi}$. Therefore, (3.41) may be written as

Since $\left|e^{j\frac{\Psi}{2}(M-1)}\right| = 1$, it is seen that

$$|\text{AF}| = |E'(\alpha,\beta)| = \left|\frac{\sin\left(\frac{M\Psi}{2}\right)}{\sin\left(\frac{\Psi}{2}\right)}\right| \tag{3.43}$$

will establish the shape of the group pattern. Furthermore, if the unit pattern is circular – as is the case, for example, for $\lambda/2$ vertical dipoles with $\alpha = \phi$ and $\theta = 90°$ – $|\text{AF}|$ establishes the resultant pattern for the array.

Equation (3.43) is worthy of closer scrutiny:

1. Notice that when $\Psi = 0$, (3.43) becomes the indeterminate form $\frac{0}{0}$. However, on using L'Hopital's rule, it readily verified that

$$\lim_{\Psi\to 0} |\text{AF}| = M.$$

It may indeed be verified that $\Psi = 0$ produces the *principal* (or absolute) *maximum* (check it out numerically). Thus, a normalized form of (3.43) having a principal maximum of unity may be written as

$$|\text{AF}|_n = \frac{1}{M}\left|\frac{\sin\left(\frac{M\Psi}{2}\right)}{\sin\left(\frac{\Psi}{2}\right)}\right| \tag{3.44}$$

2. *Secondary maxima* occur when, in (3.43),

$$M\Psi = \pm(2i+1)\pi \ , \ \ i \in \ \mathrm{N}$$

or

Therefore from (3.42)

which gives

$$\boxed{\cos\alpha = -\left(\frac{\beta}{kd}\right) \pm (2i+1)\frac{\pi}{Mkd}} \tag{3.45}$$

$i = 1$ produces the first secondary maximum, $i = 2$ produces the second secondary maximum, and so on. It is easy to show that for large $M$, the first secondary maximum is about -13.5 dB down from the principal maximum. (To be completed on a tutorial).

3. Nulls occur when, in (3.43), the numerator is zero but the denominator is not zero. With this caution, we set

$$\frac{M\Psi}{2} = \pm i\pi \ , \ \ i \in \ \mathrm{N} \ \ (\text{but } i \neq M, 2M, 3M, \ldots)$$

which implies

or

$$\boxed{\cos\alpha = -\left(\frac{\beta}{kd}\right) \pm \left(\frac{2i\pi}{Mkd}\right)} \tag{3.46}$$

Remember that in (3.45) and (3.46), $\alpha = \phi$ in the horizontal plane and $\alpha = 90^\circ - \theta$ in the vertical plane. This analysis shows that, in general, a linear array will produce an array factor with a shape something like

Case 1a. The Uniform Broadside Array

Consider $M$ vertical equally spaced dipoles along the $x$-axis as shown. We are going to examine the array factor in the $x$-$y$ plane – i.e. $\alpha = \phi$.

Recall that the normalized unit pattern is a unit circle in the $x$-$y$ plane. In a *broadside* linear array, the principal maximum is in a direction 90° to the line containing the array elements – that is, $\phi = 90°$ is the direction of the *principal maximum.* Let's check the required phase for the currents:

We know from (3.42) that

$$\Psi = \beta + kd \cos \alpha$$

and note in passing that $\Psi$ is an even function of $\phi$ for the horizontal plane, making the AF an even function also. For $\alpha = \phi$ and $\phi = \pi/2$ along with the principal maximum occurring at $\Psi = 0$ (see Note (1) above) we have

Since $k$ and $d$ are constants, $\beta = 0$. Therefore, when the elements are fed "in-phase", the array operates in "broadside" mode.

Typical Result:

Width of the Main Lobe – i.e. Beamwidth

Let's consider the width of the main lobe in terms of the $E$-field pattern. With reference to the sketch above,

$$\text{Width of Main Lobe } = 2\Delta\phi$$

where $\Delta\phi$ is the angle between the principal maximum and the first null either side of it. The direction of the first null is readily obtained from (3.46) as

Therefore,

$$\text{Width of Main Lobe } = 2\sin^{-1}\left[\frac{\lambda}{Md}\right] \tag{3.47}$$

We make the important observation from (3.47) that if the array is long (i.e. $Md$ is large), the "beam" will be narrow. In fact, if we assume $(\lambda/Md)$ is small so that

$$\sin^{-1}\left[\frac{\lambda}{Md}\right] \approx \frac{\lambda}{Md}\ ,$$

then

$$\text{Width of Main Lobe } = 2\Delta\phi = \frac{2\lambda}{Md}.$$

Half-Power Beam Width

We now attempt to find the half-power beamwidth of a uniform broadside linear array of dipoles for which, in the horizontal plane as considered above, the unit pattern is a circle. Recall that for this situation $\beta = 0$ and $\alpha = \phi$ so that (3.42) is

$$\Psi = kd\cos\phi$$

and that the $E$-field pattern has array factor with maximum value of $M$. Recalling that

$$\text{power } \propto (E\text{-field})^2,$$

a normalized power pattern may be found by simply squaring the array factor:

For the situation under consideration,

$$|\mathrm{AF}|^2 = \left| \frac{\sin\left(\frac{Mkd}{2}\cos\phi\right)}{\sin\left(\frac{kd}{2}\cos\phi\right)} \right|^2$$

whose maximum value is obviously $M^2$. To determine where this quantity reaches half its maximum value (i.e. the direction in which the power is half its maximum), we set

$$\tag{3.48}$$

and "solve" for $\phi$ eventually. Here we have used $\Psi_{1/2}$ to indicate the $\Psi$ at the half-power positions. It is not hard to convince oneself that for a long array (say, several elements spaced by half wavelengths) $\Psi_{1/2}/2$ is small so that $\sin(\Psi_{1/2}/2) \approx \Psi_{1/2}/2$. Also, recall that

and the numerator in the left member of equation (3.48) gives (using the first two terms of the expansion)

$$\sin\left(\frac{M\Psi_{1/2}}{2}\right) = \frac{M\Psi_{1/2}}{2} - \frac{1}{6}M^3\left(\frac{\Psi_{1/2}}{2}\right)^3 \tag{3.49}$$

Rearranging (3.48)

and using the above approximations gives

This reduces to

$$\Psi_{1/2} = \frac{2.65}{M}$$

Since $\beta = 0$, using (3.42) with $\alpha = \phi$ gives

$$\tag{3.50}$$

where $\phi_{1/2}$ is the angle associated with the half-power point. (Note that in (3.50), $(Md)$ is approximately the array length.) To see how this analysis leads to the half-power beamwidth, consider the following:

From this we see that for large array lengths, $\Delta\phi_{1/2}$ will be small. To a good approximation, therefore,

and the half-power beamwidth, $\mathrm{BW}_{1/2}$, is

$$\boxed{\mathrm{BW}_{1/2} = 2\Delta\phi_{1/2} = \frac{2.65\lambda}{\pi M d}} \tag{3.51}$$

Notice that the half-power beam width is inversely proportional to the array length (approximately).

Directivity

From our deliberations in Section 2.3.3, it is not too difficult to deduce that

$$D(\phi) = \frac{U_n(\phi)}{U_{0n}} = 4\pi \frac{U_n(\phi)}{\int\int_\Omega U_n d\Omega} = 4\pi \frac{|AF|_n^2}{\int_0^{2\pi}\int_0^{\pi} |AF|_n^2 \sin\theta d\theta d\phi} \tag{3.52}$$

where the subscript $n$ is used to represent normalized values. In general, of course, $D$ and $U$ are functions of both $\theta$ and $\phi$. However, here we are considering the array factor in the horizontal plane (for a broadside array) so that $\Psi$ in equation (3.44) for the case at hand is from (3.42) $\Psi = kd\cos\phi$. While the numerator in (3.52) follows directly from (3.44), we need to consider the denominator more carefully. Note the following:

(1.) For a linear array, the main beam is relatively "narrow" and, therefore, for our array orientation, in the region where most of the power is concentrated, $\sin\theta \approx 1$ (i.e., $\theta$ is close to $\pi/2$). Therefore,

This takes care of the $\theta$-integral in the denominator.

(2.) For the broadside array, the beam maximum is in the $\phi = \pi/2$ direction. Since most of the power is concentrated in a small solid angle about this direction, it is reasonable to approximate

Using (3.44), facts (1) and (2), and the relation $\Psi = kd\cos\phi$, the denominator in (3.52) becomes

Now, let $Z = \dfrac{Mkd\cos\phi}{2}$ which implies and

$$\text{Denominator} = \frac{1}{Mkd} \int_{-\frac{Mkd}{2}}^{\frac{Mkd}{2}} \left[ \frac{\sin Z}{Z} \right]^2 dz$$

where we have approximated $\sin\phi$ as unity (the beam is narrow and $\phi$ is close to $\pi/2$ in the region where most of the power is concentrated). Finally, if the array is "long", as is the case for a narrow beam, then to a good approximation the integral limits may be extended to "infinity" (it should be noted that "long" depends on wavelength and spacing as well as the number of elements). Then,

$$\text{Denominator} = \frac{1}{Mkd} \int_{-\infty}^{\infty} \left[ \frac{\sin Z}{Z} \right]^2 dz \ .$$

It is well-known (look it up in a table) that

$$\int_{-\infty}^{\infty} \left[ \frac{\sin Z}{Z} \right]^2 dz = \pi \ .$$

Therefore,

$$\text{Denominator} = \frac{\pi}{Mkd} \ .$$

Putting this into (3.52), the directivity for a "long" broadside array becomes, to a reasonable approximation,

$$D(\phi) \approx \frac{Mkd}{\pi} \left[ \frac{\sin\left(\frac{Mkd}{2}\cos\phi\right)}{\frac{Mkd}{2}\cos\phi} \right]^2 \tag{3.53}$$

The maximum directivity, $D_0$, occurring when $\phi = \pi/2$, is easily shown to be

(3.54)

where $L$ is the approximate length of the array and $\lambda$ is the wavelength of the radiation.

[Aside: If the array is aligned along the $z$-axis instead of the $x$-axis, our analysis would end up with $\theta$'s instead of $\phi$'s – many books do this.]

Case 1b. The Uniform Endfire Array

We now wish to determine the phasing condition such that the maximum radiation occurs along the axis of the array in either direction. Such an array is referred to as an *end-fire* array and for uniform element spacing and phase shifts it is a *uniform end-fire* array.

From equations (3.42) and (3.43) and note (1) following the latter, we know that

or since $\alpha = \phi$ in the $x$-$y$ plane,

$$\cos\phi = -\frac{\beta}{kd}$$

For the end-fire condition, $\phi = 0, \pi$. This implies that

$$\beta = -kd \quad \text{for a maximum in the } {}^{+}x \text{ direction}$$

$$\beta = kd \quad \text{for a maximum in the } {}^{-}x \text{ direction.}$$

Let's consider the case $\beta = -kd$. A typical pattern might be.

Beamwidth (Main Lobe) $\rightarrow 2\Delta\phi$

From equation (3.46), the first null ($i = 1$) occurs when (with reference to the above diagram)

(Why do we use $-\frac{2\pi}{Mkd}$ instead of $+\frac{2\pi}{Mkd}$ ?)

Therefore, for end-fire along the ${}^{+}x$ direction, we have, in general,

$$2\Delta\phi = 2\cos^{-1}\left[1 - \frac{\lambda}{Md}\right] \; . \tag{3.55}$$

As before, if the array is long so that $\lambda/Md$ is small and

It is easily verified that if $d = \dfrac{\lambda}{2}$, an end-fire maximum occurs in both the "forward" and "backward" directions. Furthermore, as is also the case for broadside operation, if $d = n\lambda$, additional lobes equal in magnitude to the principal maximum will appear in the radiation pattern. These lobes, which are usually undesirable, are referred to as *grating lobes.*

<u>Half-Power Beamwidth</u>

Making approximations in a similar manner (but not exactly the same) as for the broadside array, the half-power beamwidth of the end-fire array may be given as

$$\boxed{\mathrm{BW}_{1/2} = 2\Delta\phi_{1/2} = 3.26\left[\frac{\lambda}{\pi Md}\right]^{1/2}} \tag{3.56}$$

<u>Directivity</u>

Carrying out a procedure similar to that for the broadside case, we find that the directivity for a "long" end-fire array becomes

$$D(\phi) \approx \frac{2Mkd}{\pi}\left[\frac{\sin\left(\frac{Mkd(\cos\phi-1)}{2}\right)}{\frac{Mkd(\cos\phi-1)}{2}}\right]^2 \tag{3.57}$$

The maximum clearly occurs at $\phi = 0$ and we write

$$D_0 = \tag{3.58}$$

Compare this $D_0$ with that for the broadside case in equation (3.54).

Case 2. The Binomial Array (A Non-uniform Linear Array)

Consider that an array of $M$ equally spaced vertical elements are aligned along the $x$-axis. We are interested in observation of the beam in the $x$-$y$ plane. In this array, we stipulate the magnitude of the array factor from equation (3.39) as

$$|E'(\phi)| = |\mathrm{AF}| = \tag{3.59}$$

and here $\alpha = \phi$. Secondly, we stipulate that the currents have the same phase ($\beta = 0$), but have amplitudes related as follows:

Thus, there is nonuniform excitation of the array elements in the magnitude sense. The relationship shown above is clearly indicative of Pascal's triangle whose elements are the coefficients of the binomial series

$$(a+b)^N = \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} a^{N-n} b^n \tag{3.60}$$

where

$$\begin{pmatrix} N \\ n \end{pmatrix} = \qquad .$$

Thus, in this so-called *binomial array*,

$$\frac{I_n}{I_0} = \begin{pmatrix} N \\ n \end{pmatrix} .$$

In this context, we seek a simplification of the array factor in (3.59). Using the last result above, equation (3.59) becomes on letting $N = M - 1$ (i.e. there are $N + 1$ elements)

$$|E'(\phi)| = |\mathrm{AF}| = \tag{3.61}$$

In seeking a simplification, consider first the case of $N = 1$ – i.e., there are 2 elements. Then, from the coefficients in (3.60), (3.61) becomes

Next suppose $N = 2$ – i.e., there are 3 elements. It is easy to show that

$$|E'(\phi)| = 2^2 \left| \cos^2 \frac{kd \cos\phi}{2} \right|$$

Notice that the exponent on the 2 and the cosine is $N$ for both cases. This idea may be generalized to give

$$|\mathrm{AF}| = |E'(\phi)| = 2^N \left| \cos^N \left( \frac{kd \cos\phi}{2} \right) \right| \tag{3.62}$$

Also, rememeber that $\dfrac{I_n}{I_0} = \begin{pmatrix} N \\ n \end{pmatrix}$. The greater the value of $N$, the larger will be the directivity.

It may be checked that if $d \leq \lambda/2$, the binomial array exhibits no sidelobes. The trade-off is that its directivity may be seen to be less than that of a uniform linear array and the half-power beamwidth is correspondingly broader. These disadvantages are not surprising since the elements near the ends of the binomial array are weakly excited compared with those at the centre – this is essentially equivalent, electrically, to an overall reduction in array length, thus providing a broader beam than the uniform linear array of comparable physical dimensions. We note in passing that the Dolph-Chebyshev array offers a compromise between sidelobe and directivity issues. We will not address this here, but you are encouraged to read the reference text.

### 3.3.3 The Planar Array

In the case of the linear arrays of Section 3.3.2, "phase-scanning" (pointing the main beam in various directions by changing the excitation current phases) can be accomplished only in a plane containing the line of the elements' centres. Perpendicular to this, the beamwidth is determined by the individual beamwidth in that plane. This usually limits the realizable gain. To help alleviate this problem and to increase symmetry and reduce sidelobes, multidimensional arrays may be formed. A subset of these may be be realized by placing elements such that, for example, their centres lie on a rectangle, square, or circle – i.e. the array centres lie in a plane, and hence the name, *planar array*. In this section, we consider the rectangular case (it is common to deal with circular rays separately, but they are not treated in this course).

For the purpose of this discussion, we will consider the array elements to be isotropic radiators. However, this is easily extended to any palanar array of *identical*, but not necessarily isotropic, elements by noting that, as with the linear array, the total field may be obtained by applying the multiplication rule of equation (3.39).

In keeping with our earlier notation, consider a rectangular planar array of $M$ elements along the $x$-axis and $N$ elements along the $y$-axis as shown

For the far-field, $\gamma$ is approximately constant and analogous to "FACT 2" for the two-element case,

$$r_1 = r_0 - d\cos\gamma$$

$$r_2 = r_1 - d\cos\gamma \quad = \quad r_0 - 2d\cos\gamma \quad , \quad \text{etc.}$$

As far as an array row along the $x$-axis is concerned, the array factor may be given by (3.43), using $x$-subscripts to emphasize this component, as

$$|(\text{AF})_x| = \left| \frac{\sin\left(\frac{M\Psi_x}{2}\right)}{\sin\left(\frac{\Psi_x}{2}\right)} \right| \tag{3.63}$$

where $\Psi_x = \beta_x + kd_x\cos\gamma$ (see equation (3.42)) where $\beta_x$ is the successive phase increment in the $x$-direction. In terms of the standard angles we note

$$\Psi_x = \beta_x + kd_x\sin\theta\cos\phi \tag{3.64}$$

A similar analysis along the $y$-axis gives

$$|(\text{AF})_y| = \left| \frac{\sin\left(\frac{N\Psi_y}{2}\right)}{\sin\left(\frac{\Psi_y}{2}\right)} \right| \tag{3.65}$$

where

$$\Psi_y = \beta_y + kd_y\sin\theta\sin\phi \tag{3.66}$$

$\beta_y$ is the successive phase increment in the $y$-direction and $\beta_y$ is independent of $\beta_x$.

Next, assume that the current magnitude is constant so that the current, $I_{mn}$, in the $mn^{\text{th}}$ element is

Compare this with equation (3.40) for the linear array. In keeping with the array factor for the linear array in equation (3.39), we may write the corresponding planar array factor as the product of the $x$ and $y$ array factors. Normalized to $I_0$

Thus, observing, (3.41)–(3.43) we write

$$|\mathrm{AF}| = |(\mathrm{AF})_x(\mathrm{AF})_y| = \left|\left(\frac{\sin\left(\frac{M\Psi_x}{2}\right)}{\sin\left(\frac{\Psi_x}{2}\right)}\right)\left(\frac{\sin\left(\frac{N\Psi_y}{2}\right)}{\sin\left(\frac{\Psi_y}{2}\right)}\right)\right| \tag{3.67}$$

and normalizing as before

$$|\mathrm{AF}|_n = \frac{1}{MN}\left|\left(\frac{\sin\left(\frac{M\Psi_x}{2}\right)}{\sin\left(\frac{\Psi_x}{2}\right)}\right)\left(\frac{\sin\left(\frac{N\Psi_y}{2}\right)}{\sin\left(\frac{\Psi_y}{2}\right)}\right)\right| \tag{3.68}$$

Observations and Notes:

1. Now, grating lobes may be formed when $d_x \geq \lambda/2$ or $d_y \geq \lambda/2$. (Any suggestion for the difference between this and the case for the linear array?)

2. To get only one "main beam" (principal maximum), and that being directed along $\theta = \theta_0$ , $\phi = \phi_0$,

$$\beta_x = -kd_x \sin\theta_0 \cos\phi_0$$
$$\beta_y = -kd_y \sin\theta_0 \sin\phi_0$$

   (i.e. $\Psi_x = \Psi_y = 0$). In general, $\beta_x$ and $\beta_y$ are used to "scan" the beam in the desired direction. In most practical applications, the $x$ and $y$ array factors should have maxima in the same direction. An example would be: if $(\mathrm{AF})_x$ is broadside, then $(\mathrm{AF})_y$ should be endfire – what would $\theta_0$ and $\phi_0$ have to be to direct the main beam along the $y$-axis?

3. Our theory has not included mutual coupling between elements. In fact, not all array elements need to be excited – some may be passively terminated (*parasitic*

elements). These parasitic elements are excited by the near-field coupling with the *driven* elements. In this case, additional nulls, not accounted for here, appear. The width and minimum power values of these nulls depends on the size of the array. The Yagi-Uda antenna (see the Balanis reference) is such an antenna which is widely used as a TV receiving antenna.

## 3.4 Long Wire Antennas

In the 2-30 MHz frequency band, efficient antennas may consist of long wires (several wavelengths in length). These may take a variety of shapes, including a single horizontal wire. Such antennas may be operated as narrow-band *resonant* structures or as broadband *travelling wave* structures. For ionospheric transmission of shortwave radio, various types of long-wire antennas are employed. Depending on the required ranges of long distance communication, the optimum angle of radiation is usually between 10° and 30° to a horizontal line in the direction of the receiving station.

Case 1. Resonant Antenna – Current Standing Wave on Wire

Consider a long-wire antenna whose length is $\ell = n\lambda/2$ where $n$ is a natural number and $\lambda$ is the wavelength of current and on which a standing wave current pattern, whose phasor form is

$$I(x') = I_0 \sin(kx') \ ,$$

exists. That is, the antenna is resonant and operates satisfactorily at a particular frequency (or one of its harmonics). As we already know, more power can be radiated by a long antenna than by a shorter one. Of course, on such an antenna, because of phase variations in current along its length, there may be regions of both constructive and destructive superposition in the resulting far-fields. These lead to a series of maxima and minima.

Referring to the above diagram, $\Psi$ is the polar angle relative to the direction of the wire and $E_\Psi$ is the radiated field ($\vec{E} = E_\Psi \hat{\Psi}$). By analogy to equation (3.5), we write for the vector potential

$$(3.69)$$

Starting with this expression, using $\vec{H} = \frac{1}{\mu}\vec{\nabla} \times \vec{A}$ and $\vec{E} = \eta\vec{H} \times \hat{r}$ for the far field and retaining only terms of order $1/r$, it is possible to show that

$$(3.70)$$

where, as usual, the primes are used to locate source points. It is not too difficult to show (TRY IT) that (3.70) evaluates to

from which

$$E_\Psi = \frac{\eta I_0}{2\pi r} e^{-jkr} e^{j\frac{n\pi}{2}(1+\cos\Psi)} \cdot \begin{cases} \dfrac{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\cos\Psi\right)}{\sin\Psi}, & n \text{ odd} \\ \dfrac{\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}\cos\Psi\right)}{\sin\Psi}, & n \text{ even} \end{cases} \tag{3.71}$$

The expression in the brace is, of course, the normalized $E$-field pattern.

From equation (3.71), the time-averaged power density, $|\vec{\mathcal{P}}_a|$, may be formalized in the usual way and the normalized magnitude of the radiation (power) pattern may be observed to be

$$|F(\Psi)|^2 = \begin{cases} \dfrac{\cos^2\left(\frac{n\pi}{2}\cos\Psi\right)}{\sin^2\Psi}, & n \text{ odd} \\ \dfrac{\sin^2\left(\frac{n\pi}{2}\cos\Psi\right)}{\sin^2\Psi}, & n \text{ even} \end{cases} \tag{3.72}$$

Plotting (3.72) reveals:

1. there are $n$ lobes or cones formed as the two-dimensional pattern is revolved around the wire axis;

2. The pattern is symmetric with respect to a plane perpendicular to the midpoint of the wire;

3. for $n$ even, nulls appear at $\Psi = \pm\pi/2$ ;

4. increasing $n$ results in narrower lobes; and

5. as $n$ increases the angle between the first major lobe maximum and the wire axis decreases.

Finally we note that the average radiated power, $P_r$ and radiation resistance, referred to the current maximum, may be determined as before from

$$P_r = \tag{3.73}$$

and

$$R_r = \tag{3.74}$$

Of course, the integral in (3.73) will not, in general, yield to a closed form result. These ideas as well as the procedure for finding the directivity etc. may be found in Sections 2.3.2 and 2.3.3.

Case 2. The Travelling Wave Antenna

The resonant antenna of Case 1 is by definition a narrow bandwidth antenna – i.e. it resonates for only a small band of frequencies. A similar structure of length $\ell$, which this time is terminated in a matched load, is the travelling wave antenna (see illustration):

The phasor form of the travelling wave current (see Term 6 notes) may be written as where $\alpha$ is the attenuation constant along the wire. If $\alpha$ is very small, to a good approximation, we may write

$$I = I_0 e^{-jkx'} \; . \qquad (3.75)$$

Recall that this is a travelling wave in time since the time-varying form is

$$\underset{\sim}{I} =$$

These antennas, which are often referred to as *Beverage* antennas when operated in the presence of imperfect ground, have an input impedance which is largely resistive. Thus, they are not greatly frequency dependent – i.e. they are *broadband* antennas.

Rewriting equation (3.70) for this case – i.e. with the new current expression – gives

(3.76)

This may be readily shown to reduce to

(3.77)

Maxima

It is possible to show that maxima occur at

$$\frac{\tan\beta}{\beta} = 2 - \frac{2\beta}{k\ell} \qquad (3.78)$$

where $\beta = k\ell \sin^2(\Psi/2)$ (it is good for practice to try this). The utility of this equation is that when $k\ell \gg 1$,

and

$$\sin^2\left(\frac{\Psi}{2}\right) = \frac{\beta}{k\ell}$$

from which $\Psi$ for maximum radiation follows directly.

Example: Determine the direction of maximum radiation for a travelling wave antenna with $\ell = 10\lambda$.

Notice that for a long structure, the antenna becomes like a near end-fire array. If $k\ell \ngtr 1$, then (3.78) must be solved numerically to determine maxima.

Nulls

From equation (3.77), nulls occur when

It is easy to establish that $\Psi = 0, \pi$ produces nulls. Using the last expression, it is easily verified that nulls occur when

$$\Psi = \cos^{-1}\left(1 \mp \frac{n\lambda}{\ell}\right) \tag{3.79}$$

As before, the techniques of Sections 2.3.2 and 2.3.3 may be used to determine radiated power, radiation resistance, directivity, etc..