Unit 2

Time-Varying Fields and Maxwell's Equations

While the Gauss law forms for the *static* electric and *steady* magnetic field equations remain essentially unchanged for the case of time-varying fields, the remaining two equations (see summary on page 11 of Unit 1 in these notes) must be revised. Once the proper forms are attained, we then have a general set of field equations which may be applied, in their broadest sense, to any classical electromagnetics problem. The modification of the $\oint_C \vec{E} \cdot d\vec{L} = 0$ equation is a result of Faraday's work, while Maxwell provided the required adjustment to Ampère's law.

2.1 Faraday's Law

We have earlier intimated that the form $\displaystyle\oint_C \vec{E} \cdot d\vec{L} = 0$ is NOT valid for the time-varying electric field. Note that we shall use the symbolism $\underset{\sim}{\vec{E}}$ to indicate such a field, remembering, of course, that the field will, in general, be a function of position $\vec{r}$ and time, t – i.e. $\underset{\sim}{\vec{E}} \equiv \vec{E}(\vec{r}, t)$.

In 1831, Faraday showed that whenever there is relative "motion" between a closed conducting path and a magnetic field – i.e. one or the other or both may be "moving" – an electromotive force (emf) is establised in the closed path. The statement of this fact, known as *Faraday's law*, is customarily written

$$\text{emf} = -\frac{d\Phi}{dt} \tag{2.1}$$

where $\dfrac{d\Phi}{dt}$ is the rate of change of magnetic flux. The minus sign indicates that the "induced" emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement is known as *Lenz's law*. It should be noted that Faraday's law implies <u>a closed path</u> and a magnetic flux Φ which passes through <u>a surface</u> whose boundary is the closed path. This surface is not unique. The orientation of the contour (path) and the flux direction are related by the right-hand rule as illustrated.

<u>Illustration:</u>

Here, $\Phi = \int_S \underset{\sim}{\vec{B}} \cdot d\vec{S}$ where, as usual, $d\vec{S} = \hat{n} dS$. Then the emf, in volts, from equation (2.1) is the *generated* emf which causes a *positive* charge to move (i.e. causes a current flow) in a closed path with the positive direction being indicated by the arrow. If the contour is conducting, we may think of $\underset{\sim}{\vec{E}}$ as being generated at at each point on C by the $\underset{\sim}{\vec{B}}$ field.

<u>Example 1: Emf due to changing field:</u> In the illustration below, $\underset{\sim}{\vec{B}} = B_0 \sin \omega_0 t \, \hat{z}$. If the closed path is conducting, determine the induced current, $i(t)$, assuming that the self-inductance of the loop is negligible.

[Pages 3 and 4 of Unit 2 should consist of illustrative examples, the second addressing an emf induced due to a steady field "cut" by a moving conducting path – done on looseleaf.]

We next define emf, in terms of a time-varying electric field intensity as the work per unit charge *done by the field* in moving charge around the path. This is leads to

$$\text{emf} = \oint_C \underset{\sim}{\vec{E}} \cdot d\vec{L} \tag{2.2}$$

and we notice that (1) there is no minus sign on the integral as there was for the definition of potential difference (why?) and (2) in general, different paths give different emfs (quite *unlike* the static case). In electrostatics, we use the term "potential difference" to indicate the result of the integral (not closed – the closed integral was 0 there), while with time-varying fields, the result is an emf or voltage.

Let us now seek an integral form of Faraday's law which incudes both the electric and magnetic fields. Again, we write Gauss' law (magnetic) for the non-closed surface as

$$\Phi = \int_S \underset{\sim}{\vec{B}} \cdot d\vec{S} \tag{2.3}$$

where the B-field now varies with time. Remembering that the closed contour C surrounds this surface we have from equations (2.1), (2.2) and (2.3)

$$\oint_C \underset{\sim}{\vec{E}} \cdot d\vec{L} = -\frac{d}{dt} \int_S \underset{\sim}{\vec{B}} \cdot d\vec{S} \tag{2.4}$$

If we assume a stationary path and a changing flux, then the magnetic flux density is the only time-varying quantity on the RHS of (2.4). Thus, we may use partial differentiation w.r.t. time and write

$$\oint_C \underset{\sim}{\vec{E}} \cdot d\vec{L} = -\int_S \frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \cdot d\vec{S} \tag{2.5}$$

This is one of Maxwell's equations (namely, Faraday's law) in integral form for time-varying fields.

Next we seek a "point form" of equation (2.5). Applying Stokes' law (recall, $\oint_C \vec{A}\cdot d\vec{L} = \int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}$) leads to

$$\int_S (\vec{\nabla}\times \underset{\sim}{\vec{E}})\cdot d\vec{S} = -\int_S \frac{\partial \underset{\sim}{\vec{B}}}{\partial t}\cdot d\vec{S} \tag{2.6}$$

These integrals are perfectly general and the surfaces and $d\vec{S}$ are the same on both sides of the equation. This leads to

$$\vec{\nabla}\times \underset{\sim}{\vec{E}} = -\frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \tag{2.7}$$

which is the corresponding "point" form of equation (2.5). We see that, in general, for time-varying fields, the $\vec{E}$-field is not conservative (i.e. $\oint_C \underset{\sim}{\vec{E}}\cdot d\vec{L} \neq 0$ or $(\vec{\nabla}\times \underset{\sim}{\vec{E}}) \neq 0$).

Finally, consider the case where both the field and the circuit are changing with time (i.e. a combination of the sort of things occurring in the previous two examples may be happening simultaneously). We know from equation (1.40) that the force per unit charge acting on charges moving with velocity $\vec{v}$ in a $\vec{B}$ field is given by $(\vec{v}\times\vec{B})$. This is defined as the *motional* electric field intensity, $\underset{\sim}{\vec{E}}_m$. Of course, as we have said, the B-field itself may be changing, in which case we use $\underset{\sim}{\vec{B}}$. We write, in general,

$$\underset{\sim}{\vec{E}}_m = \vec{v}\times \underset{\sim}{\vec{B}} \tag{2.8}$$

It is not difficult to argue (see text, pp 326–328) that the overall emf arising from motion of the circuit and changing of the field may be obtained directly from equations (2.2), (2.5) and (2.8) as

$$\text{emf} = \oint_C \underset{\sim}{\vec{E}}\cdot d\vec{L} = -\int_S \frac{\partial \underset{\sim}{\vec{B}}}{\partial t}\cdot d\vec{S} + \oint_C (\vec{v}\times \underset{\sim}{\vec{B}})\cdot d\vec{L} \tag{2.9}$$

2.2 Ampère's Law Revisited

The Problem with Ampère's Law for Time-varying Fields:

For the steady magnetic field we had Ampère's law as

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{2.10}$$

Taking the divergence on both sides yields

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

It is easy to show (see Appendix A.3 of the text) that for any vector, $\vec{A}$, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$. Therefore, the last expression applied to equation (2.10) implies

$$\vec{\nabla} \cdot \vec{J} = 0$$

However, from the conservation of charge, and using the time-varying current density, $\underset{\sim}{\vec{J}}$,

$$\vec{\nabla} \cdot \underset{\sim}{\vec{J}} + \frac{\partial \underset{\sim}{\rho_v}}{\partial t} = 0 \tag{2.11}$$

Thus, $\vec{\nabla} \cdot \underset{\sim}{\vec{J}} = 0$ can be true only if $\frac{\partial \underset{\sim}{\rho_v}}{\partial t} = 0$. That is, the form of Ampère's law in equation (2.10) is only valid if "things" are NOT time-varying. For the time-varying case, the law must be "FIXED"! We now propose such an adjustment.

The FIX

We have contended that Gauss' law (electric) holds for time-varying fields. That is, $\vec{\nabla} \cdot \vec{D} = \rho_v$ may be written as

$$\vec{\nabla} \cdot \underset{\sim}{\vec{D}} = \underset{\sim}{\rho_v} \tag{2.12}$$

where $\underset{\sim}{\vec{D}} \equiv \vec{D}(\vec{r}, t)$ and $\underset{\sim}{\rho_v} \equiv \rho_v(\vec{r}, t)$. Furthermore, the continuity equation (equation (2.11)) becomes

$$\vec{\nabla} \cdot \underset{\sim}{\vec{J}} + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \underset{\sim}{\vec{D}}) = 0 \tag{2.13}$$

Now, if we write

$$\vec{\nabla} \times \underset{\sim}{\vec{H}} = \underset{\sim}{\vec{J}} + \frac{\partial \underset{\sim}{\vec{D}}}{\partial t}, \tag{2.14}$$

Ampère's law agrees with the conservation of charge since

$$0 = \vec{\nabla} \cdot \vec{\nabla} \times \underset{\sim}{\vec{H}} = \vec{\nabla} \cdot [\underset{\sim}{\vec{J}} + \frac{\partial \underset{\sim}{\vec{D}}}{\partial t}]$$

where we have allowed the interchange of the $\vec{\nabla}\cdot$ and $\dfrac{\partial}{\partial t}$ on the displacement flux density.

Thus, equation (2.14) appears to be a valid adjustment to the non-time-varying case of Ampère's law. Equations (2.12) and (2.14) together imply that charge is conserved. To date, equation (2.14) has not been refuted!! Using Stokes' theorem again, the integral form of (2.14) becomes

<u>Displacement Current</u>

The additional term, $\dfrac{\partial \underset{\sim}{\vec{D}}}{\partial t}$, in equation (2.14) is referred to as *displacement current density*, $\underset{\sim}{\vec{J}}_d$ in units of A/m^2. This was Maxwell's big contribution to unifying the existing theory governing electromagnetics.

To help with the concept, consider the following filamentary loop through which passes a time-varying B-field. The loop ends are terminated by the plates of a parallel-plate capacitor.

It is not difficult to deduce that the current between the plates of the capacitor has the same value as the conduction current in the filament (see text, pp 332–333). The former is referred to as the <u>displacement current</u>, $\underset{\sim}{I}_d$.

$$\underset{\sim}{I}_d = S \left| \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} \right| = S \underset{\sim}{J}_d \tag{2.15}$$

where S is the surface area of the capacitor. (We have assumed that the D-field is distributed uniformly between the plates in this simple discussion).

<u>A Final Note on Current Density:</u>

Equation (2.14) has three different kinds of current density which may appear on the right hand side. One of these is clearly the displacement current density discussed above. However, $\underset{\sim}{\vec{J}}$ itself may be comprised of (1) *conduction current* density, $\underset{\sim}{\vec{J}}_{\mathrm{c}}$ and (2) *convection current* density, $\underset{\sim}{\vec{J}}_{\mathrm{cnv}}$. The former is given by the so-called point form of Ohm's law as

$$\underset{\sim}{\vec{J}}_{\mathrm{c}} = \sigma \underset{\sim}{\vec{E}} \tag{2.16}$$

where σ is conductivity of the material in question in mhos/metre (℧/m). The convection current density, such as is associated with beams of charged particles, is given by the familiar

$$\underset{\sim}{\vec{J}}_{\mathrm{cnv}} = \underset{\sim}{\rho_v} \vec{v} \tag{2.17}$$

where we have now allowed the quantities to be time-varying in all cases. In total,

$$\underset{\sim}{\vec{J}} = \underset{\sim}{\vec{J}}_{\mathrm{c}} + \underset{\sim}{\vec{J}}_{\mathrm{cnv}} \tag{2.18}$$