1.A Examples for the Sketching of Parametric Curves

A curve in $\mathbb{R}^3$ is a one-dimensional object. To locate any point on that curve requires the value of just one parameter (a real number). The Cartesian parametric equations of any curve are therefore

$$x = x(t), \quad y = y(t), \quad z = z(t), \text{ where } t \text{ is any real number.}$$

The Cartesian vector parametric equation is

$$\vec{\mathbf{r}}(t) \;=\; x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}, \text{ where } t \text{ is any real number.}$$

If the parameter t is the time, then $\mathbf{r}(t)$ describes the location of a particle at any time t.

The velocity of the particle is just $\vec{\mathbf{v}}(t) \;=\; \dfrac{d\vec{\mathbf{r}}}{dt} \;=\; \dfrac{dx}{dt}\,\hat{\mathbf{i}} + \dfrac{dy}{dt}\,\hat{\mathbf{j}} + \dfrac{dz}{dt}\,\hat{\mathbf{k}}$.

The acceleration is $\vec{\mathbf{a}}(t) \;=\; \dfrac{d\vec{\mathbf{v}}}{dt} \;=\; \dfrac{d^2\vec{\mathbf{r}}}{dt^2} \;=\; \dfrac{d^2x}{dt^2}\,\hat{\mathbf{i}} + \dfrac{d^2y}{dt^2}\,\hat{\mathbf{j}} + \dfrac{d^2z}{dt^2}\,\hat{\mathbf{k}}$.

Example 1.A.1

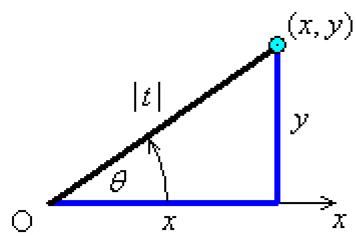
Sketch the curve in $\mathbb{R}^2$ whose Cartesian equation in parametric form is

$$x = t\cos t\,, \quad y = t\sin t$$

$$x^2 + y^2 \;=\; t^2\left(\cos t + \sin t\right) \;=\; t^2$$

(distance from O) $=$ $|\,t\,|$

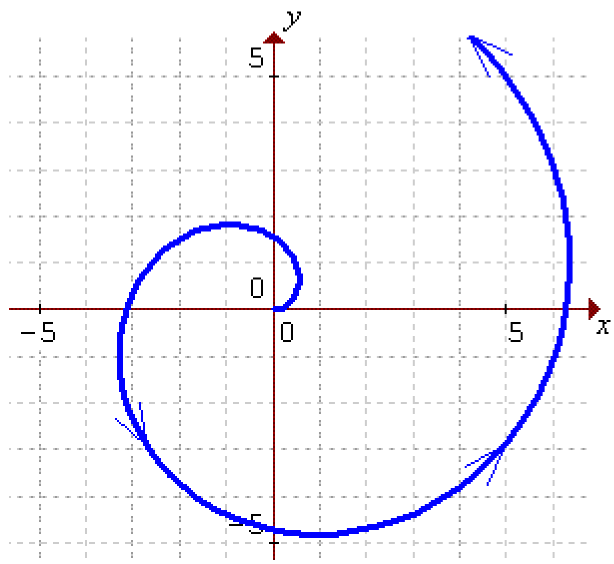
$$\frac{y}{x} \;=\; \tan t$$



$\theta \;=\; \text{angle } t \quad (\text{for } t > 0)$

<u>Example 1.A.1</u> (continued)

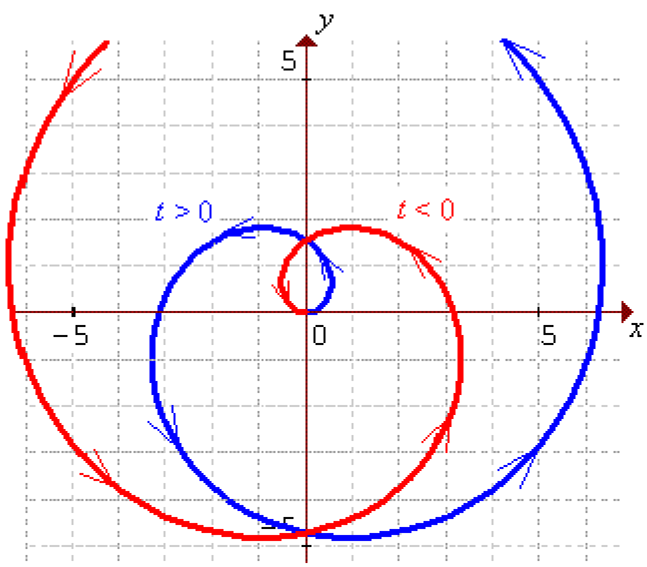
Therefore the curve is a spiral outwards from O, with period 2π.



Incorporating negative values of the parameter *t* yields a mirror image of this curve:

$$t \to -t \qquad \Rightarrow \quad x \to -t\cos(-t) = -x\,, \quad y \to -t\sin(-t) = +y$$

$\Rightarrow$ reflection in *y* axis of $t > 0$ is $t < 0$.



In polar coordinates (section 1.2), the equation of this spiral is just $r = \theta$.

<u>Example 1.A.2</u>

The parametric form of the Cartesian equation of a curve in $\mathbb{R}^2$ is

$$x = t^2, \quad y = t^3$$

(a) Sketch the curve.

(b) What happens to the principal unit tangent $\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$ at the origin?

(a) $\dfrac{dx}{dt} = 2t , \qquad \dfrac{dy}{dt} = 3t^2$

The only value of t at which any of $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ is zero is $t = 0$.

$t > 0 \quad \Rightarrow \quad x > 0, \quad y > 0, \quad \dfrac{dx}{dt} > 0 \quad \text{and} \quad \dfrac{dy}{dt} > 0$

$t = 0 \quad \Rightarrow \quad x = 0, \quad y = 0, \quad \dfrac{dx}{dt} = 0 \quad \text{and} \quad \dfrac{dy}{dt} = 0$

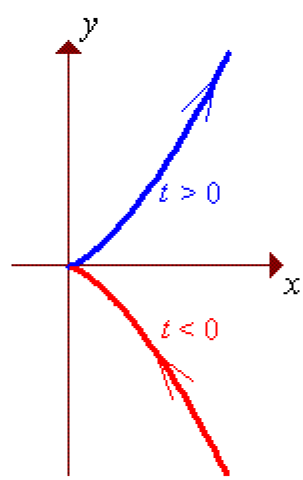
$t < 0 \quad \Rightarrow \quad x > 0, \quad y < 0, \quad \dfrac{dx}{dt} < 0 \quad \text{and} \quad \dfrac{dy}{dt} > 0$

Therefore, for $t < 0$, the curve is moving up and left through the fourth quadrant, arriving at the origin at $t = 0$. Thereafter, the curve is moving up and right through the first quadrant. No part of the curve is to the left of the y-axis.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3t^2}{2t} = \frac{3}{2}t, \quad (t \neq 0) \qquad \Rightarrow \quad \lim_{t \to 0} \frac{dy}{dx} = 0$$

There must therefore be a horizontal tangent (and a cusp) at the origin.

Sketch of $x = t^2, \quad y = t^3$:



Example 1.A.2 (continued)

Examination of concavity will help to confirm the behaviour near the origin.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\cdot\frac{dt}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dt}\div\frac{dx}{dt}\right)\div\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{3t^2}{2t}\right)\div(2t) = \frac{3}{2}\left(\frac{d}{dt}(t)\right)\div(2t) = \frac{3}{4t}$$

The curve is therefore concave down everywhere in the fourth quadrant ($t < 0$) and concave up everywhere in the first quadrant ($t > 0$).

(b) The tangent vector is

$$\frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 2t, 3t^2 \right\rangle = t\left\langle 2, 3t\right\rangle \quad \Rightarrow \quad \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = |t|\sqrt{4+9t^2}$$

The unit tangent vector, everywhere on the curve except at the origin, is

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{t}{|t|}\cdot\frac{1}{\sqrt{4+9t^2}}\left\langle 2, 3t\right\rangle$$

$$t > 0 \quad \Rightarrow \quad |t| = t \quad \Rightarrow \quad t/|t| = +1$$

$$\lim_{t\to 0^+} \hat{\mathbf{T}} = +\frac{1}{\sqrt{4+0}}\left\langle 2, 0\right\rangle = +\left\langle 1, 0\right\rangle = +\hat{\mathbf{i}}$$

$$t < 0 \quad \Rightarrow \quad |t| = -t \quad \Rightarrow \quad t/|t| = -1$$

$$\lim_{t\to 0^-} \hat{\mathbf{T}} = (-1)\frac{1}{\sqrt{4+0}}\left\langle 2, 0\right\rangle = -\left\langle 1, 0\right\rangle = -\hat{\mathbf{i}} \neq \lim_{t\to 0^+} \hat{\mathbf{T}}$$

The unit tangent therefore is undefined at the origin.
It flips direction abruptly, from **–i** to +**i**, as the curve passes through the cusp at the origin;

the curve reverses direction suddenly at the cusp.

Example 1.A.3

For the curve whose Cartesian equation in parametric form is

$$x = t^2, \quad y = t^3 - 3t$$

(a) Find the tangents at the point $(x, y) = (3, 0)$.
(b) Sketch the curve.

(a)

$$x = t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 - 3$$

$$x = 3 \quad \Rightarrow \quad t = \pm\sqrt{3}$$

and

$$y = 0 \quad \Rightarrow \quad t\left(t^2 - 3\right) = 0 \quad \Rightarrow \quad t = 0, \ \pm\sqrt{3}$$

$$\therefore \quad \text{at } (x, y) = (3, 0), \quad t = \pm\sqrt{3}$$

The curve therefore passes through the point (3, 0) *twice*.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{(3,0)} = \frac{3\left(\left(\pm\sqrt{3}\right)^2 - 1\right)}{2\left(\pm\sqrt{3}\right)} = \frac{3 \times 2}{\pm 2\sqrt{3}} = \pm\sqrt{3}$$

The two slopes are distinct.
The curve therefore crosses itself at (3, 0).
In Cartesian coordinates, the equations of the two tangents to the curve at the point (3, 0) are

$$\boxed{y = \pm\sqrt{3}\left(x - 3\right)}$$

Example 1.A.3 (continued)

(b)

$x = 0$ at $t = 0$

$x > 0$ elsewhere.

$y = t^3 - 3t = t\left(t^2 - 3\right) = 0$ at $t = 0, \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{3\left(t^2 - 1\right)}{2t} = 0 \quad \text{at} \quad t = \pm 1$$

$\dfrac{dy}{dx}$ is undefined at $t = 0$.

The values of t, at which at least one of $x, y, \dfrac{dx}{dt}, \dfrac{dy}{dt}$ are zero, are

$t = -\sqrt{3}, -1, 0, +1, +\sqrt{3}$:

Construct a table to aid in sketching the curve:

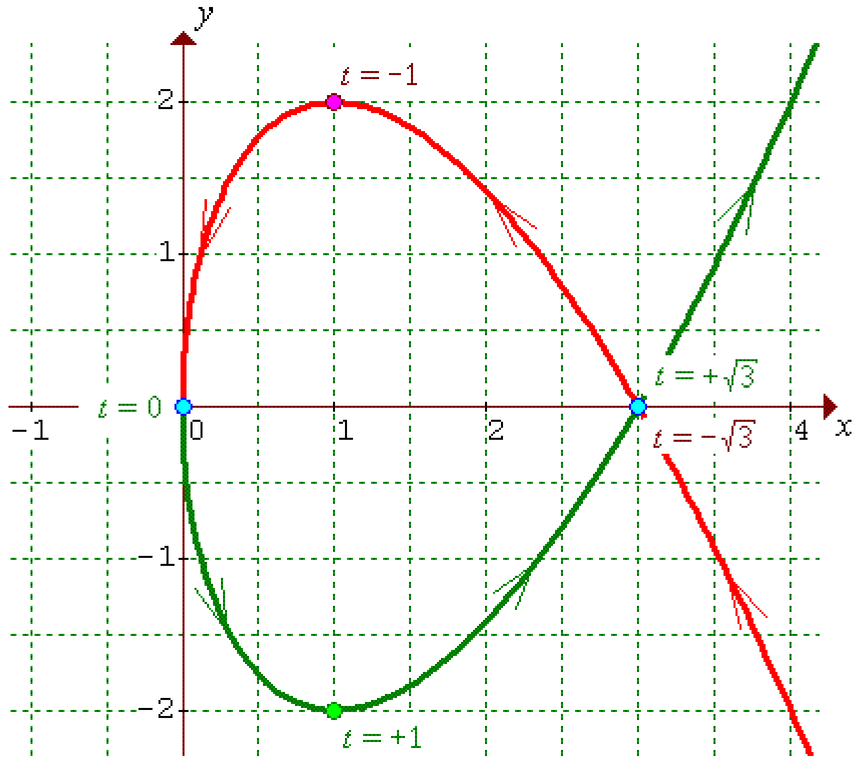
t		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
x	+		+		+	0	+		+		+
y	−	0	+		+	0	−		−	0	+
$\frac{dy}{dt}$	+		+	0	−		−	0	+		+
$\frac{dx}{dt}$	−		−		−	0	+		+		+
curve	↖		↖	←	↙	↓	↘	→	↗		↗
				MAX				MIN			

$t = \pm 1 \quad \Rightarrow \quad (x, y) = (1, \mp 2)$

$t = \pm\sqrt{3} \quad \Rightarrow \quad (x, y) = (3, 0)$

Example 1.A.3 (continued)

Sketch of $x = t^2, \quad y = t^3 - 3t$:



Concavity:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\cdot\frac{dt}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dt}\div\frac{dx}{dt}\right)\div\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{3}{2}t - \frac{3}{2}t^{-1}\right)\div(2t) = \left(\frac{3}{2} + \frac{3}{2}t^{-2}\right)\div(2t)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{3\left(t^2+1\right)}{4t^3}$$

The curve is therefore concave up for $t > 0$
and concave down for $t < 0$.
This confirms the sketch.

<u>Example 1.A.4</u>

Determine the shape of the curve whose Cartesian equation in parametric form is

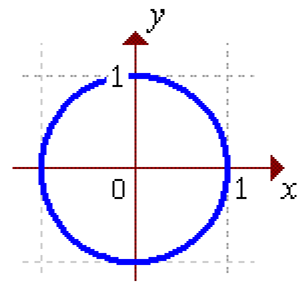
$$x = \cos t, \quad y = \sin t, \quad z = t$$

Examine the projections of the curve onto the three coordinate planes:

In the x-y plane $z = 0$ and $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

which is a circle, centre O, radius 1.

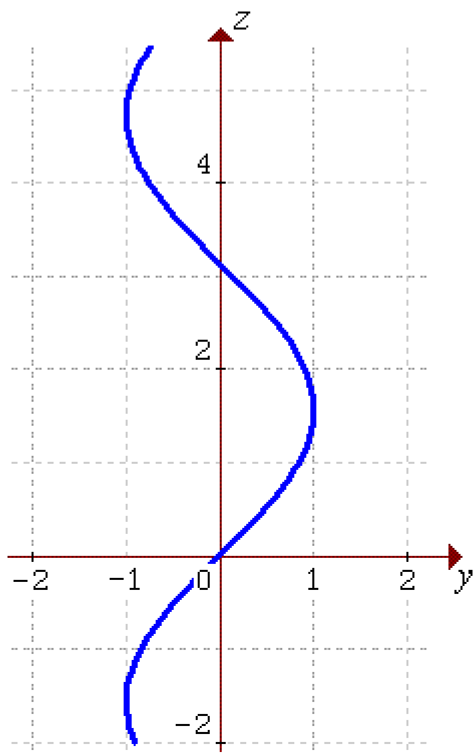
Top view:

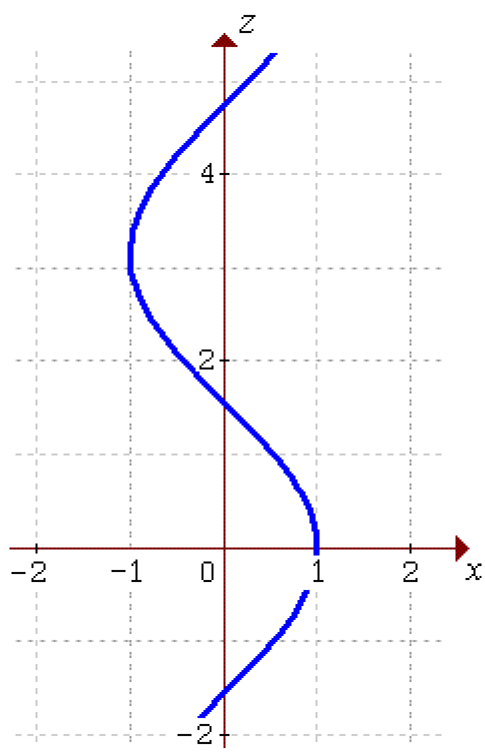


In the y-z plane $x = 0$ and $y = \sin t = \sin z$

In the x-z plane $y = 0$ and $x = \cos t = \cos z$

Side views:

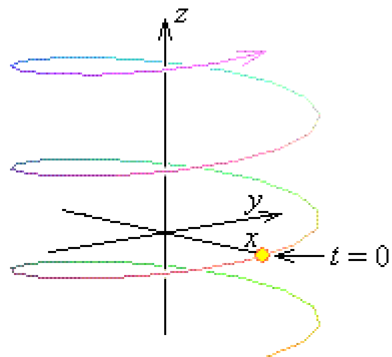




The curve is therefore a helix, centred on the z axis, radius 1, rotating once around the z axis for every change of 2π in z.

Example 1.A.4 (continued)

Modified Maple plot:



A Maple file that generates a plot of this helix is available from the course web site, in the programs directory:
"`http://www.engr.mun.ca/~ggeorge/2422/programs/`".

1.B Tangential and Normal Components of Velocity and Acceleration

The tangent vector to a curve $\mathbf{r}(t)$ is $\vec{\mathbf{T}}(t) = \dfrac{d\vec{\mathbf{r}}}{dt} = \dfrac{dx}{dt}\hat{\mathbf{i}} + \dfrac{dy}{dt}\hat{\mathbf{j}} + \dfrac{dz}{dt}\hat{\mathbf{k}}$

If the parameter t is the time, then the tangent vector is also the velocity vector $\mathbf{v}(t)$.
The tangential component v_T of velocity $\mathbf{v}(t)$ is just the speed $v(t)$.
There is no component of velocity in the normal plane.

The speed $v(t)$ is a scalar quantity: $v(t) = \left|\vec{\mathbf{T}}(t)\right| = \left|\dfrac{d\vec{\mathbf{r}}}{dt}\right| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$

But the arc length (distance measured along the curve) s is defined by

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

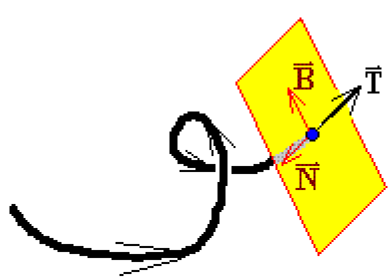
Therefore the speed $v(t) = \left|\dfrac{d\vec{\mathbf{r}}}{dt}\right| = \dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$

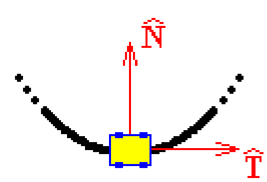
and the unit tangent vector is

$$\boxed{\hat{\mathbf{T}} = \frac{\vec{\mathbf{T}}}{\left|\vec{\mathbf{T}}\right|} = \frac{\vec{\mathbf{v}}}{v} = \frac{d\vec{\mathbf{r}}}{dt} \div \frac{ds}{dt} = \frac{d\vec{\mathbf{r}}}{ds}}.$$

As seen in Example 1.A.2 above, $\hat{\mathbf{T}}$ is ill-defined where $\dfrac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{0}}$.

As a curve travels through $\mathbb{R}^3$, its tangent vector points straight ahead, defining a normal plane at right angles to that tangent vector.





Imagine that a roller coaster car is travelling along the curve, with the front in the direction of travel and oriented so that the side doors are in the direction in which the car is instantaneously turning. Then the direction in which $\hat{\mathbf{T}}$ is changing defines the principal unit normal $\hat{\mathbf{N}}$, (except where the curve is straight or has a point of inflexion).

By definition, the magnitude of any unit vector is 1 and therefore is absolutely constant. Only the direction of a unit vector can change. The natural parameter to use for any curve (though usually not the most convenient in practice) is the distance travelled along the curve: the arc length *s*. Therefore define the **principal normal vector** to be the derivative of the unit tangent vector with respect to arc length:

$$\boxed{\vec{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \frac{ds}{dt}}$$

from which it follows that the **unit principal normal vector** is

$$\boxed{\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds} \div \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{d\hat{\mathbf{T}}}{dt} \div \left|\frac{d\hat{\mathbf{T}}}{dt}\right|}$$

The magnitude of the principal normal vector is a measure of how sharply the curve is turning. It is therefore the **curvature**,

$$\boxed{\kappa = \left|\vec{\mathbf{N}}\right| = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\frac{d\hat{\mathbf{T}}}{dt}\right| \div \frac{ds}{dt}}$$

and $\vec{\mathbf{N}} = \kappa \hat{\mathbf{N}}$.

The tangent and principal normal vectors are orthogonal to each other everywhere on the curve. A third unit vector, orthogonal everywhere to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$, is the **unit binormal vector**, defined simply as

$$\boxed{\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}}$$

These three unit vectors form an orthonormal set of vectors at every point on the curve where they are defined.

Example 1.B.1

Find the unit tangent, normal and binormal vectors everywhere on the helix

$$\vec{\mathbf{r}}(t) = \langle x, y, z \rangle = \langle \cos t, \sin t, t \rangle$$

$$\vec{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\langle \cos t, \sin t, t \rangle = \langle -\sin t, \cos t, 1 \rangle$$

$$\frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\Rightarrow \quad \hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \frac{ds}{dt} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

$$\vec{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \frac{ds}{dt} = \frac{d}{dt}\left(\frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}} \right) \times \frac{1}{\sqrt{2}} = \frac{1}{2}\langle -\cos t, -\sin t, 0 \rangle$$

$$\left| \vec{\mathbf{N}} \right| = \frac{1}{2} \qquad \Rightarrow \quad \hat{\mathbf{N}} = \frac{\vec{\mathbf{N}}}{\left| \vec{\mathbf{N}} \right|} = \langle -\cos t, -\sin t, 0 \rangle$$

[The unit principal normal therefore points directly towards the z axis at all times.]

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1 \rangle$$

One can easily show that

$\hat{\mathbf{T}} \bullet \hat{\mathbf{N}} = \hat{\mathbf{N}} \bullet \hat{\mathbf{B}} = \hat{\mathbf{B}} \bullet \hat{\mathbf{T}} = 0$ [all three vectors are orthogonal]

and that

$\hat{\mathbf{T}} \bullet \hat{\mathbf{T}} = \hat{\mathbf{N}} \bullet \hat{\mathbf{N}} = \hat{\mathbf{B}} \bullet \hat{\mathbf{B}} = 1$ [all three vectors are unit vectors]

We know that the velocity vector is purely tangential: $\vec{\mathbf{v}} = v\hat{\mathbf{T}}$.

The acceleration vector is therefore

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{dv}{dt}\hat{\mathbf{T}} + v\frac{d\hat{\mathbf{T}}}{dt}$$

But $\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{dt} \div \left|\frac{d\hat{\mathbf{T}}}{dt}\right| \quad \Rightarrow \quad \frac{d\hat{\mathbf{T}}}{dt} = \left|\frac{d\hat{\mathbf{T}}}{dt}\right|\hat{\mathbf{N}} = \left|\frac{d\hat{\mathbf{T}}}{ds}\cdot\frac{ds}{dt}\right|\hat{\mathbf{N}} = \kappa v\,\hat{\mathbf{N}}$

$$\Rightarrow \quad \vec{\mathbf{a}} = \frac{dv}{dt}\hat{\mathbf{T}} + \kappa v^2\hat{\mathbf{N}}$$

The tangential and normal components of acceleration are therefore

$$\boxed{a_T = \frac{dv}{dt}} \quad \text{and} \quad \boxed{a_N = \kappa v^2}$$

An alternative form for the normal component of acceleration is

$$\boxed{a_N = v\left|\frac{d\hat{\mathbf{T}}}{dt}\right| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| \cdot \left|\frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right|\right)\right|}$$

There is no binormal component of acceleration.

Example 1.B.2

Find the tangential and normal components of velocity and acceleration everywhere on the helix

$$\vec{\mathbf{r}}(t) = \langle x, y, z \rangle = \langle \cos t, \sin t, t \rangle$$

$$v_T = v = |\vec{\mathbf{v}}| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$a_T = \frac{dv}{dt} = 0$$

$$\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{v} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{T}}}{dt} = \frac{\langle -\cos t, -\sin t, 0 \rangle}{\sqrt{2}}$$

$$\Rightarrow \quad a_N = v\left|\frac{d\hat{\mathbf{T}}}{dt}\right| = \sqrt{2}\,\frac{\sqrt{\cos^2 t + \sin^2 t + 0}}{\sqrt{2}} = 1$$

OR

$$\Rightarrow \quad \kappa = \left|\frac{d\hat{\mathbf{T}}}{dt}\right| \div v = \frac{\sqrt{\cos^2 t + \sin^2 t + 0}}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \quad a_N = \kappa v^2 = \frac{1}{2}\left(\sqrt{2}\right)^2 = 1$$

OR

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \langle -\cos t, -\sin t, 0 \rangle \quad \Rightarrow \quad a = |\vec{\mathbf{a}}| = 1$$

$$a_T^{\,2} + a_N^{\,2} = a^2 \quad \text{and} \quad a_T = 0 \quad \Rightarrow \quad a_N = a = 1$$

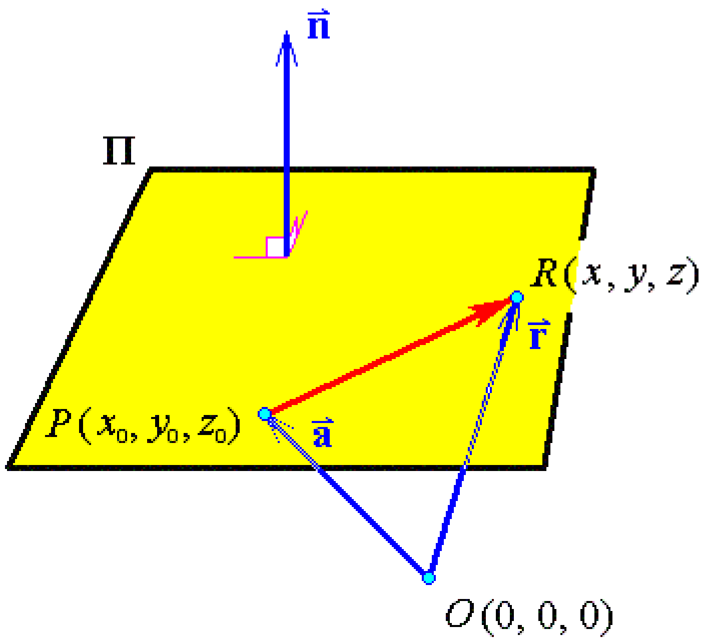
Therefore

$$\boxed{\vec{\mathbf{v}} = \sqrt{2}\,\hat{\mathbf{T}}} \quad \text{and} \quad \boxed{\vec{\mathbf{a}} = 0\,\hat{\mathbf{T}} + 1\,\hat{\mathbf{N}}}$$

1.1 Equation of a Plane

A non-zero vector $\vec{\mathbf{n}}$ in $\mathbb{R}^3$ will fix the orientation of a plane, to be at right angles to $\vec{\mathbf{n}}$.
Let P be a point that is known to be on the plane. Let the Cartesian coordinates of P be (x_0, y_0, z_0). Its position vector, $\vec{\mathbf{a}} = \overrightarrow{OP} = \langle x_o, y_o, z_o \rangle$, allows us to pick out exactly one plane from the infinite set of parallel planes that share the same orientation defined by the plane normal vector $\vec{\mathbf{n}}$.
The two vectors, together, allow us to define a single plane completely.

Let $\vec{\mathbf{r}} = \overrightarrow{OR} = \langle x, y, z \rangle$ be the position vector of a general point R, with Cartesian coordinates (x, y, z), in $\mathbb{R}^3$.



But $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} \quad \Rightarrow \quad \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{\mathbf{r}} - \vec{\mathbf{a}}$

Note that the normal vector **n** is at right angles to any vector lying in the plane Π.
R on $\Pi \quad \Rightarrow \quad \overrightarrow{PR} \perp \vec{\mathbf{n}}$ **("perpendicular to")**

$\Rightarrow \quad \overrightarrow{PR} \bullet \vec{\mathbf{n}} = 0$
$\Rightarrow \quad (\vec{\mathbf{r}} - \vec{\mathbf{a}}) \bullet \vec{\mathbf{n}} = 0$
OR

$$\boxed{\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = \vec{\mathbf{a}} \bullet \vec{\mathbf{n}}}$$

which is the vector equation of a plane.

Let $\vec{\mathbf{n}} = \langle A, B, C \rangle$, (so that A, B, C are the Cartesian components of the normal vector), then

$$\vec{\mathbf{a}} \bullet \vec{\mathbf{n}} = Ax_o + By_o + Cz_o = -D \quad (\text{a constant})$$

$$\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = Ax + By + Cz$$

The vector equation of the plane, $\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = \vec{\mathbf{a}} \bullet \vec{\mathbf{n}}$, then leads to the
Cartesian equation of the plane:

$$\boxed{Ax + By + Cz + D = 0}$$

Example 1.1.1

Find an equation for the plane through the point (3, –2, 4), which is normal to the vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

$$\vec{\mathbf{n}} = \langle A, B, C \rangle = \langle 2, 3, 1 \rangle$$

$$\vec{\mathbf{a}} = \langle x_o, y_o, z_o \rangle = \langle 3, -2, 4 \rangle$$

$$\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = \vec{\mathbf{a}} \bullet \vec{\mathbf{n}} \quad \Rightarrow \quad 2x + 3y + 1z = 2 \times 3 + 3 \times (-2) + 1 \times 4$$

Therefore the Cartesian equation of the required plane is

$$\boxed{2x + 3y + z = 4}$$

Example 1.1.2

Find a unit vector orthogonal to the plane

$$x - 2y + 2z = 7$$

From the Cartesian equation of a plane, one may read off the Cartesian coordinates of a vector that is at right angles to that plane.

$$\vec{\mathbf{n}} = \langle 1, -2, 2 \rangle$$

$$\Rightarrow \quad n = |\vec{\mathbf{n}}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

Therefore a unit normal is

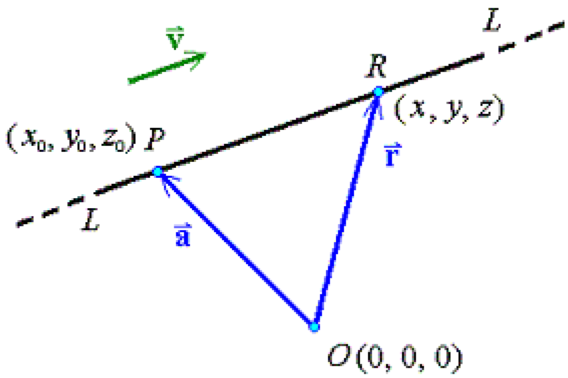
$$\boxed{\hat{\mathbf{n}} = \tfrac{1}{3}\langle 1, -2, 2 \rangle}$$

[Note that there is one other acceptable final answer, namely the unit vector that points in the opposite direction, $\hat{\mathbf{n}} = \tfrac{1}{3}\langle -1, 2, -2 \rangle$.]

Equations of a Line

A line L is determined uniquely by two vectors, a line direction vector, **v**, which orients the line in $\mathbb{R}^3$, and the position vector, **a**, of a point P known to be on that line.

Let R be a general point, with Cartesian coordinates (x, y, z), that is constrained to lie on the line L.



$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} \quad \Rightarrow \quad \vec{\mathbf{r}} = \vec{\mathbf{a}} + \overrightarrow{PR}$$

But the line L is parallel to the direction vector **v**.

$\Rightarrow \quad \overrightarrow{PR} = t\,\vec{\mathbf{v}}$ for some value of t.

Therefore the **parametric vector equation of the line** is

$$\boxed{\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}}} \qquad (t \in \mathbb{R})$$

[Note that there is one free parameter, t, which can be any real number. The number of free parameters (one) matches the dimension of the geometric object (the line is a one-dimensional object).]

The Cartesian equations of the line can be found from the vector parametric form.

Example 1.1.3

Find the Cartesian equations of the line that passes through the point (3, 1, 2) and that is parallel to the vector $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

$\vec{\mathbf{a}} = \langle 3, 1, 2 \rangle$, $\quad \vec{\mathbf{v}} = \langle 1, -2, 3 \rangle$

The vector parametric form is

$$\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}}$$

or

$$\langle x, y, z \rangle = \langle (3+t), (1-2t), (2+3t) \rangle$$

Making t the subject of all three simultaneous equations

$x = 3 + t$,

$y = 1 - 2t$

and $z = 2 + 3t$,

we obtain the **Cartesian symmetric form** for the equations of the line:

$$\boxed{t = \underline{\underline{\frac{x-3}{1} = \frac{y-1}{-2} = \frac{z-2}{3}}}}$$

General case:

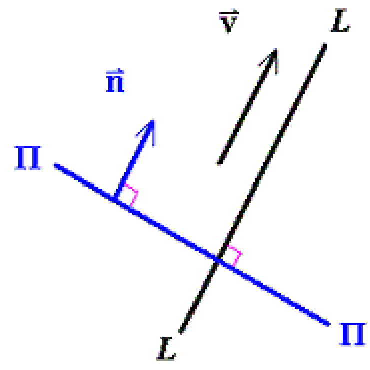
A line $\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}}$ which passes through the point (x_o, y_o, z_o) and is parallel to the vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, (where v_1, v_2, v_3 are all non-zero) has a Cartesian symmetric form

$$\boxed{\frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3}}$$

If $v_1 = 0$, then separate out the equation $x = x_o$.
If $v_2 = 0$, then separate out the equation $y = y_o$.
If $v_3 = 0$, then separate out the equation $z = z_o$.

Example 1.1.4

Find the Cartesian equations of the line through (1, 1, −1) that is perpendicular to the plane $2x + 3z = 1$.



The normal vector $\mathbf{n}$ to the plane is at right angles to the plane Π.
The line L is also at right angles to the plane Π.
Therefore the line's direction vector **v** must be parallel to **n**.

$$\vec{\mathbf{v}} = \vec{\mathbf{n}} = \langle 2, 0, 3 \rangle$$

$$\vec{\mathbf{a}} = \langle 1, 1, -1 \rangle$$

The vector parametric form is

$$\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}}$$

or

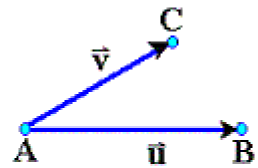
$$\langle x, y, z \rangle = \langle (1+2t),\ 1,\ (-1+3t) \rangle$$

Making t the subject of the equations where possible (for x and z), the Cartesian symmetric form then follows:

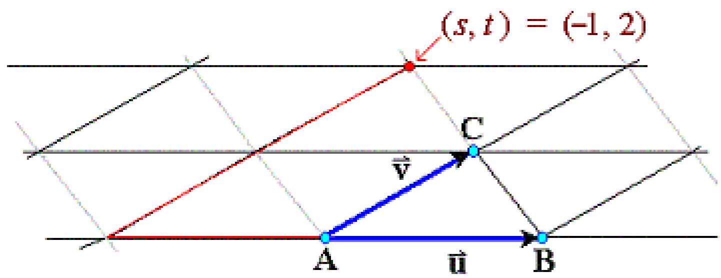
$$\boxed{y = 1, \quad \frac{x-1}{2} = \frac{z+1}{3}}$$

More Equations of Lines and Planes

Three non-collinear points, A, B, C, define a plane.



The three vectors joining the three points to each other lie in that plane. Any two of them may be used as the basis for a coordinate grid that can be laid out on the entire plane.



The **vector parametric equation of the plane** then follows:

$$\boxed{\vec{\mathbf{r}} = \vec{\mathbf{a}} + s\vec{\mathbf{u}} + t\vec{\mathbf{v}}} \qquad s \in \mathbb{R}, t \in \mathbb{R}$$

where

$\vec{\mathbf{a}} =$ any one of $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$;

$\vec{\mathbf{u}}, \vec{\mathbf{v}} =$ any two of $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$;

and $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{n}}$.

The other vector equation for the plane can be recovered from this form:
Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both in the plane
$\Rightarrow \quad \vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AC}$ is normal to the plane.
Let R be a general point in space, with Cartesian coordinates (x, y, z).
If and only if point R lies in the plane, then

$$\overrightarrow{AR} \bullet \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0 \qquad \Rightarrow \qquad (\vec{\mathbf{r}} - \vec{\mathbf{a}}) \bullet \vec{\mathbf{n}} = 0$$

Example 1.1.5

Find the Cartesian equation of the plane that passes through the points $A(1, 0, 0)$, $B(2, 3, 4)$ and $C(-1, 2, 1)$.

$$\overrightarrow{AB} = \langle (2-1), (3-0), (4-0) \rangle = \langle 1, 3, 4 \rangle$$

$$\overrightarrow{AC} = \langle -2, 2, 1 \rangle$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 4 \\ -2 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \vec{\mathbf{n}} = \langle -5, -9, +8 \rangle$$

Let R be a general point in $\mathbb{R}^3$, with Cartesian coordinates (x, y, z). Then

$$\vec{\mathbf{r}} - \vec{\mathbf{a}} = \overrightarrow{AR} = \langle (x-1), (y-0), (z-0) \rangle = \langle x-1, y, z \rangle$$

For R to be on the plane,

$$(\vec{\mathbf{r}} - \vec{\mathbf{a}}) \bullet \vec{\mathbf{n}} = 0 \qquad \Rightarrow \quad \overrightarrow{AR} \bullet \left(\overrightarrow{AB} \times \overrightarrow{AC} \right) = 0$$

$$\Rightarrow \quad (x-1)(-5) + y(-9) + z(8) = 0$$

$$\Rightarrow \quad -5x - 9y + 8z = 5$$

or

$$\boxed{5x + 9y - 8z = 5}$$

Alternative solution (next page):

Alternative Solution to Example 1.1.5:

The plane Π is $Ax + By + Cz + D = 0$
This equation must be true at all three points on the plane.
An under-determined linear system of three equations for the four unknown coefficients then follows:

$$
\begin{array}{l|c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
(1,0,0) \text{ on } \Pi: & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \end{array}\right] \\
(2,3,4) \text{ on } \Pi: & \left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 0 \end{array}\right] \\
(-1,2,1) \text{ on } \Pi: & \left[\begin{array}{cccc|c} -1 & 2 & 1 & 1 & 0 \end{array}\right]
\end{array}
$$

Row reduction leads to the reduced row echelon form

$$
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 9/5 & 0 \\ 0 & 0 & 1 & -8/5 & 0 \end{array}\right]
$$

$$\Rightarrow \quad (A,B,C,D) = \left(-D, -\tfrac{9}{5}D, \tfrac{8}{5}D, D\right), \quad D \in \mathbb{R}$$

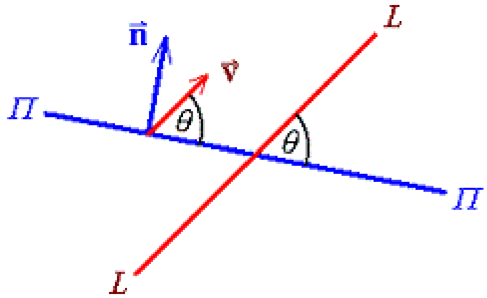
Select a convenient non-zero value for D that leaves all four coefficients as integers and makes at least two of A, B and C non-negative. Therefore select $D = -5$:

$$\boxed{5x + 9y - 8z - 5 = 0}$$

<u>Example 1.1.6</u>

Find the angle between the line $L\ :\ x = 1\,,\quad \dfrac{y-2}{3} = \dfrac{z-3}{2}$

and the plane $\Pi\ :\ x - 2y + z = 4\,.$



$$|\sin\theta| = \frac{|\vec{\mathbf{n}}\bullet\vec{\mathbf{v}}|}{n\,v} = \frac{|\langle 1,-2,1\rangle\bullet\langle 0,3,2\rangle|}{\sqrt{1+4+1}\sqrt{0+9+4}} = \frac{|-4|}{\sqrt{6\times 13}} \approx .4529$$

Therefore

$$\boxed{\theta \approx 26.9°}$$

<u>Example 1.1.7</u>

Find the intersection of the planes

$\Pi_1:\quad x - 2y + z = 1$ and

$\Pi_2:\quad 3x - 5y + 2z = 4.$

Solve the under-determined linear system

$$\begin{array}{ll} & \quad x \quad\ y \quad z \\ \Pi_1 & [\ 1\ \ -2\ \ \ 1\ |\ 1\ ] \\ \Pi_2 & [\ 3\ \ -5\ \ \ 2\ |\ 4\ ] \end{array}$$

$$R_2 \leftarrow R_2 - 3R_1: \quad \begin{array}{l} [\ 1\ \ -2\ \ \ \ 1\ |\ 1\ ] \\ [\ 0\ \ \ \ 1\ \ -1\ |\ 1\ ] \end{array}$$

$$R_1 \leftarrow R_1 + 2R_2: \quad \begin{array}{l} [\ 1\ \ \ 0\ \ -1\ |\ 3\ ] \\ [\ 0\ \ \ 1\ \ -1\ |\ 1\ ] \end{array}$$

$$\Rightarrow \quad \begin{array}{l} x = t + 3\,, \\ y = t + 1\,, \\ z = t\,, \qquad t \in \mathbb{R} \end{array}$$

Example 1.1.7 (continued)

$$\boxed{\Rightarrow \quad t = \frac{x-3}{1} = \frac{y-1}{1} = \frac{z-0}{1}}$$

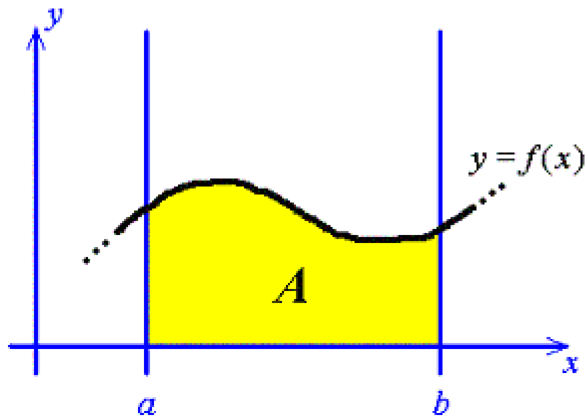
The intersection is a LINE.
In vector parametric form, it is

$$\mathbf{r} = \mathbf{a} + t\,\mathbf{v}, \quad \text{with} \quad \mathbf{a} = \langle 3, 1, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 1, 1, 1 \rangle .$$

<u>Summary for Lines and Planes</u>:

	Vector forms	Cartesian form
Line	$\vec{\mathbf{r}} = \vec{\mathbf{a}} + t\,\vec{\mathbf{v}}$ **a** = point on line, **v** = direction vector	$\frac{x-x_o}{v_1} = \frac{y-y_o}{v_2} = \frac{z-z_o}{v_3}$ where $\vec{\mathbf{a}} = \langle x_o, y_o, z_o \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$
Plane	$\vec{\mathbf{r}} = \vec{\mathbf{a}} + s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, **a** = point on plane, **u**, **v** = vectors in plane OR $\vec{\mathbf{r}}\bullet\vec{\mathbf{n}} = \vec{\mathbf{a}}\bullet\vec{\mathbf{n}}$, where $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$	$Ax + By + Cz + D = 0$, $\vec{\mathbf{n}} = \langle A, B, C \rangle$

1.3 Area, Arc Length, Tangents and Normals, Curvature



$$A = \int_a^b f(x)\,dx$$

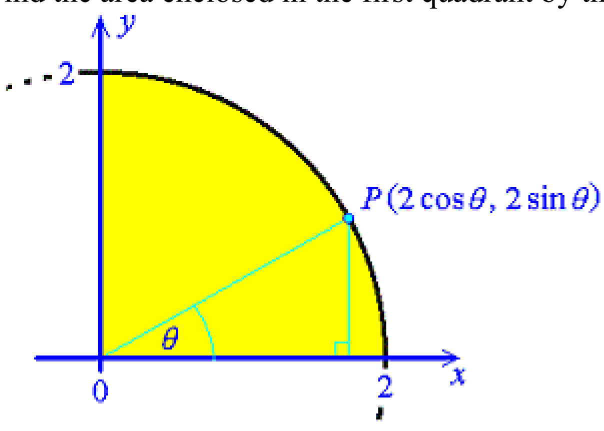
With parameterization:

$$\boxed{A = \int_{t_a}^{t_b} y\frac{dx}{dt}\,dt}$$

where $x(t_a) = a$, $x(t_b) = b$, $a < b$ and $f(x) \geq 0$ on $[a, b]$.

<u>Example 1.3.1</u>

Find the area enclosed in the first quadrant by the circle $x^2 + y^2 = 4$.



$x = 2\cos\theta, \quad y = 2\sin\theta.$

$x = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

$x = 2 \quad \Rightarrow \quad \theta = 0$

$$x = 2\cos\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -2\sin\theta$$

Example 1.3.1 (continued)

$$\Rightarrow \quad A = \int_{\pi/2}^{0} (2\sin\theta)(-2\sin\theta)\,d\theta = +4\int_{0}^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4\int_{0}^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2\left\{\left(\frac{\pi}{2}-0\right) - (0-0)\right\}$$

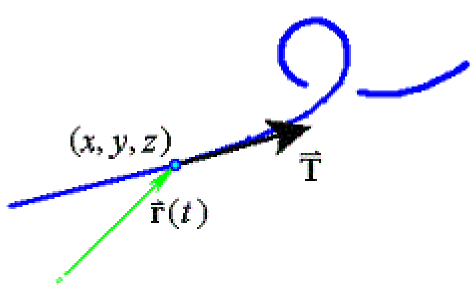
Therefore

$$\boxed{A = \pi}$$

Check: The area is the interior of a quarter-circle.

$$\frac{1}{4}\left(\pi r^2\right) = \frac{1}{4}\left(\pi 2^2\right) = \pi \quad \checkmark$$

Review of the Tangent:

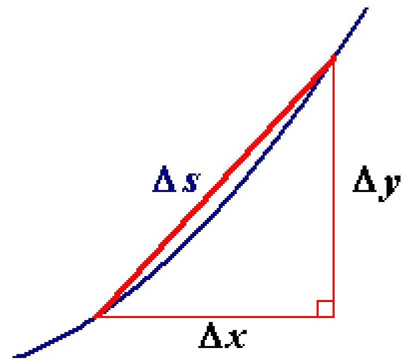


$$\vec{\mathbf{T}} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{d\vec{\mathbf{r}}}{dt}$$

The unit tangent is

$$\boxed{\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right|}$$

Arc Length



In $\mathbb{R}^2$:

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

In $\mathbb{R}^3$:

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$\boxed{\Rightarrow \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left|\frac{d\vec{\mathbf{r}}}{dt}\right|}$$

The vector $\dfrac{d\vec{\mathbf{r}}}{dt}$ points in the direction of the tangent $\vec{\mathbf{T}}$ to the curve defined parametrically by $\mathbf{r} = \mathbf{r}(t)$.

$$\Rightarrow \quad \hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{d\vec{\mathbf{r}}}{dt} \div \frac{ds}{dt} = \frac{d\vec{\mathbf{r}}}{dt} \times \frac{dt}{ds}$$

$$\therefore \quad \hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds}$$

Example 1.3.2

(a) Find the arc length along the curve defined by $\vec{\mathbf{r}} = \left\langle \tfrac{1}{4}(2t - \sin 2t),\ \tfrac{1}{4}\cos 2t,\ \sin t \right\rangle$, from the point where $t = 0$ to the point where $t = 4\pi$.

(b) Find the unit tangent $\hat{\mathbf{T}}$.

(a) $$\frac{dx}{dt} = \frac{1}{4}(2 - 2\cos 2t) = \frac{2}{4}(1 - \cos 2t) = \frac{1}{2}(2\sin^2 t) = \sin^2 t$$

$$\frac{dy}{dt} = -\frac{1}{4}(2\sin 2t) = -\frac{2}{4}(2\sin t\cos t) = -\sin t\cos t$$

$$\frac{dz}{dt} = \cos t$$

$$\Rightarrow \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \sin^2 t,\ -\sin t\cos t,\ \cos t \right\rangle$$

$$\Rightarrow \frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{\sin^4 t + \sin^2 t\cos^2 t + \cos^2 t} = \sqrt{\sin^2 t\left(\sin^2 t + \cos^2 t\right) + \cos^2 t}$$

$$= \sqrt{1\sin^2 t + \cos^2 t} = 1$$

$$\frac{ds}{dt} = 1 \qquad \Rightarrow \quad s = \int_0^{4\pi} 1\, dt = [t]_0^{4\pi}$$

Therefore

$$\boxed{s = 4\pi}$$

(b)

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$$

But $\left| \frac{d\vec{\mathbf{r}}}{dt} \right| = 1 \quad \forall t \quad$ ("for all t"). Therefore

$$\boxed{\hat{\mathbf{T}} = \left\langle \sin^2 t,\ -\sin t\cos t,\ \cos t \right\rangle}$$

[Note that, in this example, $|\, t \,|$ itself is the arc length from the point (0, 1/4, 0), (where $t = 0$). There are very few curves for which the parameterization is this convenient!]

<u>Review of Normal and Binormal</u>:

$$\left|\hat{\mathbf{T}}\right| = 1 \quad \Rightarrow \quad \hat{\mathbf{T}} \bullet \hat{\mathbf{T}} = 1$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{T}}}{ds} \bullet \hat{\mathbf{T}} + \hat{\mathbf{T}} \bullet \frac{d\hat{\mathbf{T}}}{ds} = 0 \quad \Rightarrow \quad 2\hat{\mathbf{T}} \bullet \frac{d\hat{\mathbf{T}}}{ds} = 0$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{T}}}{ds} = \vec{\mathbf{0}} \quad \text{or} \quad \frac{d\hat{\mathbf{T}}}{ds} \perp \hat{\mathbf{T}}$$ (Note: a unit vector can never be zero).

Select the principal normal $\vec{\mathbf{N}}$ to be the direction in which the unit tangent is, instantaneously, changing:

$$\vec{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \frac{ds}{dt}$$

The **unit principal normal** then follows:

$$\boxed{\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{dt} \div \left|\frac{d\hat{\mathbf{T}}}{dt}\right|}$$

The **binormal** is at right angles to both tangent and principal normal:

$$\boxed{\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}}$$

<u>Curvature</u>:

The derivative of the unit tangent with respect to distance travelled along a curve is the principal normal vector, $\vec{\mathbf{N}} = \dfrac{d\hat{\mathbf{T}}}{ds}$. Its direction is the unit principal normal $\hat{\mathbf{N}}$. Its magnitude is a measure of how rapidly the curve is turning and is defined to be the **curvature:**

$$\boxed{\kappa = \left|\vec{\mathbf{N}}\right| = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\frac{d\hat{\mathbf{T}}}{dt}\right| \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right|}$$

$$\Rightarrow \quad \vec{\mathbf{N}} = \kappa \hat{\mathbf{N}}$$

The **radius of curvature** is $\rho = \dfrac{1}{\kappa}$.
The radius of curvature at a point on the curve is the radius of the circle which best fits the curve at that point.

<u>Example 1.3.2</u> (continued)

(c) Find the curvature $\kappa(s)$ at any point for which $s > 0$, for the curve
$\vec{\mathbf{r}}(s) = \left\langle \tfrac{1}{4}(2s - \sin 2s),\ \tfrac{1}{4}\cos 2s,\ \sin s \right\rangle$
where s is the arc length from the point (0, 1/4, 0).

From part (b):
$$\hat{\mathbf{T}} = \left\langle \sin^2 s,\ -\sin s \cos s,\ \cos s \right\rangle$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{T}}}{ds} = \left\langle 2\sin s \cos s,\ -\cos^2 s + \sin^2 s,\ -\sin s \right\rangle = \left\langle \sin 2s,\ -\cos 2s,\ -\sin s \right\rangle$$

$$\boxed{\Rightarrow \quad \kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \underline{\underline{\sqrt{1+\sin^2 s}}}}$$

$$\Rightarrow \quad \rho = \frac{1}{\kappa} = \frac{1}{\sqrt{1+\sin^2 s}}$$

Another formula for curvature:

Let $\dot{\vec{\mathbf{r}}} = \dfrac{d\vec{\mathbf{r}}}{dt}$ and $\dot{s} = \dfrac{ds}{dt} = \left|\dfrac{d\vec{\mathbf{r}}}{dt}\right| = \left|\dot{\vec{\mathbf{r}}}\right|$, then

$$\dot{\vec{\mathbf{r}}} = \dot{s}\,\hat{\mathbf{T}}$$

$$\Rightarrow \quad \ddot{\vec{\mathbf{r}}} = \ddot{s}\,\hat{\mathbf{T}} + \dot{s}\,\frac{d\hat{\mathbf{T}}}{dt}$$

But $\dfrac{d\hat{\mathbf{T}}}{dt} = \dfrac{d\hat{\mathbf{T}}}{ds}\cdot\dfrac{ds}{dt} = \left(\kappa\hat{\mathbf{N}}\right)\dot{s}$

$$\Rightarrow \quad \dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}} = \left(\dot{s}\,\hat{\mathbf{T}}\right) \times \left(\ddot{s}\,\hat{\mathbf{T}} + \kappa\dot{s}^{2}\hat{\mathbf{N}}\right) = \vec{\mathbf{0}} + \kappa\dot{s}^{3}\hat{\mathbf{B}}$$

$\Rightarrow \quad \left|\dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}}\right| = \kappa\dot{s}^{3}$, but $\dot{s} = \left|\dot{\vec{\mathbf{r}}}\right|$. Therefore

$$\boxed{\kappa = \frac{\left|\dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}}\right|}{\left|\dot{\vec{\mathbf{r}}}\right|^{3}}}$$

Example 1.3.3

Find the curvature and the radius of curvature for the curve, given in parametric form by $x = \cos t\,, \quad y = \sin t\,, \quad z = t\,.$ Assume SI units.

Let $c = \cos t\,, \quad s = \sin t\,.$

$$\vec{\mathbf{r}} = \langle c,\ s,\ t \rangle$$
$$(\vec{\mathbf{v}} =)\ \dot{\vec{\mathbf{r}}} = \langle -s,\ c,\ 1 \rangle$$
$$(\vec{\mathbf{a}} =)\ \ddot{\vec{\mathbf{r}}} = \langle -c,\ -s,\ 0 \rangle$$

$$\left| \dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}} \right| = \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -s & c & 1 \\ -c & -s & 0 \end{matrix} \right\| = \left| \langle +s,\ -c,\ s^2 + c^2 \rangle \right| = \left| \langle +s,\ -c,\ 1 \rangle \right|$$

$$= \sqrt{s^2 + c^2 + 1} = \sqrt{2}$$

$$\left| \dot{\vec{\mathbf{r}}} \right| = \sqrt{s^2 + c^2 + 1} = \sqrt{2}$$

$$\Rightarrow \quad \kappa = \frac{\left| \dot{\vec{\mathbf{r}}} \times \ddot{\vec{\mathbf{r}}} \right|}{\left| \dot{\vec{\mathbf{r}}} \right|^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

Therefore

$$\boxed{\kappa = \tfrac{1}{2}\,\mathrm{m}^{-1} \quad (\text{constant})}$$

and

$$\boxed{\rho = \frac{1}{\kappa} = 2\,\mathrm{m}}$$

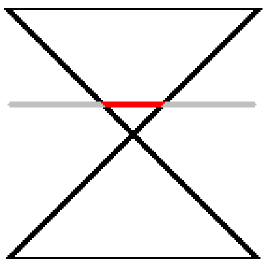
1.4 Conic Sections

All members of the family of curves known as conic sections can be generated, (as the name implies), from the intersections of a plane and a double cone. The Cartesian equation of any conic section is a second order polynomial in x and y. The only cases that we shall consider in this course are such that any axis of symmetry is lined up along a coordinate axis. For all such cases, the Cartesian equation is of the form

$$A x^2 + C y^2 + D x + E y + F = 0$$

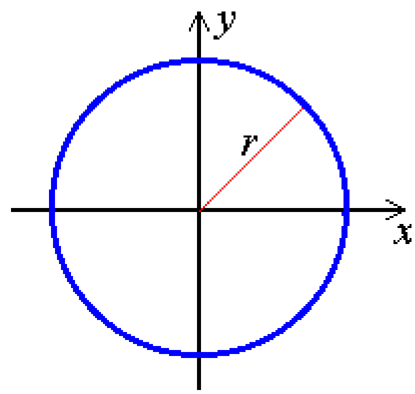
where A, C, D, E and F are constants. There is no "xy" term, so $B = 0$.

The slope of the intersecting plane is related to the **eccentricity**, e of the conic section.

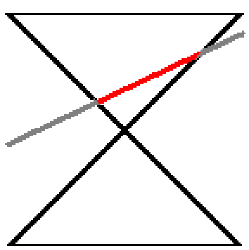


$e = 0$
circle

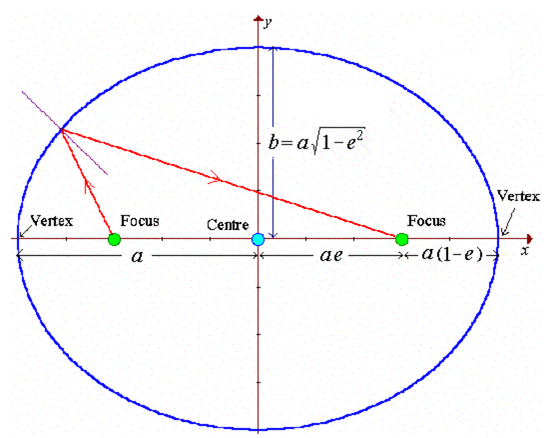
$$x^2 + y^2 = r^2$$



A parametric form is
$(x, y) = (r \cos \theta, r \sin \theta)$, $(0 \le \theta < 2\pi)$.



$0 < e < 1$
ellipse

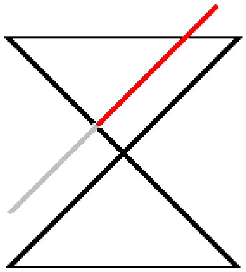


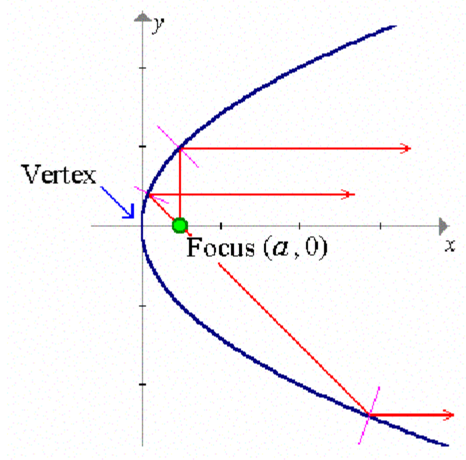
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(1 - e^2\right)$

The circle is clearly a special case of the ellipse, with $e = 0$ and $b = a = r$.
The longest diameter is the major axis ($2a$). The shortest diameter is the minor axis ($2b$).
If a mirror is made in the shape of an ellipse, then all rays emerging from one focus will, after reflection, converge on the other focus.

A parameterization for the ellipse is $\vec{\mathbf{r}}(\theta) = a\cos\theta\,\hat{\mathbf{i}} + b\sin\theta\,\hat{\mathbf{j}}$, $\left(0 \le \theta < 2\pi\right)$.

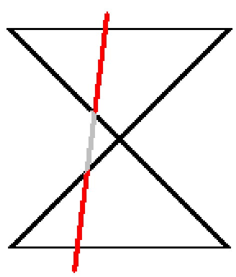


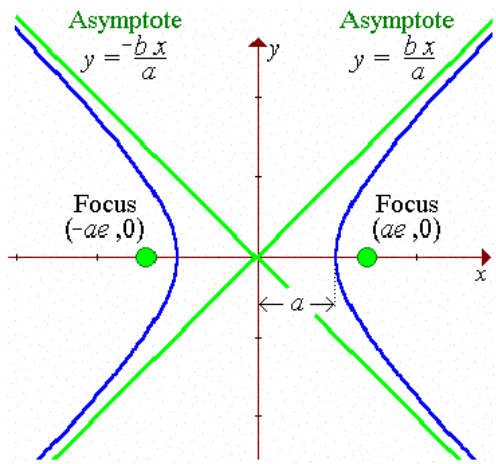


$y^2 = 4ax$

One vertex is at the origin.
The centre and the other vertex and focus are at infinity.

If a mirror is made in the shape of a parabola, then all rays emerging from the focus will, after reflection, travel in parallel straight lines to infinity (where the other focus is). The primary mirrors of most telescopes follow a paraboloid shape.





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(e^2 - 1\right)$

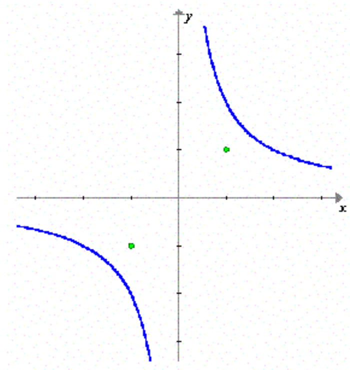
The hyperbola has two separate branches.
As the curve retreats towards infinity, the curve approaches the asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \left(\Rightarrow \quad y = \pm\frac{bx}{a}\right).$$

The distance between the two vertices is the major axis ($2a$).
If a mirror is made in the shape of an hyperbola, then all rays emerging from one focus will, after reflection, appear to be diverging from the other focus.

Circles and ellipses are closed curves. Parabolas and hyperbolas are open curves.

A special case of the hyperbola occurs when the eccentricity is $e = \sqrt{2}$ and it is rotated $45°$ from the standard orientation. The asymptotes line up with the coordinate axes, the graph lies entirely in the first and third quadrants and the Cartesian equation is $xy = k$.



This is the **rectangular hyperbola**.

Degenerate conic sections arise when the intersecting plane passes through the apex of the cone. Two cases are:

$0 \le e < 1$: $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \qquad$ point at the origin.

$e > 1$: $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \qquad$ line pair through the origin.

Another degenerate case is

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \qquad$ nothing !

Example 1.4.1

Classify the conic section whose Cartesian equation is $3y^2 = x^2 + 3$.

Rearranging into standard form,

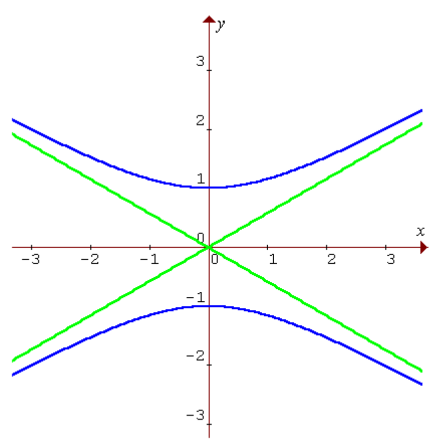
$$3y^2 - x^2 = 3 \quad \Rightarrow \quad \frac{y^2}{1} - \frac{x^2}{3} = 1$$

Compare with

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1: \quad X = y\,, \quad Y = x\,, \quad a = 1\,, \quad b = \sqrt{3}$$

Therefore this is an **hyperbola**,
rotated through 90°,
with vertices at (0, ±1) and

asymptotes $x = \pm\sqrt{3}y$ or $y = \pm\dfrac{\sqrt{3}}{3}x$.



[The asymptotes make angles of 30° with the x axis.]

Example 1.4.2

Classify the conic section whose Cartesian equation is $21x^2 + 28y^2 = 168x + 168y$.

Rearranging into standard form, first complete the square.

$$21x^2 + 28y^2 - 168x - 168y = 0$$

$$\Rightarrow \quad 21(x^2 - 8x) + 28(y^2 - 6y) = 0$$

$$\Rightarrow \quad 21((x-4)^2 - 16) + 28((y-3)^2 - 9) = 0$$

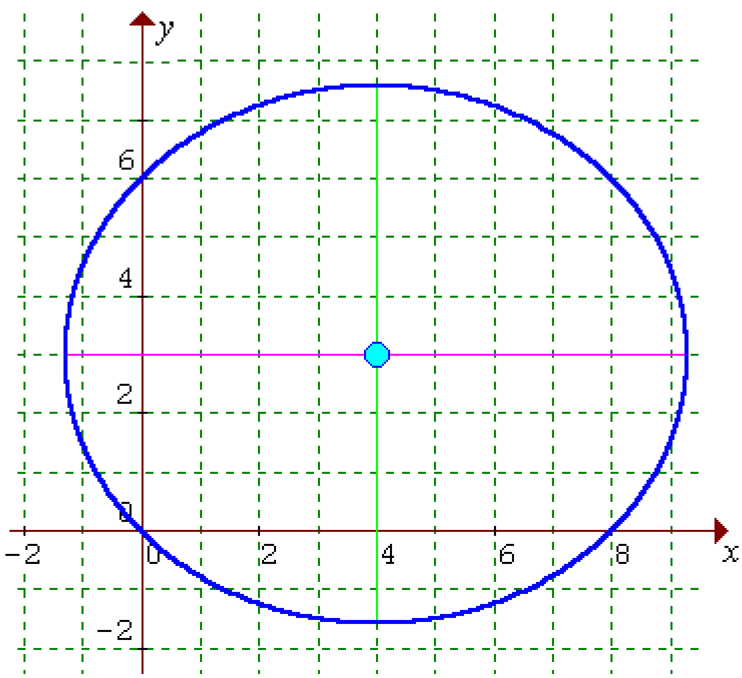
$$\Rightarrow \quad 21(x-4)^2 + 28(y-3)^2 = 588$$

$$\Rightarrow \quad \frac{(x-4)^2}{28} + \frac{(y-3)^2}{21} = 1$$

Compare this with the standard form

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1: \quad \Rightarrow \quad X = x-4, \quad Y = y-3, \quad a = \sqrt{28}, \quad b = \sqrt{21}$$

The conic section is therefore an **ellipse**,
semi major axis $\sqrt{28}$, semi minor axis $\sqrt{21}$,
(from which the eccentricity can be found to be exactly $e = 1/2$),
centre (4, 3).
It happens to pass exactly through the origin.



Moving up to three dimensions, we have the family of quadric surfaces.

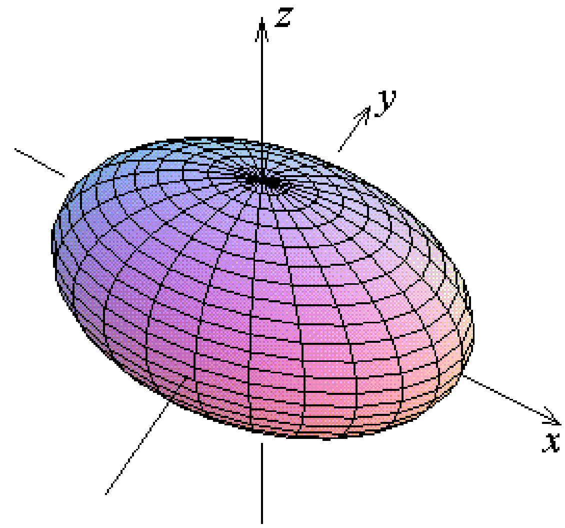
1.5 Classification of Quadric Surfaces

Again, we shall consider only the simplest cases, where any planes of symmetry are located on the Cartesian coordinate planes. In nearly all cases, this eliminates "cross-product terms", such as xy, from the Cartesian equation of a surface. Except for the paraboloids, the Cartesian equations involve only x^2, y^2, z^2 and constants.

The five main types of quadric surface are:

The **ellipsoid** (axis lengths a, b, c)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



The axis intercepts are at
$(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$.

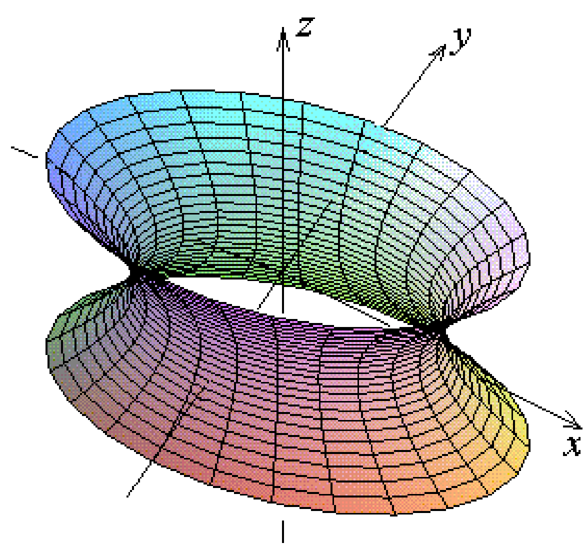
All three coordinate planes are planes of symmetry.

The cross-sections in the three coordinate planes are all ellipses.

Special cases:
$a = b > c$: oblate spheroid (a "squashed sphere")
$a = b < c$: prolate spheroid (a "stretched sphere" or cigar shape)
$a = b = c$: sphere

Hyperboloid of One Sheet (Ellipse axis lengths a, b; aligned along the z axis)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



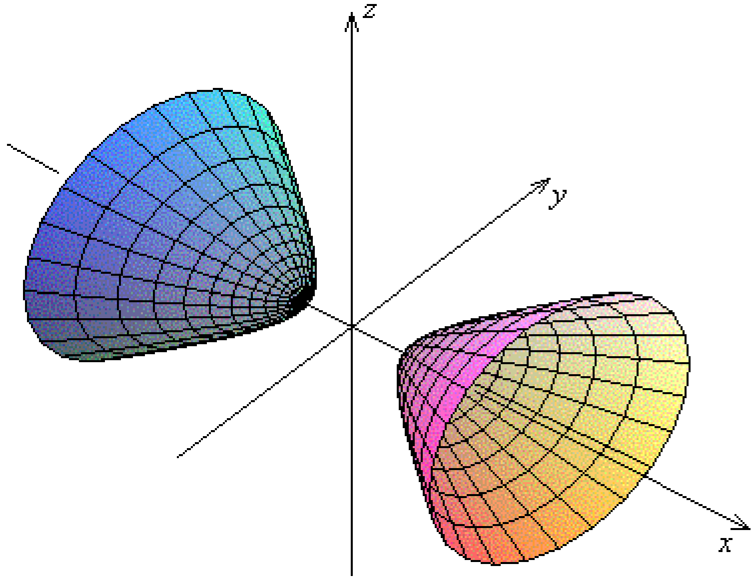
For hyperboloids, the central axis is associated with the "odd sign out".

In the case illustrated, the hyperboloid is aligned along the z axis.

The axis intercepts are at
$(\pm a, 0, 0)$ and $(0, \pm b, 0)$.
The vertical cross sections in the x-z and y-z planes are hyperbolae.
All horizontal cross sections are ellipses.

Hyperboloid of Two Sheets (Ellipse axis lengths b , c ; aligned along the x axis)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



For hyperboloids, the central axis is associated with the "odd sign out".

In the case illustrated, the hyperboloid is aligned along the x axis.

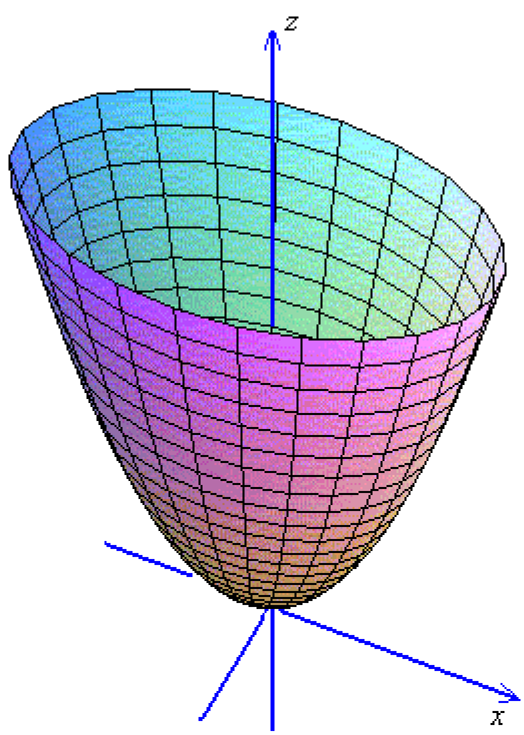
The axis intercepts are at $(\pm a, 0, 0)$ only.

Vertical cross sections parallel to the y-z plane are either ellipses or null.

All cross sections containing the x axis are hyperbolae.

Elliptic Paraboloid
(Ellipse axis lengths a , b ;
aligned along the z axis)

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



For paraboloids, the central axis is associated with the "odd exponent out".

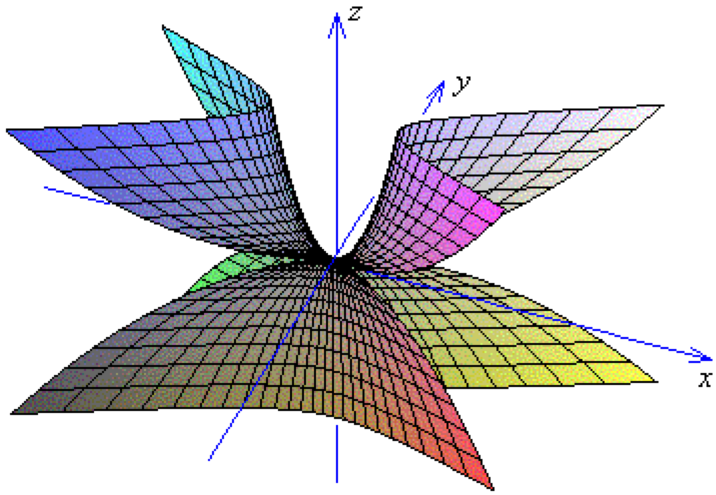
In the case illustrated, the paraboloid is aligned along the z axis.

The only axis intercept is at the origin.

The vertical cross sections in the x-z and y-z planes are parabolae.
All horizontal cross sections are ellipses (for $z > 0$).

Hyperbolic Paraboloid (Hyperbola axis length a ; aligned along the z axis)

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



For paraboloids, the central axis is associated with the "odd exponent out".

In the case illustrated, the paraboloid is aligned along the z axis.

The only axis intercept is at the origin.

The vertical cross section in the x-z plane is an upward-opening parabola.
The vertical cross section in the y-z plane is a downward-opening parabola.
All horizontal cross sections are hyperbolae, (except for a point at $z = 0$).

The plots of the five standard quadric surfaces shown here were generated in the software package Maple. The Maple worksheet is available from a link at "`http://www.engr.mun.ca/~ggeorge/2422/programs/index.html`".

Degenerate Cases:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$: A single POINT at the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$: Elliptic CONE, aligned along the z axis;

[asymptote to both types of hyperboloid].

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$$: NOTHING

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$: ELLIPTIC CYLINDER, aligned along the z axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$: HYPERBOLIC CYLINDER, aligned along the z axis.

$$\frac{y}{b} = \frac{x^2}{a^2}$$: PARABOLIC CYLINDER, vertex line on the z axis.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$: LINE (the z axis)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$: NOTHING

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$: PLANE PAIR (intersecting along the z axis)

$\frac{x^2}{a^2} = 1$: Parallel PLANE PAIR

$\frac{x^2}{a^2} = 0$: Single PLANE (the y-z coordinate plane)

$\frac{x^2}{a^2} = -1$: NOTHING

Example 1.5.1

Classify the quadric surface, whose Cartesian equation is $2x = 3y^2 + 4z^2$.

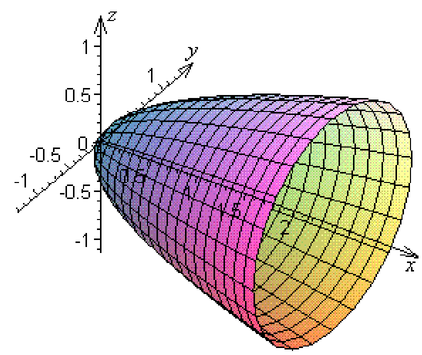
Rearranging into standard form,

$$\frac{x}{6} = \frac{y^2}{4} + \frac{z^2}{3}$$

Compare to the standard form

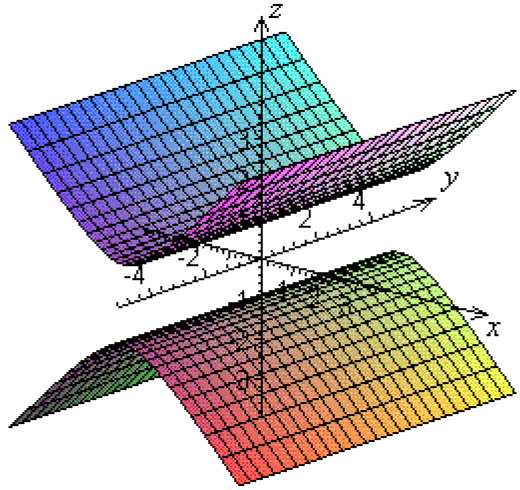
$$\frac{Z}{c} = \frac{X^2}{a^2} + \frac{Y^2}{b^2}: \quad Z = x\,,\ X = y\,,\ Y = z\,,\quad a = 2\,,\ b = \sqrt{3}\,,\ c = 6$$

The quadric surface is therefore an **elliptic paraboloid**, aligned along the x axis, with its vertex at the origin.



Example 1.5.2

Classify the quadric surface, whose Cartesian equation is $z^2 = 1 + x^2$.

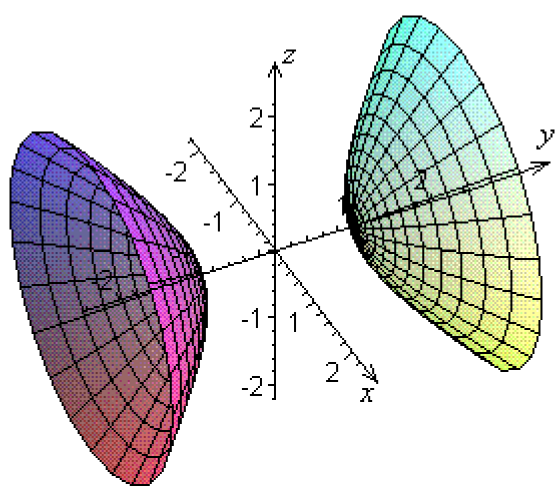


Rearranging into standard form,
$z^2 - x^2 = 1$
Compare to the standard form
$\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1: \quad X = z\,,\; Y = x\,,\; a = b = 1$
The quadric surface is therefore one of the degenerate cases.
It is a **hyperbolic cylinder**, centre at the origin, aligned along the y axis.

Example 1.5.3

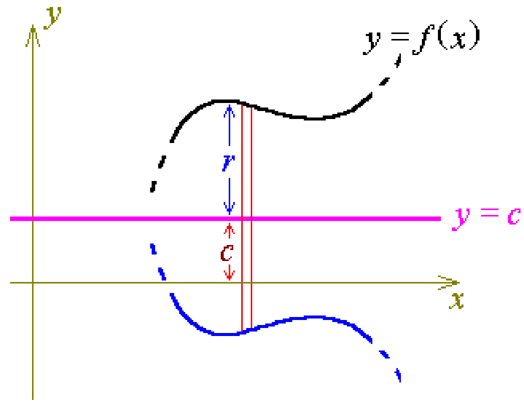
Classify the quadric surface, whose Cartesian equation is $x^2 - y^2 + z^2 + 1 = 0$.

Rearranging into standard form,
$y^2 - x^2 - z^2 = 1$
Compare to the standard form
$\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} - \dfrac{Z^2}{c^2} = 1: \quad X = y\,,\; Y = x\,,\; Z = z\,,\; a = b = c = 1$
The quadric surface is therefore an
hyperboloid of two sheets,
centre at the origin,
aligned along the y axis.



1.6 Surfaces of Revolution

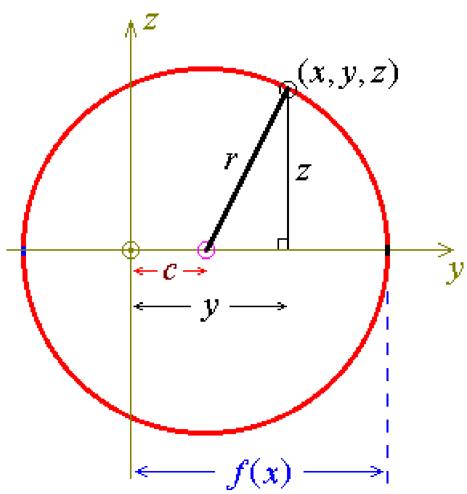
Consider a curve in the x-y plane, defined by the equation $y = f(x)$.
If it is swept once around the line $y = c$, then it will generate a surface of revolution.



At any particular value of x, a thin cross-section through that surface, parallel to the y-z plane, will be a circular disc of radius r, where

$$\boxed{r = \left| f(x) - c \right|}$$

Let us now view the circular disc face-on, (so that the x axis and the axis of rotation are both pointing directly out of the page and the page is parallel to the y-z plane).



Let (x, y, z) be a general point on the surface of revolution.

From this diagram, one can see that

$$r^2 = (y - c)^2 + z^2$$

Therefore, the equation of the surface generated, when the curve $y = f(x)$ is rotated once around the axis $y = c$, is

$$\boxed{(y-c)^2 + z^2 = \left(f(x) - c\right)^2}$$

Special case: When the curve $y = f(x)$ is rotated once around the x axis, the equation of the surface of revolution is

$$\boxed{y^2 + z^2 = \left(f(x)\right)^2} \quad \text{or} \quad \boxed{\sqrt{y^2 + z^2} = \left| f(x) \right|}$$

Example 1.6.1

Find the equation of the surface generated, when the parabola $y^2 = 4ax$ is rotated once around the x axis.

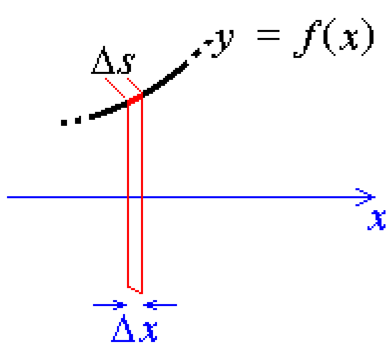
The solution is immediate:

$$y^2 + z^2 = 4ax \,,$$

which is the equation of an **elliptic paraboloid**,
(actually a special case, a circular paraboloid).

The Curved Surface Area of a Surface of Revolution

For a rotation around the x axis,



the curved surface area swept out by the element of arc length Δs is approximately the product of the circumference of a circle of radius y with the length Δs.

$$\Delta A \approx 2\pi |y| \Delta s$$

Integrating along a section of the curve $y = f(x)$ from $x = a$ to $x = b$, the total curved surface area is

$$\boxed{A = 2\pi \int_{x=a}^{x=b} |f(x)| \, ds}$$

For a rotation of $y = f(x)$ about the axis $y = c$, the curved surface area is

$$A = 2\pi \int_{x=a}^{x=b} |f(x) - c| \, ds = 2\pi \int_a^b |f(x) - c| \frac{ds}{dx} dx \quad \text{and} \quad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore

$$\boxed{A = 2\pi \int_a^b |f(x) - c| \sqrt{1 + (f'(x))^2} \, dx}$$

Example 1.6.2

Find the curved surface area of the circular paraboloid generated by rotating the portion of the parabola $y^2 = 4cx$ $(c > 0)$ from $x = a\ (\geq 0)$ to $x = b$ about the x axis.

$$A = 2\pi \int_{x=a}^{x=b} |y|\, ds$$

$$y^2 = 4cx \quad \Rightarrow \quad 2yy' = 4c \quad \Rightarrow \quad y' = \frac{2c}{y}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4c^2}{y^2}} = \sqrt{1 + \frac{4c^2}{4cx}} = \sqrt{\frac{x+c}{x}}$$

$$\Rightarrow \quad A = 2\pi \int_a^b 2\sqrt{cx}\sqrt{\frac{x+c}{x}}\, dx = 4\pi\sqrt{c}\int_a^b (x+c)^{1/2}\, dx$$

$$= 4\pi\sqrt{c}\left[\frac{(x+c)^{3/2}}{\frac{3}{2}}\right]_a^b$$

Therefore

$$A = \underline{\underline{\frac{8\pi\sqrt{c}}{3}\left((b+c)^{3/2} - (a+c)^{3/2}\right)}}$$

1.7 Hyperbolic Functions

When a uniform inelastic (unstretchable) perfectly flexible cable is suspended between two fixed points, it will hang, under its own weight, in the shape of a catenary curve. The equation of the standard catenary curve is most concisely expressed as the hyperbolic cosine function, $y = \cosh x$, where

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The solutions to some differential equations can be expressed conveniently in terms of hyperbolic functions.

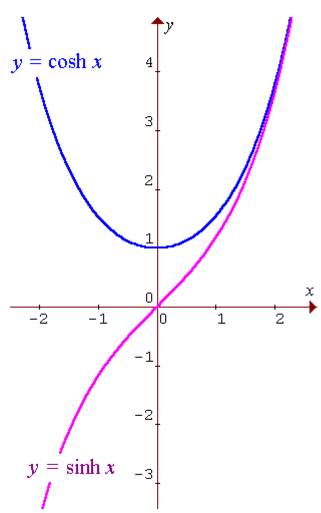
The other five hyperbolic functions are

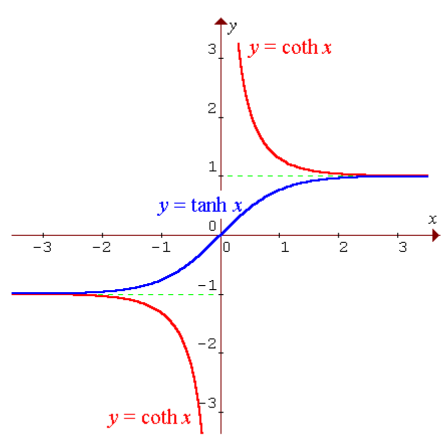
$$\sinh x = \frac{e^x - e^{-x}}{2},$$

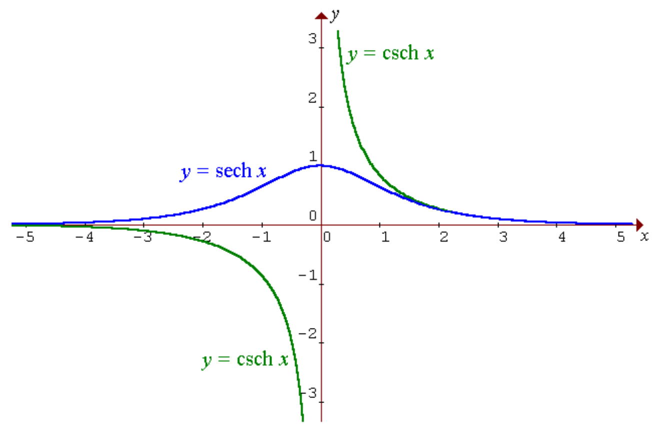
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

Unlike the trigonometric functions, the hyperbolic functions are not periodic.
However, parity is preserved:
Of the six trigonometric function, only $\cos\theta$ and $\sec\theta$ are even functions.
Of the six hyperbolic functions, only $\cosh\theta$ and $\operatorname{sech}\theta$ are even functions.
The other functions are odd.







There is a close relationship between the hyperbolic and trigonometric functions.

From the Euler form for $e^{j\theta}$, $e^{j\theta} = \cos\theta + j\sin\theta$,

$$\Rightarrow \quad e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{1}{j}\sinh(j\theta) = -j\sinh(j\theta)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} = \cosh(j\theta)$$

$$\Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sinh(j\theta)}{j\cosh(j\theta)} = \frac{1}{j}\tanh(j\theta) = -j\tanh(j\theta)$$

Identities:

Let $x = j\theta$:

$$\sin^2\theta + \cos^2\theta \equiv 1 \quad \Rightarrow \quad \left(\frac{\sinh^2 x}{j^2}\right) + \cosh^2 x \equiv 1 \quad \Rightarrow$$

$$\cosh^2 x - \sinh^2 x \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta \quad \Rightarrow \quad 1 + \left(\frac{\tanh^2 x}{j^2}\right) \equiv \left(\frac{1}{\cosh^2 x}\right) \quad \Rightarrow$$

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

etc.

Derivatives

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2}$$

Therefore

$$\boxed{\frac{d}{dx}(\sinh x) = \cosh x}$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2}$$

Therefore

$$\boxed{\frac{d}{dx}(\cosh x) = +\sinh x}$$

[Note the different sign from the trigonometric version, $(\cos x)' = -\sin x$.]

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

Therefore

$$\boxed{\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x}$$

OR

Let $x = j\theta$, then $\theta = -jx$

$$\tanh(j\theta) = j\tan\theta$$

$\Rightarrow$ $\tanh x = j\tan(-jx)$

$$\Rightarrow \quad \frac{d}{dx}(\tanh x) = j\left(\sec^2(-jx)\right)\times(-j) = +\sec^2(-jx) = \sec^2\theta$$

$$= \operatorname{sech}^2(j\theta) = \operatorname{sech}^2 x .$$

Therefore $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$$\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}\left((\sinh x)^{-1}\right) = -(\sinh x)^{-2}(\cosh x)$$

$$= -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x}$$

Therefore

$$\boxed{\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x}$$

etc.

A list of identities and derivatives for hyperbolic functions is presented in the suggestions for a formula sheet, in Appendix A to these lecture notes.

1.8 Integration by Parts

Review:
Let $u(x)$ and $v(x)$ be functions of x. Then, by the product rule of differentiation,

$$\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

Integrating with respect to x :

$$[uv] = \int u'v \, dx + \int uv' \, dx$$

This leads to the formula for integration by parts:

$$\boxed{\int uv' \, dx = [uv] - \int u'v \, dx}$$

Example 1.8.1

Find $I = \int x^3 e^{-x^2} \, dx$.

[v' must be identified with a factor of the integrand that can be antidifferentiated easily.]

$$u = -\tfrac{1}{2} x^2 \qquad v' = -2x \, e^{-x^2}$$

$$\Rightarrow \quad u' = -x \qquad v = e^{-x^2}$$

$$\Rightarrow \quad I = [uv] - \int u'v \, dx = \left[-\tfrac{1}{2} x^2 e^{-x^2}\right] - \int -x \, e^{-x^2} \, dx$$

$$= e^{-x^2}\left(-\tfrac{1}{2} x^2 - \tfrac{1}{2}\right) + C$$

Therefore

$$I = \underline{\underline{-\tfrac{1}{2}\left(x^2 + 1\right) e^{-x^2} + C}}$$

Example 1.8.2 [Repeated use of integration by parts]

Find $I = \int x^2 \cos x \, dx$.

$$u = x^2 \qquad v' = \cos x$$

$$\Rightarrow \quad u' = 2x \qquad v = \sin x$$

$$\Rightarrow \quad I = x^2 \sin x - \int \underbrace{2x}_{u} \cdot \underbrace{\sin x}_{v'} \, dx$$

Using integration by parts again, with $u = 2x$ and $v' = \sin x$,

$$u' = 2 \qquad v = -\cos x$$

$$\Rightarrow \quad I = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x \, dx \right)$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Therefore

$$I = \int x^2 \cos x \, dx = \left(x^2 - 2 \right) \sin x + 2x \cos x + C$$

Check:

$$I' = 2x \sin x + (x^2 - 2) \cos x + 2 \cos x - 2x \sin x + 0$$

$$= x^2 \cos x \quad ✓$$

Shortcut (a tabular form for repeated integrations by parts):

$I = \int x^2 \cos x \, dx$:

D		***I***
x^2		$\cos x$
	$+$	
$2x$		$\sin x$
	$-$	
2		$-\cos x$
	$+$	
0		$-\sin x$

Reading off the diagonals,

$I = x^2 \sin x + 2x \cos x - 2 \sin x + C$

Note:

There are three ways the table can end:

1) column 'D' reduces to 0 (as in the examples on this page);

2) the product across a row is easy to integrate;

3) the product across a row is a constant multiple of the original integrand. (ex. 1.8.4 next page)

Example 1.8.3

Find $I = \int x^4 e^x \, dx$.

D		***I***
x^4		e^x
	$+$	
$4x^3$		e^x
	$-$	
$12x^2$		e^x
	$+$	
$24x$		e^x
	$-$	
24		e^x
	$+$	
0		e^x

Therefore

$I = (x^4 - 4x^3 + 12x^2 - 24x + 24)\, e^x + C$

Check:

$$\begin{aligned} I' &= (x^4 - 4x^3 + 12x^2 - 24x + 24 \\ &\quad + 4x^3 - 12x^2 + 24x - 24 + 0)\, e^x \\ &= x^4 e^x \quad \checkmark \end{aligned}$$

Example 1.8.4 (recursive use of integration by parts)

Find $I = \int e^{ax} \sin bx \, dx$.

Either

D		I
e^{ax}		$\sin bx$
	$+$	
$a e^{ax}$		$-\frac{1}{b} \cos bx$
	$-$	
$a^2 e^{ax}$	$+$ INTEGRATE	$-\frac{1}{b^2} \sin bx$

$$I = \left[-e^{ax} \frac{\cos bx}{b} + a e^{ax} \frac{\sin bx}{b^2} \right] - \frac{a^2}{b^2} I$$

$$\Rightarrow \left(1 + \frac{a^2}{b^2} \right) I = \frac{e^{ax}}{b^2} \left[a \sin bx - b \cos bx \right]$$

$$\left(a^2 + b^2 \right) I = e^{ax} \left[a \sin bx - b \cos bx \right]$$

or

D		I
$\sin bx$		e^{ax}
	$+$	
$b \cos bx$		$\frac{1}{a} e^{ax}$
	$-$	
$-b^2 \sin bx$	$+$ INTEGRATE	$\frac{1}{a^2} e^{ax}$

$$I = \left[+ \sin bx \frac{e^{ax}}{a} - b \cos bx \frac{e^{ax}}{a^2} \right] - \frac{b^2}{a^2} I$$

$$\Rightarrow \left(1 + \frac{b^2}{a^2} \right) I = \frac{e^{ax}}{a^2} \left[a \sin bx - b \cos bx \right]$$

$$\left(a^2 + b^2 \right) I = e^{ax} \left[a \sin bx - b \cos bx \right]$$

Therefore

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left(a \sin bx - b \cos bx \right) + C}$$

Check:

Let $s = \sin bx$ and $c = \cos bx$, then $s' = bc$, $c' = -bs$ and

$$I = \frac{e^{ax}}{a^2 + b^2} \left(as - bc \right) + C$$

$$\Rightarrow \frac{dI}{dx} = \frac{e^{ax}}{a^2 + b^2} \Big(a \left(as - bc \right) + \big(a \left(bc \right) - b \left(-bs \right) \big) \Big)$$

$$= \frac{e^{ax}}{a^2 + b^2} \left(a^2 s - abc + abc + b^2 s \right) = \frac{e^{ax}}{a^2 + b^2} \left(a^2 + b^2 \right) s = e^{ax} \sin bx \quad \checkmark$$

1.9 Leibnitz Differentiation of a Definite Integral

$$\boxed{\frac{d}{dx}\left(\int_{y=f(x)}^{y=g(x)} H(x,y)\,dy\right) = H\left(x,g(x)\right)\frac{dg}{dx} - H\left(x,f(x)\right)\frac{df}{dx} + \int_{y=f(x)}^{y=g(x)} \frac{\partial H}{\partial x}\,dy}$$

If the limits of integration are both constant, then just differentiate the integrand with respect to x, treating all other terms as constants.

$$\boxed{\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial f}{\partial x}\,dt}$$

Example 1.9.1

Evaluate $\dfrac{dI}{dt}$, where $I(t) = \int_t^{2t} zt\,dz$.

Using Leibnitz differentiation:

$$\frac{dI}{dt} = \left((2t)t\right)\times\frac{d}{dt}(2t) - \left((t)t\right)\times\frac{d}{dt}(t) + \int_t^{2t} z(1)\,dz$$

$$= 4t^2 - t^2 + \left[\frac{z^2}{2}\right]_{z=t}^{z=2t} = 3t^2 + \frac{3t^2}{2} = \underline{\underline{\frac{9t^2}{2}}}$$

Directly:

$$I(t) = \int_t^{2t} zt\,dz = t\left[\frac{z^2}{2}\right]_t^{2t} = t\left(\frac{4t^2-t^2}{2}\right) = \frac{3t^3}{2}$$

$$\Rightarrow \quad \frac{dI}{dt} = \frac{d}{dt}\left(\frac{3t^3}{2}\right) = \underline{\underline{\frac{9t^2}{2}}}$$

See Problem Set 3 and Section 5.10 for more practical examples of Leibnitz differentiation.

End of Chapter 1

1.2 Polar Coordinates

The description of the location of an object in $\mathbb{R}^2$ relative to the observer is not very natural in Cartesian coordinates: "the object is three metres to the east of me and four metres to the north of me", or $(x, y) = (3, 4)$. It is much more natural to state how far away the object is and in what direction: "the object is five metres away from me, in a direction approximately 53° north of due east", or $(r, \theta) = (5, 53°)$.

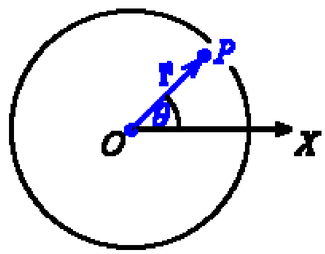
Radar also operates more naturally in plane polar coordinates.

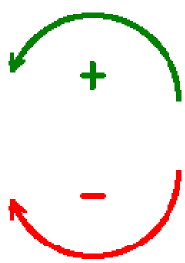
r = range

θ = azimuth

O is the pole

OX is the polar axis (where $\theta = 0$)



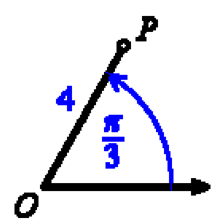


Anticlockwise rotations are positive.

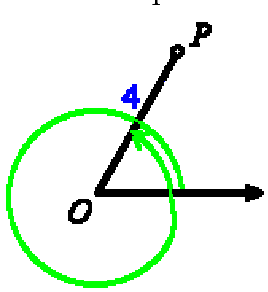
[Nautical bearings are very different: positive rotation is measured *clockwise*, from zero at due north !]

Example 1.2.01

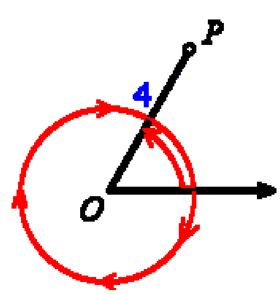
The point P with the polar coordinates $(r, \theta) = (4, \pi / 3)$



also has the polar coordinates



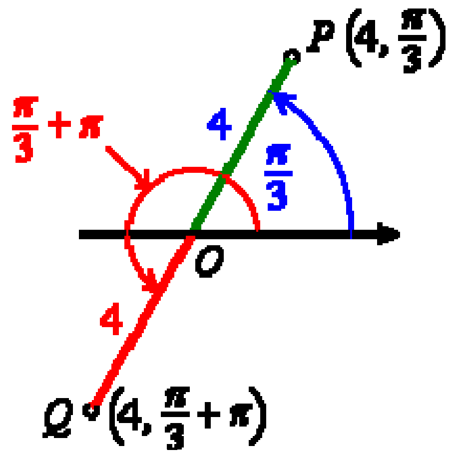
or



$\left(4, 2\pi + \frac{\pi}{3}\right)$ $\left(4, -2\pi + \frac{\pi}{3}\right)$

or $\left(4, 2n\pi + \frac{\pi}{3}\right)$, n = any integer

Example 1.2.01 (continued)



The point $\left(-4, \frac{\pi}{3} + \pi\right)$ is at P.

So also is $\left(-4, \frac{\pi}{3} + \pi + 2n\pi\right)$, n = any integer .

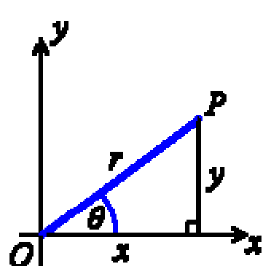
In general, if the polar coordinates of a point are (r, θ), then

$\boxed{(r,\ \theta + 2n\pi)}$ and $\boxed{(-r,\ \theta + (2n+1)\pi)}$

(n = any integer)
also describe the same point.

The polar coordinates of the pole are $(0, \theta)$ for *any* θ.

In some situations, we impose restrictions on the range of the polar coordinates, such as $r \geq 0$, $-\pi < \theta \leq +\pi$ for the principal value of a complex number in polar form.



Conversion between Cartesian and polar coordinates:

$\boxed{x = r\cos\theta}$ $\boxed{y = r\sin\theta}$

Inverse:

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\therefore\ r^2 = x^2 + y^2 \Rightarrow$

$$\boxed{r = \pm\sqrt{x^2 + y^2}}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

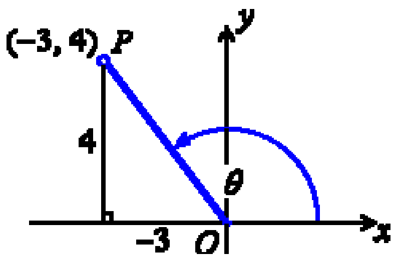
Therefore

$$\boxed{\tan\theta = \frac{y}{x}}$$

More information is needed in order to select the correct quadrant.

<u>Example 1.2.02</u>

Find the polar coordinates for the point whose Cartesian coordinates are (–3, 4).



$$x = -3\,,\quad y = 4$$

$$r^2 = 9 + 16 = 25$$

$$\Rightarrow \qquad r = \pm 5$$

$$\tan\theta = \frac{4}{-3} = -\frac{4}{3}$$

(–3, 4) is in the second quadrant.

If we choose $r > 0$, then one value of θ is $\theta = -\mathrm{Tan}^{-1}(4/3) + \pi \approx 2.21$ rad.

One possibility: $(r, \theta) = (5, 2.21)$ (to 3 s.f.)

Therefore, to 3 s.f.,

$$(r, \theta) = (5, 2.21 + 2n\pi) \text{ or } (-5, 2.21 + (2n+1)\pi)\,, \quad (n \in \mathbb{Z})$$

Example 1.2.03

Find the Cartesian coordinates for $(r, \theta) = \left(2, -\frac{11\pi}{3}\right)$.

$$-\frac{11\pi}{3} = \frac{(1-12)\pi}{3} = \frac{\pi}{3} + (-2)2\pi$$

$$x = r\cos\theta = 2\cos\tfrac{\pi}{3} = 1$$

$$y = r\sin\theta = 2\sin\tfrac{\pi}{3} = \sqrt{3}$$

Therefore

$$(x, y) = \left(1, \sqrt{3}\right)$$

Polar Curves $\boldsymbol{r = f(\theta)}$

The representation (x, y) of a point in Cartesian coordinates is unique. For a curve defined implicitly or explicitly by an equation in x and y, a point (x, y) is on the curve if and only if its coordinates (x, y) satisfy the equation of the curve.

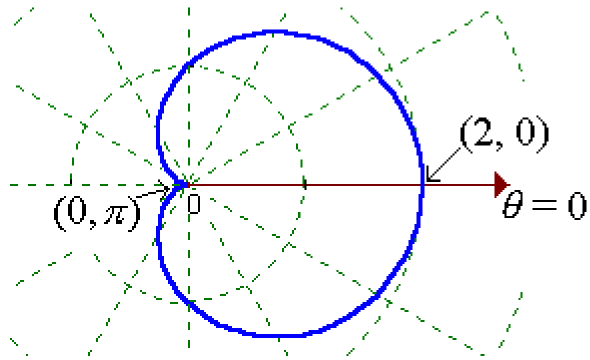
The same is *not* true for plane polar coordinates. Each point has infinitely many possible representations, $(r,\ \theta + 2n\pi)$ and $(-r,\ \theta + (2n+1)\pi)$ (where n is any integer). A point lies on a curve if and only if at least one pair (r, θ) of the infinitely many possible pairs of polar coordinates for that point satisfies the polar equation of the curve. It doesn't matter if other polar coordinates for that same point do not satisfy the equation of the curve.

Example 1.2.04

The curve whose polar equation is

$$r = 1 + \cos\theta$$

is a **cardioid**
(literally, a "heart-shaped" curve).



$\{ r = 2,\ \theta = 2n\pi \}$
(where n is any integer)
satisfies the equation $r = 1 + \cos\theta$.

$\Rightarrow$ $(r, \theta) = (2, 2n\pi)$ is on the cardioid curve.

But $(2, 2n\pi)$ is the same point as $(-2, (2n+1)\pi)$.

$\theta = (2n+1)\pi \Rightarrow 1 + \cos\theta = 0 \neq r$.

Yet the point whose polar coordinates are $(-2, (2n+1)\pi)$ is on the curve!

Example 1.2.05

Convert to polar form the equation

$$x^2 + y^2 = \sqrt{x^2 + y^2} + 3y \ .$$

$$r^2 = r + 3r\sin\theta$$

$$\Rightarrow \quad r\,(r - 1 - 3\sin\theta) = 0$$

$$\Rightarrow \quad r = 0 \quad \text{or} \quad r = 1 + 3\sin\theta$$

But $\left(0, \arcsin\left(-\frac{1}{3}\right)\right)$ is a solution of $r = 1 + 3\sin\theta$

$\Rightarrow$ $r = 0$ is included in $r = 1 + 3\sin\theta$.

Therefore the polar equation of the curve is

$$r = 1 + 3\sin\theta$$

(which is a limaçon).

Note that there is no restriction on the sign of r ; it can be negative.

Example 1.2.06

Convert to Cartesian form the equation of the cardioid curve $r = 1 + \cos\theta$.

$$r = 1 + \frac{x}{r}$$

$$\Rightarrow \quad r^2 = r + x$$

$$\Rightarrow \quad r^2 - x = r$$

$$\Rightarrow \quad (r^2 - x)^2 = r^2$$

Therefore

$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

Tangents to $r = f(\theta)$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

By the chain rule for differentiation:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

This leads to a general expression for the slope anywhere on a curve $r = f(\theta)$:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

$\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$ at (r, θ) $\Rightarrow$ horizontal tangent at (r, θ).

$\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$ at (r, θ) $\Rightarrow$ vertical tangent at (r, θ).

At the pole $(r = 0)$:

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + 0}{\dfrac{dr}{d\theta}\cos\theta - 0} = \tan\theta \quad \left(\text{provided } \frac{dr}{d\theta} \neq 0\right)$$

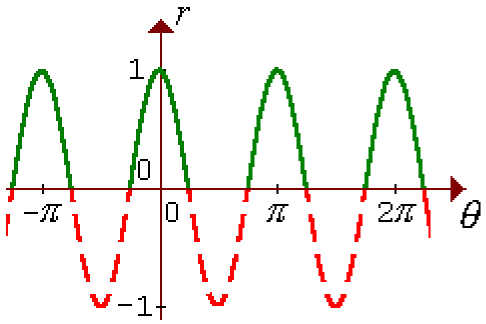
If $r \to 0$ but $\dfrac{dr}{d\theta} \not\to 0$ as $\theta \to \theta_o$, then
the radial line $\theta = \theta_o$ is a **tangent at the pole.**
[This can be of some help when sketching polar curves.]

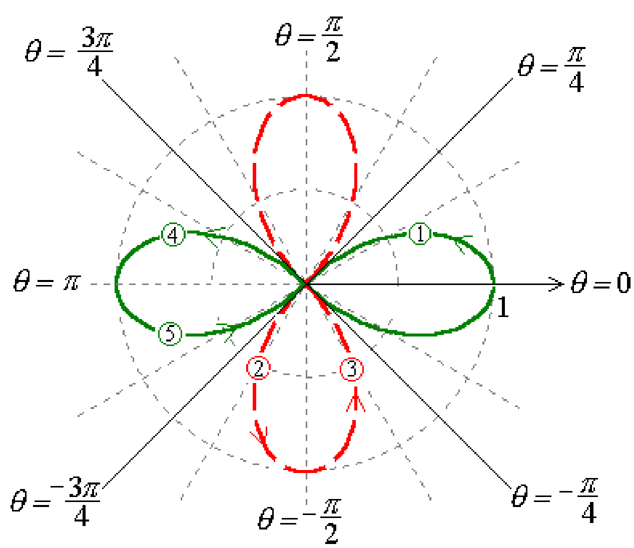
Example 1.2.07

Sketch the curve whose equation in polar form is $r = \cos 2\theta$.

Two methods will be demonstrated here. The first method is a direct transfer from a Cartesian plot of r against θ (as though the curve were $y = \cos 2x$). The second method is a systematic tabular method, involving investigation of the behaviour of the curve in intervals of θ between consecutive critical points (where r and/or its derivative is/are zero or undefined).

Method 1.





Method 2.

$r = \cos 2\theta = 0$ at $2\theta =$ (any odd multiple of $\pi/2$)
$\Rightarrow \quad r = 0$ at $\theta =$ (any odd multiple of $\pi/4$)
$\frac{dr}{d\theta} = -2\sin 2\theta = 0$ at $2\theta =$ (any integer multiple of π)
$\Rightarrow \quad r' = 0$ at $\theta =$ (any integer multiple of $\pi/2$)
Therefore tabulate in intervals bounded by $\theta =$ (consecutive integer multiples of $\pi/4$).

2θ	$0 \to \pi/2$	$\pi/2 \to \pi$	$\pi \to 3\pi/2$	$3\pi/2 \to 2\pi$	$2\pi \to 5\pi/2$	...
θ	$0 \to \pi/4$	$\pi/4 \to \pi/2$	$\pi/2 \to 3\pi/4$	$3\pi/4 \to \pi$	$\pi \to 5\pi/4$	...
r	$1 \to 0$	$0 \to -1$	$-1 \to 0$	$0 \to 1$	$1 \to 0$	...
Region in sketch	(1)	(2)	(3)	(4)	(5)	

This leads to the same sketch as in Method 1 above.

You can follow a plot of $r = \cos n\theta$ by Method 1 (for $n = 1, 2, 3, 4, 5$ and 6) on the web site. See the link at "`http://www.engr.mun.ca/~ggeorge/2422/programs/`".

The distinct polar tangents are

$$\theta = \pm\frac{\pi}{4}$$

Length of a Polar Curve

If $r = f(\theta)$ (for $\alpha \leq \theta \leq \beta$), then

$x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$

Let $r = f(\theta)$, $r' = f'(\theta)$, $c = \cos\theta$ and $s = \sin\theta$, then

$\dfrac{dx}{d\theta} = r'c - rs$ and $\dfrac{dy}{d\theta} = r's + rc$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left((r')^2 c^2 - 2rcsr' + r^2s^2\right) + \left((r')^2 s^2 + 2rcsr' + r^2c^2\right)$$

$$= (r')^2(c^2 + s^2) + 0 + r^2(s^2 + c^2)$$

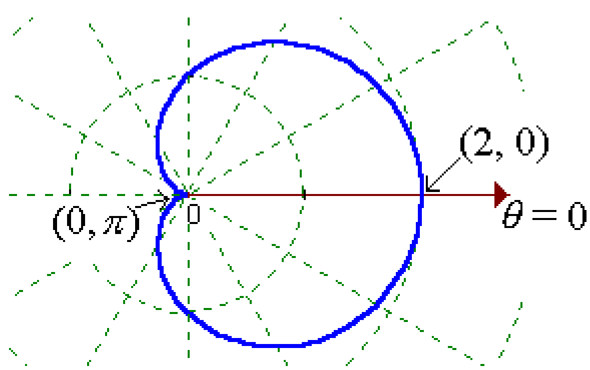
$$= r^2 + \left(\frac{dr}{d\theta}\right)^2$$

Therefore the length L along the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example 1.2.08

Find the length L of the perimeter of the cardioid $r = 1 + \cos\theta$.



$\alpha = 0$, $\beta = 2\pi$.

$r = 1 + \cos\theta = 1 + c$

$\dfrac{dr}{d\theta} = -\sin\theta = -s$

$\Rightarrow r^2 + \left(\dfrac{dr}{d\theta}\right)^2 =$
$1 + 2c + c^2 + s^2 = 2 + 2c$

Example 1.2.08 (continued)

But $1 + \cos 2x = 2\cos^2 x$. Set $\theta = 2x$.

$$\Rightarrow \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2\left(2\cos^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad L = \int_0^{2\pi} \sqrt{2^2\cos^2\frac{\theta}{2}}\, d\theta = 2\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta$$

Using symmetry in the horizontal axis,

$$L = 2\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8(1-0)$$

Therefore the perimeter of the cardioid curve is $L = 8$.

Note:

For $\pi < \theta < 2\pi$, $\sqrt{\cos^2\frac{\theta}{2}} = -\cos\frac{\theta}{2}$

and $\int_0^{2\pi} \cos\frac{\theta}{2}\, d\theta = 0$!

Example 1.2.09

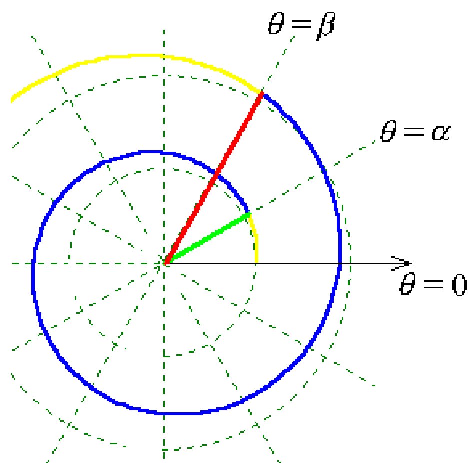
Find the arc length along the spiral curve $r = a\,e^{\theta}$ $(a > 0)$, from $\theta = \alpha$ to $\theta = \beta$.

$$r = a\,e^{\theta} \quad \Rightarrow \quad \frac{dr}{d\theta} = a\,e^{\theta}$$

$$\Rightarrow \quad \frac{ds}{d\theta} = \sqrt{\left(a\,e^{\theta}\right)^2 + \left(a\,e^{\theta}\right)^2} = a\,e^{\theta}\sqrt{2}$$

$$\Rightarrow \quad s = a\sqrt{2}\int_{\alpha}^{\beta} e^{\theta}\, d\theta = a\sqrt{2}\left[e^{\theta}\right]_{\alpha}^{\beta}$$

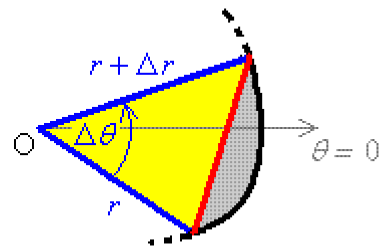
$$\therefore \quad L = \underline{\underline{a\sqrt{2}\left(e^{\beta} - e^{\alpha}\right)}}$$



Area Swept Out by a Polar Curve $r = f(\theta)$

$\Delta A \approx$ Area of triangle

$= \tfrac{1}{2} r \left(r + \Delta r \right) \sin \Delta\theta$



But the angle $\Delta\theta$ is small, so that $\sin \Delta\theta \approx \Delta\theta$ and the increment Δr is small compared to r. Therefore

$\Delta A \approx \tfrac{1}{2} r^2 \Delta\theta \quad \Rightarrow$

$$\boxed{A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta}$$

Example 1.2.10

Find the area of a circular sector, radius r, angle θ.

$r =$ constant, $\quad \beta = \alpha + \theta$

$$A = \tfrac{1}{2} \int_{\alpha}^{\alpha+\theta} r^2 \, d\phi = \tfrac{1}{2} r^2 \left[\phi \right]_{\alpha}^{\alpha+\theta} = \tfrac{1}{2} r^2 \left(\left(\alpha + \theta \right) - \alpha \right)$$

Therefore

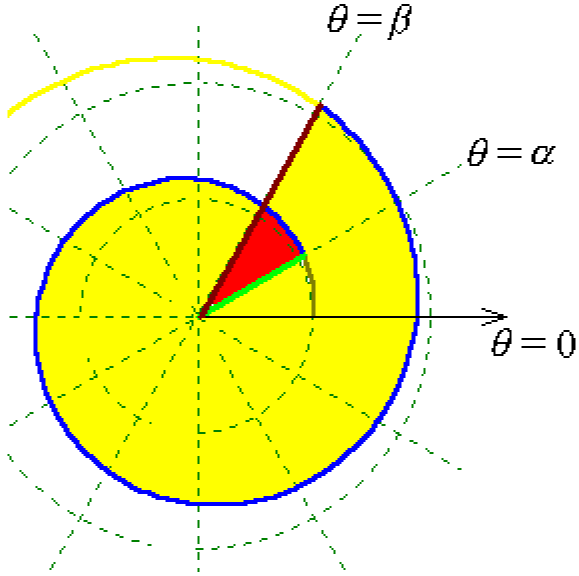
$$A = \underline{\underline{\tfrac{1}{2} \theta r^2}}$$

Full circle: $\theta = 2\pi$ and $A = \pi r^2$.

Example 1.2.11

Find the area swept out by the polar curve $r = a\,e^{\theta}$ over $\alpha < \theta < \beta$, (where $a > 0$ and $\alpha < \beta < \alpha + 2\pi$).

The condition ($\alpha < \beta < \alpha + 2\pi$) prevents the same area being swept out more than once.



If $\beta > \alpha + 2\pi$ then one needs to subtract areas that have been counted more than once [the **red** area in the diagram]

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left(a\,e^{\theta}\right)^2 d\theta = \frac{1}{2}a^2\left[\frac{e^{2\theta}}{2}\right]_{\alpha}^{\beta}$$

Therefore

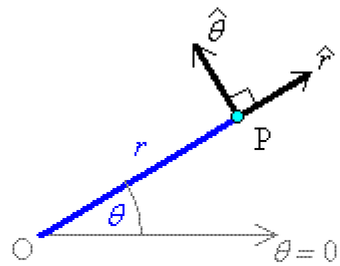
$$A = \underline{\underline{\frac{a^2}{4}\left(e^{2\beta} - e^{2\alpha}\right)}}$$

In general, the area bounded by two polar curves $r = f(\theta)$ and $r = g(\theta)$ and the radius vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta}\left(\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right)d\theta}$$

See the problem sets for more examples of polar curve sketching and the calculation of the lengths and areas swept out by polar curves.

Radial and Transverse Components of Velocity and Acceleration



At any point P (not at the pole), the unit radial vector $\hat{\mathbf{r}}$ points directly away from the pole. The unit transverse vector $\hat{\boldsymbol{\theta}}$ is orthogonal to $\hat{\mathbf{r}}$ and points in the direction of increasing θ. These vectors form an orthonormal basis for $\mathbb{R}^2$.

Only if θ is constant will $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be constant unit vectors, (unlike the Cartesian **i** and **j**).

The derivatives of these two non-constant unit vectors can be shown to be

$$\boxed{\frac{d\hat{\mathbf{r}}}{dt} = \left(\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}}} \quad \text{and} \quad \boxed{\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\left(\frac{d\theta}{dt}\right)\hat{\mathbf{r}}}$$

Using the "overdot" notation to represent differentiation with respect to the parameter t, these results may be expressed more compactly as

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}$$

The radial and transverse components of velocity and acceleration then follow:

$$\vec{\mathbf{r}} = r\hat{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\Rightarrow \quad v_{\text{radial}} = \dot{r} \quad \text{and} \quad v_{\text{transverse}} = r\dot{\theta}$$

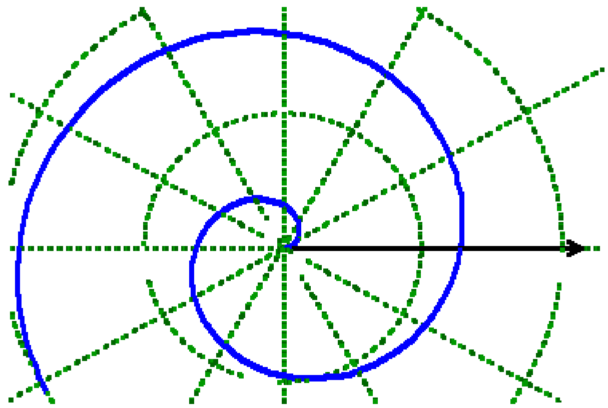
$$\vec{\mathbf{a}} = \dot{\vec{\mathbf{v}}} = \ddot{\vec{\mathbf{r}}} = \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}}\right) + \left(\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}}\right)$$

$$= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\left(\dot{\theta}\right)^2\hat{\mathbf{r}} = \underbrace{\left(\ddot{r} - r\left(\dot{\theta}\right)^2\right)}_{a_{\text{radial}}}\hat{\mathbf{r}} + \underbrace{\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)}_{a_{\text{transverse}}}\hat{\boldsymbol{\theta}}$$

The transverse component of acceleration can also be written as $a_{\text{tr}} = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

Example 1.2.12

A particle follows the path $r = \theta$, where the angle at any time is equal to the time: $\theta = t > 0$. Find the radial and transverse components of acceleration.



$$r = \theta = t$$

$$\Rightarrow \quad \dot{r} = \dot{\theta} = 1$$

$$\Rightarrow \quad \ddot{r} = \ddot{\theta} = 0$$

$$\vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} + t\,\hat{\boldsymbol{\theta}}$$

$$\left(\Rightarrow \quad v = \sqrt{1+t^2}\right)$$

$$\vec{\mathbf{a}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = -t\,\hat{\mathbf{r}} + 2\hat{\boldsymbol{\theta}}$$

$$\left(\Rightarrow \quad a = \sqrt{4+t^2}\right)$$

Therefore

$$a_r = \underline{\underline{-t}} \quad \text{and} \quad a_{tr} = \underline{\underline{2}}$$

Example 1.2.13

For circular motion around the pole, with constant radius r and constant angular velocity $\dot{\theta} = \omega$, the velocity vector is purely tangential, $\vec{\mathbf{v}} = r\,\omega\,\hat{\boldsymbol{\theta}}$, and the acceleration vector is

$$\vec{\mathbf{a}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = -r\,\omega^2\,\hat{\mathbf{r}}$$

which matches the familiar result that “centrifugal” or “centripetal” force = $r\,\omega^2$, directed radially inward.
