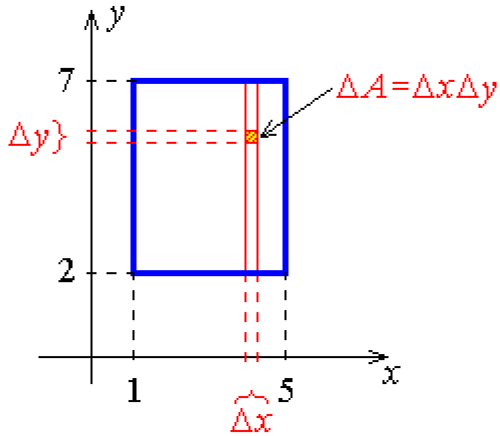
## 6. Multiple Integration

This chapter provides only a very brief introduction to the major topic of multiple integration. Uses of multiple integration include the evaluation of areas, volumes, masses, total charge on a surface and the location of a centre-of-mass.

### 6.1 Double Integrals (Cartesian Coordinates)

Example 6.1.1

Find the area shown (assuming SI units).




$$\text{Area of strip} \approx \left( \sum_{y=2}^{7} \Delta y \right) \Delta x$$

$$\text{Total Area} \approx \sum_{x=1}^{5} \left( \left( \sum_{y=2}^{7} \Delta y \right) \Delta x \right)$$

As $\Delta x \to 0$ and $\Delta y \to 0$, the summations become integrals:

$$(\text{Total Area}) \quad \to \quad A = \int_{x=1}^{x=5} \left( \int_{y=2}^{y=7} 1 \, dy \right) dx$$

The inner integral has no dependency at all on $x$, in its limits or in its integrand. It can therefore be extracted as a "constant" factor from inside the outer integral.

$$\Rightarrow \quad A = \left( \int_{y=2}^{y=7} 1 \, dy \right) \int_{x=1}^{x=5} 1 \, dx$$

$$= [y]_2^7 [x]_1^5 = (7-2) \times (5-1) = 5 \times 4 = \underline{\underline{20 \text{ m}^2}}$$

Example 6.1.1 (continued)

Suppose that the surface density on the rectangle is $\sigma = x^2 y$. Find the mass of the rectangle.

The element of mass is

$$\Delta m = \sigma \Delta A = \sigma \Delta x \, \Delta y$$

$$\rightarrow \quad m = \int_1^5 \int_2^7 \sigma \, dy \, dx = \int_1^5 \int_2^7 x^2 y \, dy \, dx$$

$$= \int_1^5 x^2 \left( \int_2^7 y \, dy \right) dx = \left( \int_2^7 y \, dy \right) \int_1^5 x^2 \, dx$$

$$= \left[ \frac{y^2}{2} \right]_2^7 \cdot \left[ \frac{x^3}{3} \right]_1^5 = \frac{49-4}{2} \times \frac{125-1}{3} = 15 \times 62$$

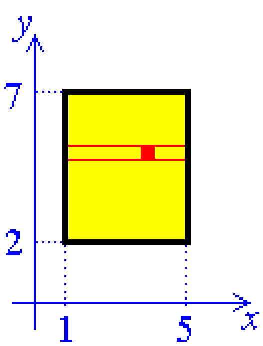
Therefore the mass of the rectangle is $m = \underline{\underline{930 \text{ kg}}}$.

**OR**

We can choose to sum horizontally first:

$$m = \int_2^7 \int_1^5 x^2 y \, dx \, dy$$

$$m = \int_2^7 y \left( \int_1^5 x^2 \, dx \right) dy$$



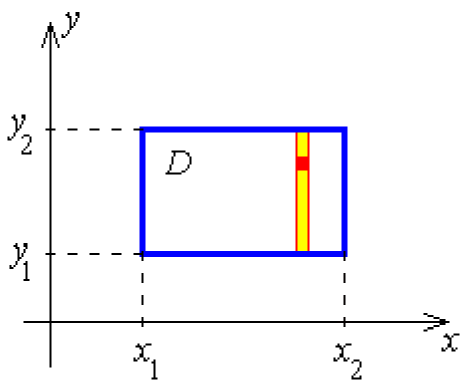

The inner integral has no dependency at all on $y$, in its limits or in its integrand. It can therefore be extracted as a "constant" factor from inside the outer integral.

$$m = \left( \int_1^5 x^2 \, dx \right) \left( \int_2^7 y \, dy \right)$$

which is exactly the same form as before, leading to the same value of 930 kg.

A double integral $\iint_D f(x,y)\,dA$ may be separated into a pair of single integrals if

- the region $D$ is a rectangle, with sides parallel to the coordinate axes; and
- the integrand is separable: $f(x,y) = g(x)\,h(y)$.




$$\iint_D f(x,y)\,dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} g(x)h(y)\,dy\,dx$$

$$= \left(\int_{x_1}^{x_2} g(x)\,dx\right)\left(\int_{y_1}^{y_2} h(y)\,dy\right)$$

This was the case in Example 6.1.1.

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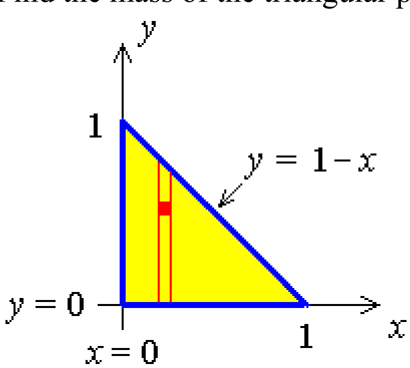
Example 6.1.2

The triangular region shown here has surface density $\sigma = x + y$.
Find the mass of the triangular plate.




Element of mass:

$$\Delta m = \sigma \Delta A = \sigma \Delta x\, \Delta y$$

$$\text{Mass of strip} \approx \left(\sum_{y=0}^{1-x} \sigma\, \Delta y\right)\Delta x$$

$$\text{Total Mass} \approx \sum_{x=0}^{1}\left(\left(\sum_{y=0}^{1-x} \sigma\, \Delta y\right)\Delta x\right)$$

$$\rightarrow \quad m = \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[ xy + \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \left( x(1-x) + \frac{(1-x)^2}{2} - 0 - 0 \right) dx$$

$$= \int_0^1 \frac{1-x^2}{2}\,dx = \left[ \frac{x}{2} - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} - 0 + 0 = \underline{\underline{\frac{1}{3}\text{ kg}}}$$

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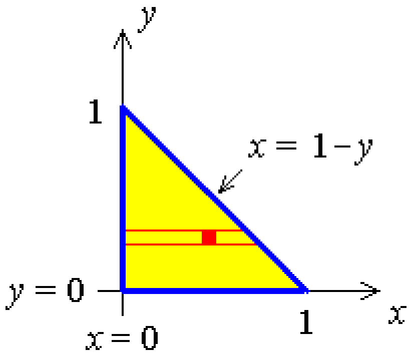
Example 6.1.2 (continued)

**OR**

We can choose to sum horizontally first (re-iterate):




$$m = \int_0^1 \int_0^{1-y} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[ \frac{x^2}{2} + xy \right]_0^{1-y} dy = \cdots = \underline{\underline{\frac{1}{3}\,\text{kg}}}$$

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**Generally:**

In Cartesian coordinates on the $xy$-plane, the rectangular element of area is

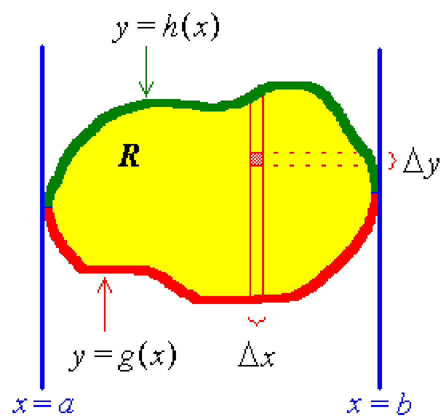
$$\Delta A = \Delta x \, \Delta y \, .$$

Summing all such elements of area along a vertical strip, the area of the elementary strip is

$$\left( \sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x$$

Summing all the strips across the region $R$, the total area of the region is:

$$A \approx \sum_{x=a}^{b} \left( \left( \sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x \right)$$




In the limit as the elements $\Delta x$ and $\Delta y$ shrink to zero, this sum becomes

$$A = \int_{x=a}^{b} \int_{y=g(x)}^{h(x)} 1\,dy\,dx$$

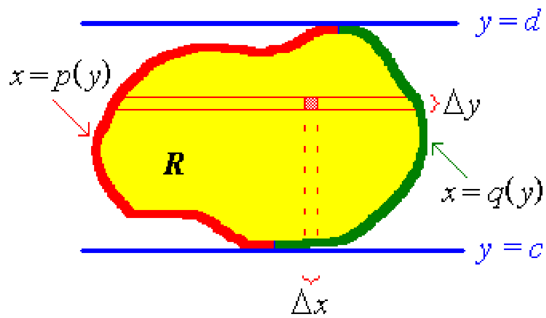
If the surface density $\sigma$ within the region is a function of location, $\sigma = f(x, y)$, then the mass of the region is

$$m = \int_{x=a}^{b} \left( \int_{y=g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

The inner integral must be evaluated first.

**Re-iteration:**

One may reverse the order of integration by summing the elements of area $\Delta A$ horizontally first, then adding the strips across the region from bottom to top. This generates the double integral for the total area of the region



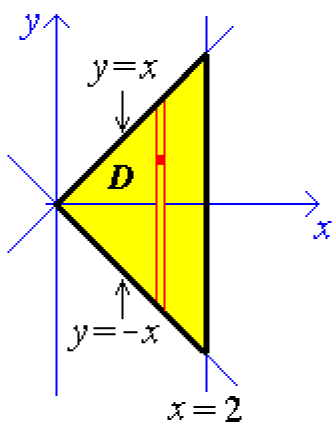

$$A = \int_{y=c}^{d} \left( \int_{x=p(y)}^{q(y)} 1 \, dx \right) dy$$

The mass becomes

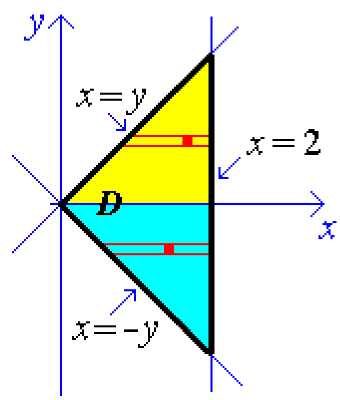
$$m = \int_{y=c}^{d} \left( \int_{x=p(y)}^{q(y)} f(x,y) \, dx \right) dy$$

Choose the orientation of elementary strips that generates the simpler double integration.

For example,




is preferable to

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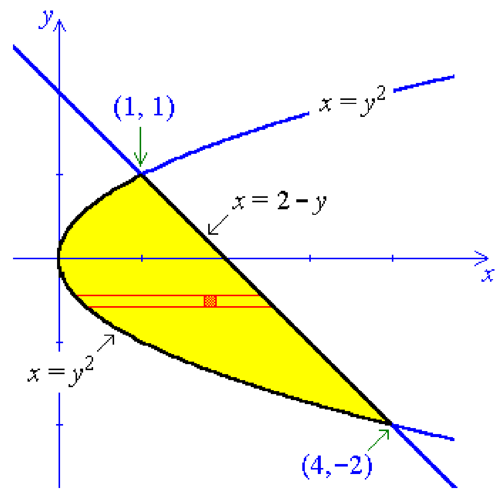
$$\int_0^2 \int_{-x}^{x} f(x,y) \, dy \, dx \quad = \quad \int_{-2}^{0} \int_{-y}^{2} f(x,y) \, dx \, dy + \int_0^2 \int_{+y}^{2} f(x,y) \, dx \, dy$$

Example 6.1.3

Evaluate $I = \iint_R \left(6x + 2y^2\right) dA$

where $R$ is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

The upper boundary changes form at $x = 1$.
The left boundary is the same throughout $R$.
The right boundary is the same throughout $R$.
Therefore choose horizontal strips.




$$I = \int_{-2}^{1} \int_{y^2}^{2-y} \left(6x + 2y^2\right) dx\, dy$$

$$I = \int_{-2}^{1} \left[3x^2 + 2xy^2\right]_{x=y^2}^{x=2-y} dy$$

$$= \int_{-2}^{1} \left(\left(3(2-y)^2 + 2(2-y)y^2\right) - \left(3y^4 + 2y^4\right)\right) dy$$

$$= \int_{-2}^{1} \left(\left(12 - 12y + 3y^2\right) + \left(4y^2 - 2y^3\right) - 5y^4\right) dy$$

$$= \int_{-2}^{1} \left(12 - 12y + 7y^2 - 2y^3 - 5y^4\right) dy$$

$$= \left[12y - 6y^2 + \frac{7}{3}y^3 - \frac{1}{2}y^4 - y^5\right]_{-2}^{+1}$$

$$= \left(12 - 6 + \frac{7}{3} - \frac{1}{2} - 1\right) - \left(-24 - 24 - \frac{56}{3} - 8 + 32\right)$$

Therefore

$$I = \underline{\underline{\frac{99}{2}}}$$

## 6.2 Polar Double Integrals

The Jacobian of the transformation from Cartesian to plane polar coordinates (Example 2.4.1 on page 2-17) is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left\| \begin{matrix} x_r & y_r \\ x_\theta & y_\theta \end{matrix} \right\| = r$$

The element of area is therefore

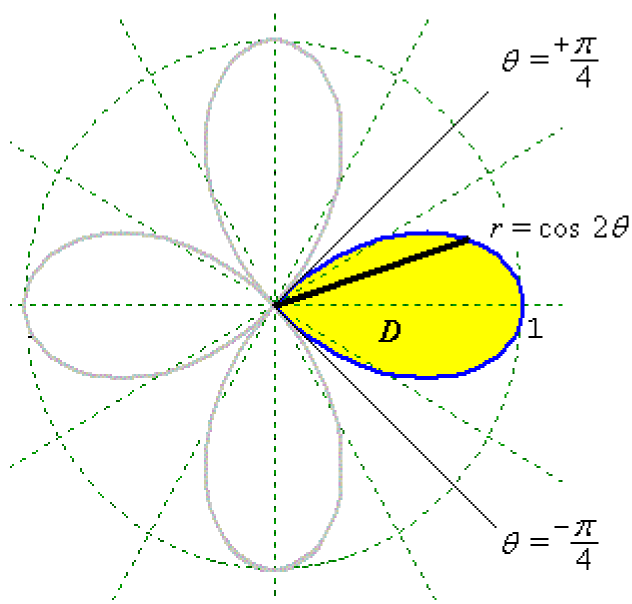
$$dA = dx\,dy = r\,dr\,d\theta$$

Example 6.2.1

Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

**Boundaries:**

$0 \le r \le \cos 2\theta \;;\quad -\frac{\pi}{4} \le \theta \le +\frac{\pi}{4}$




**Area:**

$$A = \iint_D 1\,dA = \int_{-\pi/4}^{+\pi/4} \int_0^{\cos 2\theta} 1\,r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \left(\frac{\cos^2 2\theta}{2} - 0\right) d\theta$$

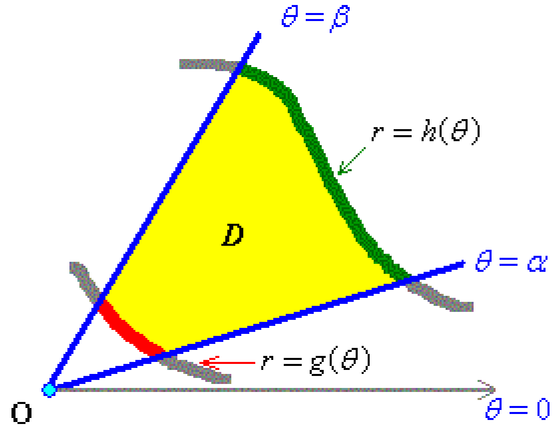
$$= \int_{-\pi/4}^{+\pi/4} \frac{\cos 4\theta + 1}{4}\, d\theta$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4}\right]_{-\pi/4}^{+\pi/4} = \left(0 + \frac{\pi}{16}\right) - \left(0 - \frac{\pi}{16}\right)$$

Therefore

$$A = \underline{\underline{\frac{\pi}{8}}}$$

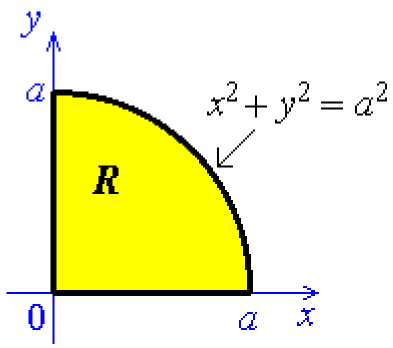
**In general**, in plane polar coordinates,




$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 6.2.2

Find the centre of mass for a plate of surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$, whose boundary is the portion of the circle $x^2 + y^2 = a^2$ that is inside the first quadrant. $k$ and $a$ are positive constants.




Use plane polar coordinates.
**Boundaries:**
The positive $x$-axis is the line $\theta = 0$.
The positive $y$-axis is the line $\theta = \pi/2$.
The circle is $r^2 = a^2$, which is $r = a$.

**Mass:**

Surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$.

$$m = \iint_R \sigma\,dA = \int_0^{\pi/2}\int_0^a \frac{k}{r}\, r\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\int_0^a 1\,dr\,d\theta = k\left(\int_0^a 1\,dr\right)\left(\int_0^{\pi/2} 1\,d\theta\right) = k\,[r]_0^a\,[\theta]_0^{\pi/2}$$

$$m = \frac{k\pi a}{2}$$

<u>Example 6.2.2</u> (continued)

**First Moments about the *x*-axis:**

$$\Delta M_x = y\,\Delta m \qquad \Rightarrow \quad M_x = \iint_R y\,\sigma\,dA$$

$$= \int_0^{\pi/2} \left( \int_0^a (r \sin\theta) \frac{k}{r} r\,dr \right) d\theta$$

$$= k \int_0^a r\,dr \int_0^{\pi/2} \sin\theta\,d\theta = k \left[ \frac{r^2}{2} \right]_0^a \left[ -\cos\theta \right]_0^{\pi/2}$$

$$= k \left( \frac{a^2}{2} - 0 \right) \left( -0 + 1 \right)$$

$$\therefore \quad M_x = \frac{k\,a^2}{2}$$

But $M_x = m\,\bar{y} \qquad \Rightarrow \quad \bar{y} = \dfrac{M_x}{m} = \dfrac{k\,a^2}{2} \cdot \dfrac{2}{k\,\pi\,a} = \dfrac{a}{\pi}$

By sym., $\bar{x} = \bar{y}$

Therefore the centre of mass is at

$$\left( \bar{x}, \bar{y} \right) = \underline{\underline{\left( \frac{a}{\pi}, \frac{a}{\pi} \right)}}$$

## 6.3 Triple Integrals

The concepts for double integrals (surfaces) extend naturally to triple integrals (volumes). The element of volume, in terms of the Cartesian coordinate system ($x$, $y$, $z$) and another orthogonal coordinate system ($u$, $v$, $w$), is

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

and

$$\iiint_V f(x,y,z)\,dV = \int_{w_1}^{w_2}\int_{v_1(w)}^{v_2(w)}\int_{u_1(v,w)}^{u_2(v,w)} f\big(x(u,v,w),\,y(u,v,w),\,z(u,v,w)\big)\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

The most common choices for non-Cartesian coordinate systems in $\mathbb{R}^3$ are:

**Cylindrical Polar Coordinates:**

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\phi,z)}\,dr\,d\phi\,dz = r\,dr\,d\phi\,dz$$

**Spherical Polar Coordinates:**

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Example 6.3.1:

Verify the formula $V = \frac{4}{3}\pi a^3$ for the volume of a sphere of radius $a$.

$$V = \iiint\limits_V 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \left(\int_0^a r^2\, dr\right)\left(\int_0^{\pi} \sin\theta\, d\theta\right)\left(\int_0^{2\pi} 1\, d\phi\right)$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \left(\frac{a^3}{3} - 0\right)(+1+1)(2\pi - 0)$$

Therefore

$$V = \tfrac{4}{3}\pi a^3$$

Example 6.3.2:

The density of an object is equal to the reciprocal of the distance from the origin. Find the mass and the average density inside the sphere $r = a$ .

Use spherical polar coordinates.
Density:

$$\rho = \frac{1}{r}$$

Mass:

$$m = \iiint_V \rho \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{1}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^a r \, dr\right)\left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} 1 \, d\phi\right)$$

$$= \left[\frac{r^2}{2}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \left(\frac{a^2}{2} - 0\right)(+1+1)(2\pi - 0)$$

Therefore

$$m = \underline{\underline{2\pi a^2}}$$

Average density =

$$\bar{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{m}{V} = \frac{2\pi a^2}{\frac{4}{3}\pi a^3} = \frac{3}{2a}$$

Therefore

$$\bar{\rho} = \frac{3}{\underline{\underline{2a}}}$$

[Note that the mass is finite even though the density is infinite at the origin!]

Example 6.3.3:

Find the proportion of the mass removed, when a hole of radius 1, tangent to a diameter, is bored through a uniform sphere of radius 2.

**Cross-section at right angles to the axis of the hole:**

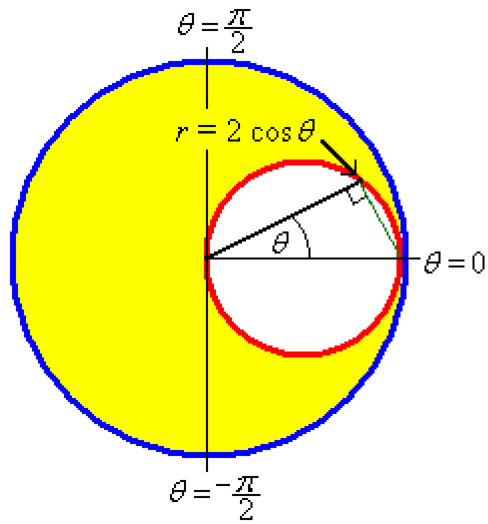
Use cylindrical polar coordinates, with the $z$-axis aligned parallel to the axis of the cylindrical hole.

The plane polar equation of the boundary of the hole is then

$$r = 2\cos\theta$$

The entire circular boundary is traversed once for

$$-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$$




**Cross-section parallel to the axis of the hole:**

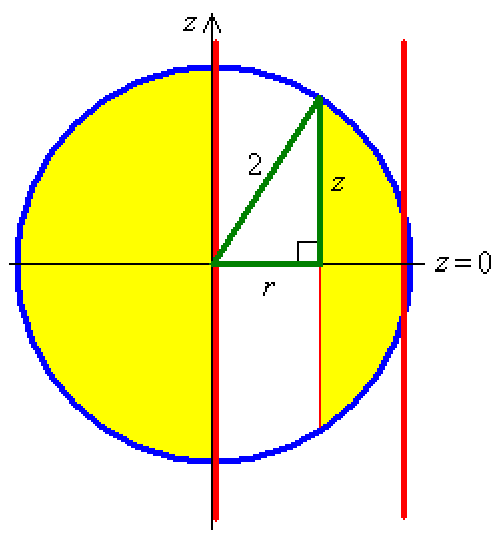
At each value of $r$, the distance from the equatorial plane to the point where the hole emerges from the sphere is

$$z = \sqrt{2^2 - r^2}$$

The element of volume for the hole is therefore

$$dV = 2z\,dA = 2\sqrt{4-r^2}\,(r\,dr\,d\theta)$$

$$V = \int_{-\pi/2}^{+\pi/2}\int_0^{2\cos\theta} 2\sqrt{4-r^2}\;r\,dr\,d\theta$$




We can**not** separate the two integrals, because the upper limit of the inner integral, ($r = 2\cos\theta$), is a function of the variable of integration in the outer integral.

The geometry is entirely symmetric about $\theta = 0$

$$\Rightarrow \quad V = 4\int_0^{+\pi/2}\int_0^{2\cos\theta} \sqrt{4-r^2}\;r\,dr\,d\theta$$

Example 6.3.3 (continued)

$$= 4 \int_0^{\pi/2} \left[ \frac{\left(4-r^2\right)^{3/2}}{\frac{3}{2}\times(-2)} \right]_0^{2\cos\theta} d\theta$$

$$= -\frac{4}{3} \int_0^{\pi/2} \left( \left(4-4\cos^2\theta\right)^{3/2} - \left(4-0\right)^{3/2} \right) d\theta = +\frac{4}{3} \int_0^{\pi/2} \left( \left(-4\sin^2\theta\right)^{3/2} + 4^{3/2} \right) d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} \left(8 - 8\sin^3\theta\right) d\theta = \frac{32}{3} \int_0^{\pi/2} \left(1 - \sin^3\theta\right) d\theta$$

$$= \frac{32}{3} \left( \int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2} \sin^2\theta \sin\theta\, d\theta \right)$$

$$= \frac{32}{3} \left( \int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2} \left(1 - \cos^2\theta\right) \sin\theta\, d\theta \right)$$

Let $u = \cos\theta$, then $du = -\sin\theta\, d\theta$.
$\theta = 0 \;\Rightarrow\; u = 1$ and $\theta = \frac{\pi}{2} \;\Rightarrow\; u = 0$

$$V = \frac{32}{3} \left( \int_0^{\pi/2} 1\, d\theta - \int_{u=1}^{u=0} \left(1-u^2\right)\left(-du\right) \right) = \frac{32}{3} \left( \int_0^{\pi/2} 1\, d\theta - \int_0^1 \left(1-u^2\right) du \right)$$

$$= \frac{32}{3} \left( \left[\theta\right]_0^{\pi/2} - \left[ u - \frac{u^3}{3} \right]_0^1 \right) = \frac{32}{3} \left( \left(\frac{\pi}{2} - 0\right) - \left(\frac{2}{3} - 0\right) \right)$$

$$= \frac{16\pi}{3} - \frac{64}{9}$$

The density is constant throughout the sphere. Therefore

$$\frac{m_{\text{hole}}}{m_{\text{sphere}}} = \frac{V_{\text{hole}}}{V_{\text{sphere}}} = \left( \frac{16\pi}{3} - \frac{64}{9} \right) \cdot \frac{3}{4\pi 2^3} = \frac{1}{2} - \frac{2}{3\pi}$$

Therefore the proportion of the sphere that is removed is

$$\frac{1}{2} - \frac{2}{3\pi} \approx 29\%$$

END OF CHAPTER 6