1. Events A and B are such that $\mathrm{P}[A] = .3$, $\mathrm{P}[B] = .2$ and $\mathrm{P}[A \cup B] = .5$. [6]
Are the events A, B
   (i) incompatible (mutually exclusive) but not independent?
   (ii) independent but not incompatible?
   (iii) both independent and incompatible? or
   (iv) neither independent nor incompatible?
[Circle whichever one of (i), (ii), (iii) or (iv) is correct **and** justify your choice.]

Note that (iii) is never possible.

$$\mathrm{P}\left[A \cup B\right] = \mathrm{P}\left[A\right] + \mathrm{P}\left[B\right] - \mathrm{P}\left[A \cap B\right]$$

$$\Rightarrow \quad \mathrm{P}\left[A \cap B\right] = \mathrm{P}\left[A\right] + \mathrm{P}\left[B\right] - \mathrm{P}\left[A \cup B\right] = .2 + .3 - .5 = 0$$

$\Rightarrow$ A, B are mutually exclusive $\Rightarrow$ A, B are dependent.

Therefore **(i) incompatible (mutually exclusive) but not independent**

---

2. The time T (hours) until the next failure of a gearbox is a random quantity that is known to follow an exponential distribution. The mean time to the next failure is 1200 hours.

(a) Find the probability that the gearbox fails during the next 1200 hours. [3]

(b) Find the median time $\tilde{\mu}$ to the next failure. [5]

(c) If a random sample of 100 such gearboxes is taken, then estimate the probability that the sample mean failure time exceeds 1400 hours, (that is, estimate $\mathrm{P}\left[\overline{T} > 1400\right]$). [4]

The c.d.f. for T is $F(t) = \begin{cases} 0 & (t<0) \\ 1-e^{-\lambda t} & (t \ge 0) \end{cases}$, where $\lambda = \frac{1}{\mu} = \frac{1}{1200}$.

(a) $\mathrm{P}[T \le 1200] = F(1200) = 1 - e^{-1200/1200} = 1 - e^{-1} \quad \Rightarrow$

$$\mathrm{P}[T \le 1200] \;=\; 1 - \frac{1}{e} \;\approx\; .632$$

(b) $F(\tilde{\mu}) = \frac{1}{2} = 1 - e^{-\tilde{\mu}/1200} \quad \Rightarrow \quad e^{-\tilde{\mu}/1200} = \frac{1}{2} \quad \Rightarrow \quad -\frac{\tilde{\mu}}{1200} = \ln\left(\frac{1}{2}\right) = -\ln 2 \quad \Rightarrow$

$$\tilde{\mu} \;=\; 1200 \ln 2 \;\approx\; 832 \text{ h}$$

(c) $n = 100 > 30 \quad \Rightarrow \quad \overline{T} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ by the central limit theorem.

$\sigma = \mu = 1200 \quad \Rightarrow \quad \overline{T} \sim \mathrm{N}\left(1200, (120)^2\right)$

$\Rightarrow \quad \mathrm{P}\left[\overline{T} > 1400\right] \;=\; \mathrm{P}\left[Z > \frac{1400-1200}{120}\right] \;=\; \mathrm{P}\left[Z > \frac{5}{3}\right] \;=\; \mathrm{P}\left[Z < -1.\dot{6}\right]$ (by sym.)

$\approx \Phi(-1.67) \;=\; .04746$ (note that $\Phi(-1.66) \;=\; .04846$)

Therefore, correct to 2 s.f.,

$$\mathrm{P}\left[\overline{T} > 1400\right] \;=\; \mathbf{.048}$$

Correct to 2 s.f., an answer of .047 is also acceptable.

A more precise answer [not required in this examination] is **.047 790** .

3. The mass M of a pallet of boxes of a product from a production line is known to be a random quantity that follows a Normal distribution. It is believed that the population mean mass is 1 000 kg and the strength of that belief is represented by the standard deviation $\sigma_o = 4\,\text{kg}$. A random sample of 36 such pallets has a mean mass of 1 006.8 kg with a sample standard deviation of 24.0 kg.

(a) Construct a Bayesian 95% confidence interval estimate for the true mean mass μ. [6]
(b) Construct a classical 95% confidence interval estimate for the true mean mass μ. [5]
(c) Is there sufficient evidence to conclude that the true mean mass μ is not 1000 kg? [2]
(d) Provide a *brief* reason for the different widths of your two confidence intervals. [2]

(a) Prior information: $\mu_o = 1000\,, \quad \sigma_o = 4 \quad \Rightarrow \quad w_o = \dfrac{1}{\sigma_o^{\,2}} = \dfrac{1}{16}$

Data: $\bar{x} = 1006.8\,, \quad s = 24.0\,, \quad n = 36 \quad \Rightarrow \quad w_d = \dfrac{1}{\left(\dfrac{s^2}{n}\right)} = \dfrac{36}{(24)^2} = \dfrac{1}{16}$

$$\Rightarrow \quad w_f = w_o + w_d = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$\Rightarrow \quad \left(\sigma^*\right)^2 = \frac{1}{w_f} = 8 \quad \text{and}$$

$$\mu^* = \frac{w_o \mu_o + w_d \bar{x}}{w_o + w_d} = 8\left(\frac{1}{16}\times 1000 + \frac{1}{16}\times 1006.8\right) = 1003.4$$

The 95% Bayesian CI for μ is $\mu^* \pm t_{.025,\nu}\,\sigma^*$, where the number of degrees of freedom is between 35 and ∞. Taking the most conservative estimate,

$\mu^* \pm t_{.025,35}\,\sigma^* = 1003.4 \pm 2.03011\sqrt{8} = 1003.4 \pm 5.742\ldots$

The CI estimate is, correct to 1 d.p., therefore

$$\boxed{[997.7, 1009.1]}$$

If the t distribution is approximated by z, then the estimate becomes

$\mu^* \pm z_{.025}\,\sigma^* = 1003.4 \pm 1.95996\sqrt{8} = 1003.4 \pm 5.543\ldots = [997.9, 1008.9]$

3 (b) Classical confidence interval for μ: [5]

$$\bar{x} \pm t_{.025,35} \frac{s}{\sqrt{n}} = 1006.8 \pm 2.03011 \times \frac{24}{6} = 1006.8 \pm 8.12044 =$$

$$\boxed{[998.7, 1014.9]}$$

(correct to 1 d.p.)

If the t distribution is approximated by z, then the estimate becomes

$$\bar{x} \pm z_{.025} \frac{s}{\sqrt{n}} = 1006.8 \pm 1.95996 \times \frac{24}{6} = 1006.8 \pm 7.83984 = [999.0, 1014.6]$$

(c) Both confidence intervals include $\mu = 1000$. Therefore [2]

$$\boxed{\textbf{NO}}$$

(d) The Bayesian CI is narrower [more precise] because it incorporates more information (the prior information) than the classical CI does. [2]

4. Two devices intended to measure tension are undergoing calibration tests. The true value of the tension in a test cable is known to be 5000 N. The reading from device A is normally distributed with a mean of 5000 N and a standard deviation of 20 N, while the reading from device B is normally distributed with a mean of 5010 N and a standard deviation of 10 N. A reading is deemed to be acceptable if it is within 20 N of the correct value (5000 N).

(a) Find the probability that a single reading from device B is greater than a single reading from device A . [7]

(b) In a single reading, which device is more likely to provide an acceptable reading? Show your working. [10]

(c) Find the probability that at least two of the next three readings from device A are acceptable. [You may quote $\mathrm{P}[4980 < A < 5020] = .68268$.] [8]

(a) Assume that all readings are independent of each other. Then

$A \sim \mathrm{N}(5000, 20^2)$ and $B \sim \mathrm{N}(5010, 10^2)$

$$\Rightarrow \quad A - B \sim \mathrm{N}\big((5000-5010), (400+100)\big) = \mathrm{N}(-10, 500)$$

$$\mathrm{P}[B > A] = \mathrm{P}[A - B < 0] = \mathrm{P}\left[Z < \frac{0-(-10)}{\sqrt{500}}\right] = \Phi(0.447\ldots) \approx \Phi(0.45)$$

Correct to two significant figures,

$$\boxed{\mathrm{P}[B > A] = .67}$$

[Using software, a more precise answer is .6726396]

(b) $\mathrm{P}[\text{reading from } A \text{ is acceptable}] = \mathrm{P}[4980 < A < 5020]$

$$= \mathrm{P}\left[\frac{4980-5000}{20} < Z < \frac{5020-5000}{20}\right] = \mathrm{P}\left[Z < \frac{20}{20}\right] - \mathrm{P}\left[Z < \frac{-20}{20}\right]$$

$\Phi(1.00) - \Phi(-1.00)$ $\quad$ [or $1 - 2\Phi(-1.00)$ by symmetry]

= .84134 – .15866 = **.68268** .

$\mathrm{P}[\text{reading from } B \text{ is acceptable}] = \mathrm{P}[4980 < B < 5020]$

$$= \mathrm{P}\left[\frac{4980-5010}{10} < Z < \frac{5020-5010}{10}\right] = \mathrm{P}\left[Z < \frac{10}{10}\right] - \mathrm{P}\left[Z < \frac{-30}{10}\right]$$

$\Phi(1.00) - \Phi(-3.00) = .84134 - .00135 =$ **.83999**

Clearly **device B is more likely to produce an acceptable reading** (even though device B is biased while device A is not biased).

4 (c) Let $p = \text{P}[4980 < A < 5020] = \mathbf{.68268}$ and [8]

X = number of readings among the next three readings that are acceptable.
Each trial has a complementary pair of outcomes:

- Each reading (trial) is either acceptable (success) or not acceptable (failure).
- The probability of success is constant (p = .68268).
- Trials are independent (readings are independent).
- The sample size is fixed (n = 3).

Therefore X follows a binomial distribution (n = 3, p = .68268).

$$\text{P}[X \geq 2] = \text{P}[X = 2] + \text{P}[X = 3] = b(2; 3, p) + b(3; 3, p)$$

$$= {}^3C_2 p^2 (1-p)^1 + {}^3C_3 p^3 (1-p)^0 = 3p^2(1-p) + p^3$$

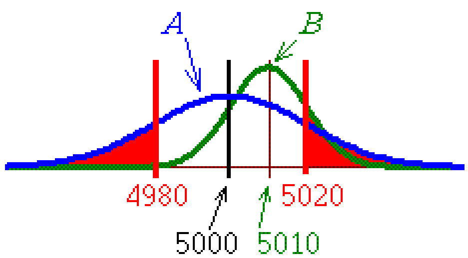
$$= 3(.68268)^2(.31732) + (.68268)^3 = .44366\ldots + .31816\ldots = .76182\ldots \Rightarrow$$

$$\boxed{\text{P}[X \geq 2] = \mathbf{.76}}$$

(correct to 2 s.f.)

Additional note for part (b):

The fact that the biased device B is more reliable than the unbiased device A becomes apparent upon viewing a plot of the two probability density functions together with the bounds for an acceptable reading.



5. A sales agent claims that a new muffler reduces the mean noise output of a particular type of machine by more than 10 dB. A random sample of 50 such machines is drawn. The noise level from each machine is tested with the existing muffler (x) and with the new muffler (y). It is known that both populations are normally distributed with a common variance. The summary statistics are:

$$\sum x = 4775.10\,, \quad \sum x^2 = 466425.01\,, \quad n_x = 50$$
$$\sum y = 4221.52\,, \quad \sum y^2 = 366791.35\,, \quad n_y = 50$$

and with d defined as $d = y - x$,

$$\sum d = -553.58\,, \quad \sum d^2 = 6533.10\,, \quad n_d = 50$$

(a) Which of the two sample t-tests (paired or unpaired) should be conducted? State the reason for your selection. [3]

(b) Conduct the appropriate hypothesis test, at a level of significance of 5%. [10]

(c) Does the evidence support the consultant's claim? [2]

(a) The same individual machines are present in both samples. Therefore use the **paired test**

(b) Let $D = X - Y$, then test $\mathcal{H}_o : \mu_D = -10$ vs. $\mathcal{H}_a : \mu_D < -10$ at $\alpha = .05$.

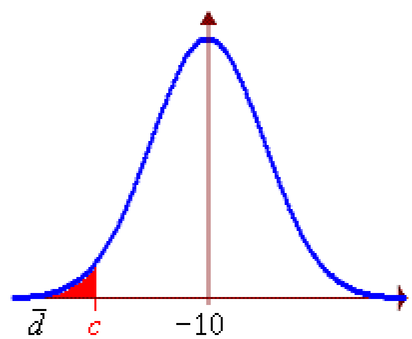
$$\bar{d} = \frac{\sum d}{n} = \frac{-553.58}{50} = -11.0716$$

$$s^2 = \frac{n\sum d^2 - \left(\sum d\right)^2}{n(n-1)} = \frac{50 \times 6533.10 - (-553.58)^2}{50 \times 49} = \frac{20204.1836}{2450} = 8.2466\ldots$$

Method 1:

$$c = \mu_o - t_{.05,\,49}\frac{s}{\sqrt{n}} \approx -10 - 1.676\sqrt{\frac{8.2466\ldots}{50}} \approx -10.68$$

$\bar{d} = -11.0716 < c \quad \therefore$ reject $\mathcal{H}_o$.



OR

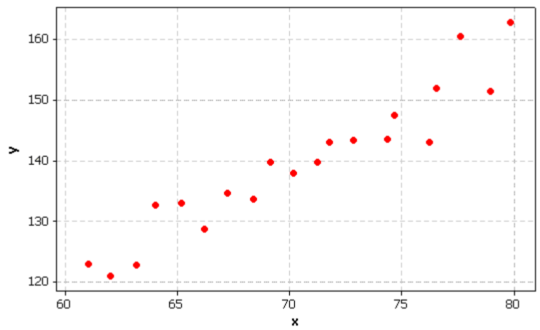
Method 2:

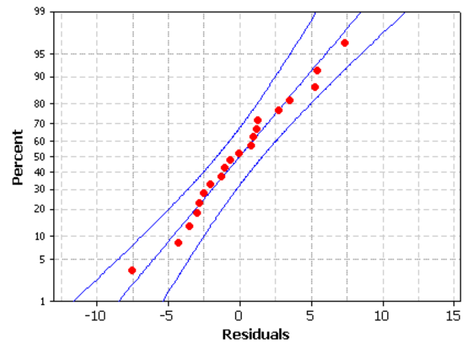
$$t_{obs} = \frac{\bar{x} - \mu_o}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{-11.0716 - (-10)}{\sqrt{\frac{8.2466\ldots}{50}}} = -\frac{1.0716}{0.406\ldots} \approx -2.64$$

$t_{.05,\,49} \approx 1.676 \qquad t_{obs} < -t_{.05,\,49} \qquad \therefore$ reject $\mathcal{H}_o$.

(c) Therefore **yes** there is sufficient evidence to conclude that the new muffler reduces the mean noise output of the particular type of machine by more than 10 dB.

6. A random sample of twenty test objects is measured before (x) and after (y) a recalibration of the measuring device. A plot of the observed values of (x, y) and a normal probability plot of the residuals from simple linear regression are shown here.





(a) State two reasons why the simple linear regression model for Y as a function of x is appropriate. [3]

Minitab® produces the regression equation $y = 5.27 + 1.905\,x$ and the following ANOVA table for these data.

Source	DF	SS	MS	F	P
Regression	1	2318.94	2318.94	164.94	0.000
Error	18	253.07	14.06		
Total	19	2572.01			

(b) To the nearest 1%, how much of the total variation in Y is explained by the simple linear regression model? [3]

(c) Use the Minitab output above to conduct an appropriate hypothesis test to determine whether or not there is a significant linear association between Y and x. [3]

(d) Use the information above, together with [3]

$$\sum x = 1410.82 \quad \text{and} \quad \sum x^2 = 100159.12,$$

to construct a 95% confidence interval for the expected value of Y when $x = 70.541$.

(a) [Any two of]

- The data have a clear linear trend
- The variance of Y appears to be independent of x
- The distribution of the residuals is consistent with a normal distribution

(b) The coefficient of determination is required here:

$$R^2 = \frac{SSR}{SST} = \frac{2318.94}{2572.01} = .9016\ldots$$

To two significant figures,

$$\boxed{R^2 = 90\%}$$

6 (c) Test $\mathcal{H}_o : \beta_1 = 0$ vs. $\mathcal{H}_a : \beta_1 \neq 0$

From the ANOVA table, $p < 0.001 <$ any reasonable α .

Therefore reject $\mathcal{H}_o : \beta_1 = 0$ in favour of $\mathcal{H}_a : \beta_1 \neq 0$.

<u>Yes</u>, there is a significant linear association between Y and x .

(d) The 95% confidence interval for $\text{E}[Y \mid x]$ is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{\frac{1}{n} + \frac{\left(x_o - \bar{x}\right)^2}{S_{xx}}}$$

$$\hat{\beta}_0 = 5.27\,, \quad \hat{\beta}_1 = 1.905\,, \quad x_o = 70.541\,, \quad n = 20\,, \quad \bar{x} = \frac{\sum x}{n} = \frac{1410.82}{20} = 70.541 = x_o$$

$$t_{.025,18} = 2.10092\,, \quad s = \sqrt{MSE} = \sqrt{14.06} = 3.749\,666\,651\ldots$$

$$n\,S_{xx} = n\sum x^2 - \left(\sum x\right)^2 = 20 \times 100159.12 - \left(1410.82\right)^2$$

$= 2\,003182.4 - 1990\,413.072\,4 = 12\,769.327\,6$ (But S_{xx} is not needed here!)

The CI is

$$\left(5.27 + 1.905 \times 70.541\right) \pm 2.100\,92\sqrt{14.06\left(\frac{1}{20} + 0\right)} = 139.6506\ldots \pm 1.7615\ldots$$

Correct to 4 s.f.,

$$\boxed{137.9 \leq \text{E}\left[Y \mid x_o\right] \leq 141.4}$$

7. A partially completed decision tree is shown here.

G = the item is good
D = the item is defective
$D = \tilde{G}$
The payoffs for accepting or rejecting items are as shown at the bottom right corner of the diagram (in units of dollars).

The test costs $50.

It is known that 90% of all items are good.

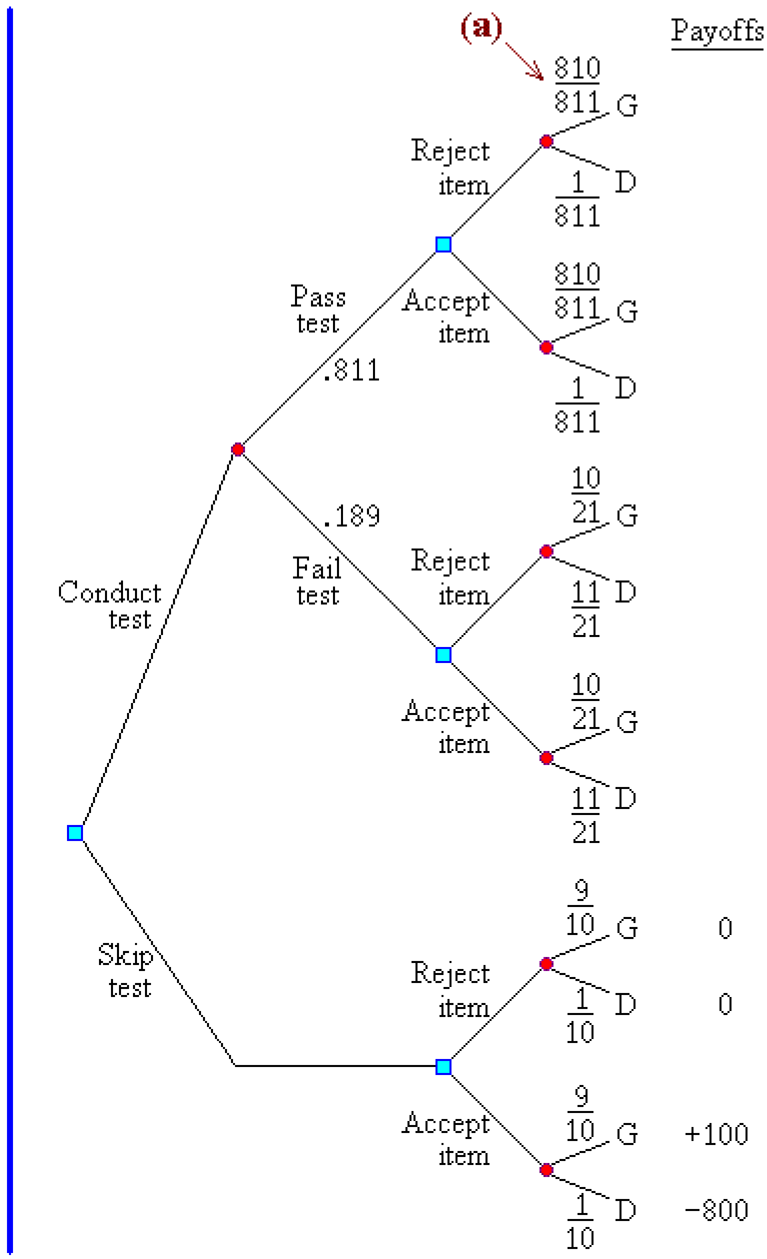
If an item is good then it will pass the test 90% of the time.

If an item is defective, then it will pass the test only 1% of the time.

(a) Verify that the probability labelled **(a)** is correct. [4]

(b) Complete the column for payoffs in the diagram. [2]

(c) Determine the optimum strategy that maximizes the expected payoff. [9]

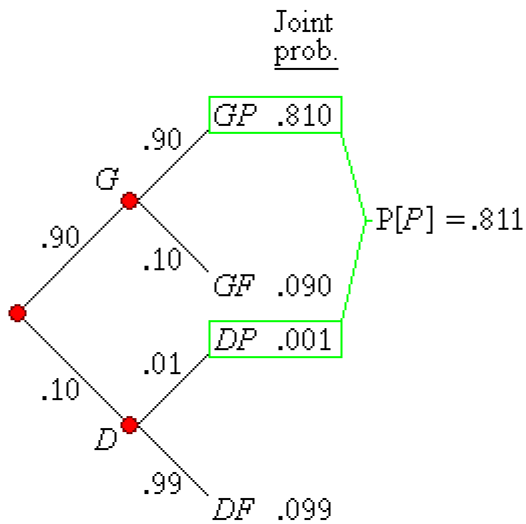


Let P = item passes test and F = item fails test = $\tilde{P}$
Given in the question:
$\text{P}[G] = .90$, $\text{P}[P \mid G] = .90$, $\text{P}[P \mid D] = .01$

(a) Using Bayes' theorem,

$$\text{P}[G \mid P] = \frac{\text{P}[GP]}{\text{P}[P]} = \frac{\text{P}[GP]}{\text{P}[GP] + \text{P}[DP]} = \frac{\text{P}[P \mid G] \cdot \text{P}[G]}{\text{P}[P \mid G] \cdot \text{P}[G] + \text{P}[P \mid D] \cdot \text{P}[D]}$$

$$= \frac{.90 \times .90}{.90 \times .90 + .01 \times .10} = \frac{.810}{.810 + .001} = \frac{810}{811}$$



OR

Use a Bayes calculation tree diagram:

$$\mathrm{P}[G \mid P] = \frac{\mathrm{P}[GP]}{\mathrm{P}[P]} = \frac{.810}{.811} = \frac{810}{811}$$

(b) For all of the missing payoffs, the test cost of \$50 must be deducted:

(c) Folding back, clearly
$a = c = -50$ and $e = 0$

$$b = 50 \times \frac{810}{811} - 850 \times \frac{1}{811} = \frac{39650}{811}$$

$$d = 50 \times \frac{10}{21} - 850 \times \frac{11}{21} = -\frac{8850}{21}$$

$$f = 100 \times \frac{9}{10} - 800 \times \frac{1}{10} = \frac{100}{10} = +10$$

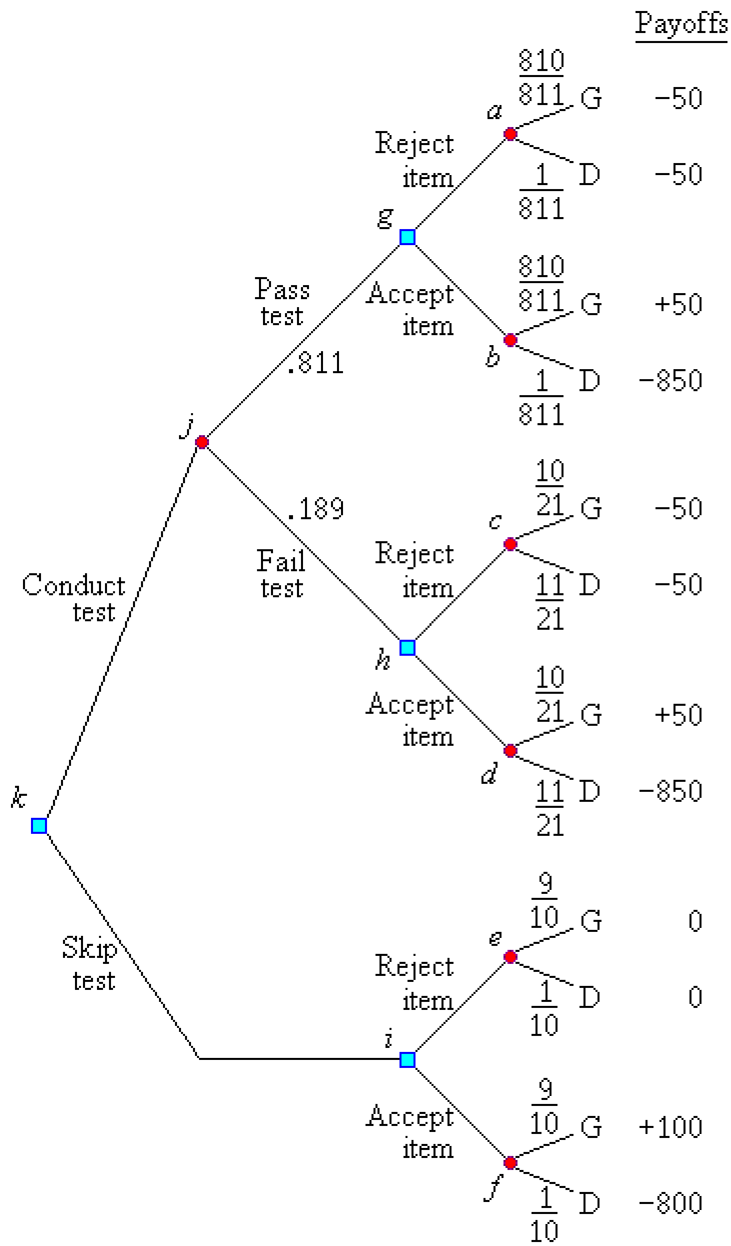
$$g = \max(a, b) = b = \frac{39650}{811}$$

$$h = \max(c, d) = c = -50$$

$$i = \max(e, f) = f = 10$$

$$j = \frac{39650}{\cancel{811}} \times \frac{\cancel{811}}{1000} + (-50) \times \frac{189}{1000}$$
$$= +30.20$$

$$k = \max(i, j) = j = 30.20$$



7 (c) (continued)

Therefore the optimum strategy is to **test the item** and

- if it passes, then accept
- if it fails, then reject

for an overall expected gain of **$30.20**

Additional notes:
The actual gain in any one instance of this process is one of the following:

- a gain of $50 (accept a good item, which occurs 81.0% of the time);
- a loss of $50 (reject an item, which occurs 18.9% of the time); or
- a loss of $850 (accept a defective item, which occurs 0.1% of the time)

The expected gains for these three parts of the optimum strategy are:

$$(\text{pass test}) \cap (\text{accept item}) \cap (\text{item is good}): \quad .811 \times \frac{810}{811} \times (+\$50) = +\$40.50$$

$$(\text{pass test}) \cap (\text{accept item}) \cap (\text{item is defective}): \quad .811 \times \frac{1}{811} \times (-\$850) = -\$0.85$$

$$(\text{fail test}) \cap (\text{reject item}): \quad .189 \times (-\$50) = -\$9.45$$

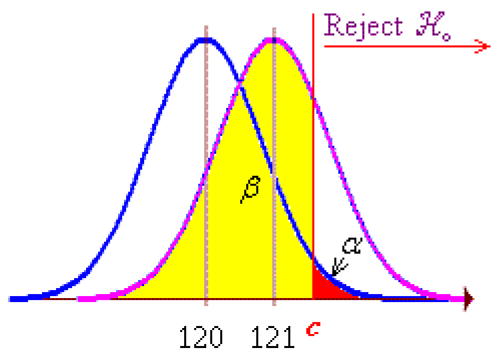
$\Rightarrow$ overall expected gain = (40.50 – 0.85 – 9.45) = **+$30.20** .
However, one must first determine which of the twelve terminal branches form the optimum strategy.

BONUS QUESTION

8. A classical hypothesis test is to be conducted on the null hypothesis $\mathcal{H}_o : \mu = 120$ vs. the alternative hypothesis $\mathcal{H}_a : \mu > 120$ at a level of significance of $\alpha = .05$, using a random sample of size 25. It is known that the population is normally distributed, with a population standard deviation $\sigma = 5.0$. Find the probability, correct to two significant figures, of committing a type II error when the true value of μ is 121.0. [+5]

We must use Method 1:

$$c = \mu_o + z_\alpha \cdot \frac{\sigma}{\sqrt{n}} = 120 + 1.64485 \cdot \frac{5.0}{\sqrt{25}} = 121.64485$$



$\mathcal{H}_o$ will be rejected iff $\bar{x} > c$.

When μ is 121.0, the probability of committing a type II error is

$$\beta(121.0) = \text{P}\left[\overline{X} < c \mid \mu = 121\right]$$

$$= \text{P}\left[Z < \frac{c - 121}{\left(\frac{\sigma}{\sqrt{n}}\right)}\right] = \Phi\left(\frac{0.64485}{1}\right) \approx \Phi(0.64) = .738\ldots$$

[Software provides a more precise value of .74049]
Therefore, correct to 2 significant figures,

$$\boxed{\beta(121.0) = .74}$$

OR using a formula,

$$\beta(\mu) = \Phi\left(\frac{\mu_o - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} + z_\alpha\right) = \Phi\left(\frac{-1}{1} + 1.64485\right) = \Phi(0.64485) \approx .74$$

☜ Back to the index of solutions