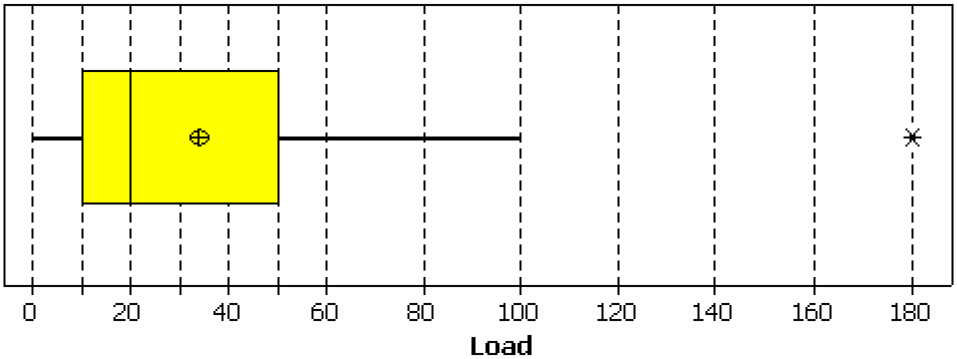
1. A sample of 43 measurements of breaking load is summarized in this boxplot.



The location of the sample mean is indicated by the symbol ⊕.

(a) Determine whether the outlier is mild or extreme. Show your working. [5]

(b) Excluding the outlier, state any two other reasons to deduce positive skew. [2]

(c) Using the summary statistics [3]

$$n = 43\,,\quad \sum x = 1462\,,\quad \sum x^2 = 107\,212$$

find the value of the sample standard deviation s for these data.

(a) $IQR = x_U - x_L = 50 - 10 = 40$

The upper outer fence is located at $x_U + 3 \times IQR = 50 + 3 \times 40 = 50 + 120 = 170$

$180 > 170 \Rightarrow$ the outlier is beyond the outer fence. Therefore the outlier is

extreme

(b) [Any two of]

- the upper quartile is much further away from the median than the lower quartile;
- mean > median
- the right whisker is much longer than left whisker

(c) $$s^2 = \frac{n\sum x^2 - \left(\sum x\right)^2}{n(n-1)} = \frac{43 \times 107\,212 - (1462)^2}{43 \times 42} = \frac{2\,472\,672}{1806} = \frac{9584}{7}$$

$$= 1369.\dot{1}4285\dot{7} \quad \Rightarrow \quad s = \sqrt{1369.\dot{1}4285\dot{7}} = 37.001930\ldots$$

Correct to 3 s.f.,

$$s = 37.0$$

2. The load W (in g) needed to break a new type of cotton thread is known to be a random quantity that follows a Normal distribution. It is believed that the population mean breaking load is 20 g and the strength of that belief is represented by the standard deviation $\sigma_o = 1\text{g}$. A random sample of 25 such threads has a mean breaking load of 18.72 g with a sample standard deviation of 2.50 g.

(a) Construct a Bayesian 95% confidence interval estimate for the true mean breaking load μ. [5]

(b) Construct a classical 95% confidence interval estimate for the true mean breaking load μ. [5]

(c) Is there sufficient evidence to conclude that the true mean breaking load μ is not 20 g? [2]

(d) Provide a *brief* reason for the different widths of your two confidence intervals. [2]

(a) Prior: $\mu_o = 20\,,\quad \sigma_o = 1 \quad \Rightarrow \quad w_o = \dfrac{1}{1^2} = 1$

Data: $\bar{x} = 18.72\,,\quad n = 25\,,\quad s = 2.50 \quad \Rightarrow \quad w_d = \dfrac{n}{s^2} = \dfrac{25}{6.25} = 4$

$$w_f = w_o + w_d = 1 + 4 = 5 \quad \Rightarrow \quad \left(\sigma^*\right)^2 = \frac{1}{w_f} = \frac{1}{5} = 0.2$$

$$\mu^* = \frac{w_o \mu_o + w_d \bar{x}}{w_o + w_d} = 5\left(1\times 20 + 4\times 18.72\right) = 18.976$$

$$\alpha = 5\% \quad \Rightarrow \quad z_{\alpha/2} = z_{.025} = 1.95996$$

The Bayesian 95% CI for μ is

$$\mu^* \pm z_{\alpha/2}\sigma^* = 18.976 \pm 1.95\ldots\times\sqrt{0.2} = 18.976 \pm 0.876\ldots =$$

$$\boxed{[18.10,\ 19.85]}$$

OR using a conservative estimate with 24 degrees of freedom,

$$\alpha = 5\% \quad \Rightarrow \quad t_{\alpha/2,\,n-1} = t_{.025,\,24} = 2.06390$$

$$\mu^* \pm t_{\alpha/2,\,\nu}\,\sigma^* = 18.976 \pm 2.06\ldots\times\sqrt{0.2} = 18.976 \pm 0.923\ldots =$$

$$\boxed{(18.05,\ 19.90)}$$

---

2 (b) $\alpha = 5\% \quad \Rightarrow \quad t_{\alpha/2,\,n-1} = t_{.025,\,24} = 2.06390$

The Classical 95% CI for μ is

$$\bar{x} \pm t_{\alpha/2,\,n-1} \frac{s}{\sqrt{n}} = 18.72 \pm 2.06\ldots \times \frac{2.5}{\sqrt{25}} = 18.72 \pm 1.03195 =$$

$$\boxed{[17.69,\ 19.75]}$$

Note that $n < 30$, so it is *not* correct to take

$$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}} = 18.72 \pm 1.95\ldots \times \frac{2.5}{\sqrt{25}} = 18.72 \pm 0.97998 = \quad (17.74,\ 19.70)$$

(c) Neither confidence interval includes 20. Therefore

YES

(d) The Bayesian confidence interval is narrower because it includes more information.

---

3. A continuous random quantity X has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{2}\right) & (-1 \le x \le +1) \\ 0 & (\text{otherwise}) \end{cases}$$

(a) Find the exact value of $\text{P}\left[\,|X| < \frac{1}{2}\right]$. [5]

(b) Find the exact value of the population variance σ^2. [5]

(a) $\text{P}\left[\,|X| < \frac{1}{2}\right] = \text{P}\left[-\frac{1}{2} < X < \frac{1}{2}\right] = \int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{1/2} \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right) dx$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi x}{2}\right)\right]_{-1/2}^{+1/2} = \frac{1}{2}\left(\sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \Rightarrow$$

$$\boxed{\text{P}\left[\,|X| < \frac{1}{2}\right] = \frac{\sqrt{2}}{2}}$$

[This probability is approximately 71%, but an exact value is required.]

Additional note for 3(a):

One may find the c.d.f. first:

$$\mathrm{P}[X \le x] = \int_{-\infty}^{x} f(t)\,dt = \int_{-1}^{x} \frac{\pi}{4}\cos\left(\frac{\pi t}{2}\right) dt \qquad (-1 \le x \le +1)$$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi t}{2}\right)\right]_{-1}^{x} = \frac{1}{2}\left(\sin\frac{\pi x}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}\left(\sin\frac{\pi x}{2} - (-1)\right) = \boxed{\frac{1}{2}\left(1 + \sin\frac{\pi x}{2}\right)}$$

Then

$$\mathrm{P}\left[\,|X| < \frac{1}{2}\right] = \mathrm{P}\left[-\frac{1}{2} < X < \frac{1}{2}\right] = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad \mathrm{P}\left[\,|X| < \frac{1}{2}\right] = \frac{\sqrt{2}}{2}$$

(b) $f(x)$ is symmetric about $x = 0 \quad \Rightarrow \quad \mu = \mathrm{E}[X] = 0$

D		I
x^2		$\frac{\pi}{4}\cos\frac{\pi x}{2}$
	+	
$2x$		$\frac{1}{2}\sin\frac{\pi x}{2}$
	−	
2		$\frac{-1}{\pi}\cos\frac{\pi x}{2}$
	+	
0		$\frac{-2}{\pi^2}\sin\frac{\pi x}{2}$

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-1}^{1} x^2 \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right) dx$$

$$= \left[\left(\frac{x^2}{2} - \frac{4}{\pi^2}\right)\sin\left(\frac{\pi x}{2}\right) + \frac{2x}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_{-1}^{+1}$$

$$= \left(\left(\frac{1}{2} - \frac{4}{\pi^2}\right)\times 1 + 0\right) - \left(\left(\frac{1}{2} - \frac{4}{\pi^2}\right)\times(-1) + 0\right) = 1 - \frac{8}{\pi^2}$$

$$\mathrm{V}[X] = \mathrm{E}\left[X^2\right] - \left(\mathrm{E}[X]\right)^2 = 1 - \frac{8}{\pi^2} - 0 \quad \Rightarrow$$

$$\boxed{\sigma^2 = 1 - \frac{8}{\pi^2}}$$

[$\sigma^2 \approx 0.189 \quad \Rightarrow \quad \sigma \approx 0.435$, but an exact value is required.]

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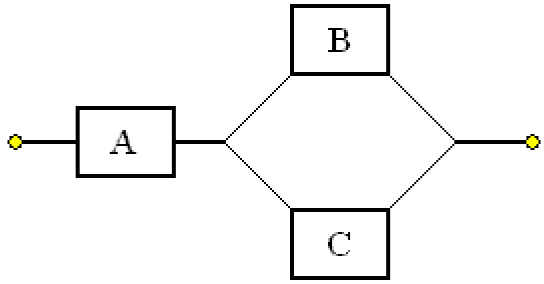
4. In a network, components B and C are connected in parallel and that subsystem is connected in series with component A as shown.



The subsystem works if at least one of the components B or C functions properly. The complete system works only if component A and the subsystem both function properly.
Let A = the event that component A functions properly;
B = the event that component B functions properly; and
C = the event that component C functions properly.
It is known that $\text{P}[A] = .80$, independently of the other two components, and that $\text{P}[B] = .50$, independently of component A. However, the event C is dependent on the event B. A failure in component B increases the load on component C and therefore reduces the probability that component C will work.

It is known that $\text{P}\left[C \mid B\right] = .90$ and $\text{P}\left[C \mid \tilde{B}\right] = .50$.

(a) Find $\text{P}\left[B \cup C\right]$. [6]

(b) Find the probability that the complete system works. [3]

(c) Find $\text{P}\left[C\right]$. [5]

(a) There are many routes to a correct solution.

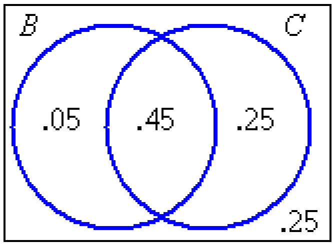
$\text{P}\left[B \cup C\right] = 1 - \text{P}\left[\sim\left(B \cup C\right)\right] = 1 - \text{P}\left[\tilde{B} \cap \tilde{C}\right]$ (by deMorgan's Laws)

$= 1 - \text{P}\left[\tilde{B}\right]\text{P}\left[\tilde{C} \mid \tilde{B}\right]$ (general multiplication law of probability)

$= 1 - \left(1 - \text{P}[B]\right)\left(1 - \text{P}\left[C \mid \tilde{B}\right]\right)$ (complementary events) [Venn diagram ***not*** required]

$= 1 - (1 - .5)(1 - .5) = 1 - \frac{1}{2} \times \frac{1}{2} = 1 - \frac{1}{4} \Rightarrow$

$$\boxed{\text{P}\left[B \cup C\right] = \frac{3}{4} = .75}$$



OR

$\text{P}\left[B \cup C\right] = \text{P}\left[B \cup \left(C \cap \tilde{B}\right)\right] = \text{P}[B] + \text{P}\left[C \cap \tilde{B}\right] = \text{P}[B] + \text{P}\left[\tilde{B}\right] \cdot \text{P}\left[C \mid \tilde{B}\right]$

$= \text{P}[B] + \left(1 - \text{P}[B]\right) \cdot \text{P}\left[C \mid \tilde{B}\right] = .5 + (1 - .5) \times .5 = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

OR

4 (a) (continued)

By the general multiplication law of probability,

$\mathrm{P}[B \cap C] = \mathrm{P}[B]\mathrm{P}[C \mid B] = .5 \times .9 = .45$ and

$\mathrm{P}[\tilde{B} \cap C] = \mathrm{P}[\tilde{B}]\mathrm{P}[C \mid \tilde{B}] = (1 - \mathrm{P}[B])\mathrm{P}[C \mid \tilde{B}] = .5 \times .5 = .25$

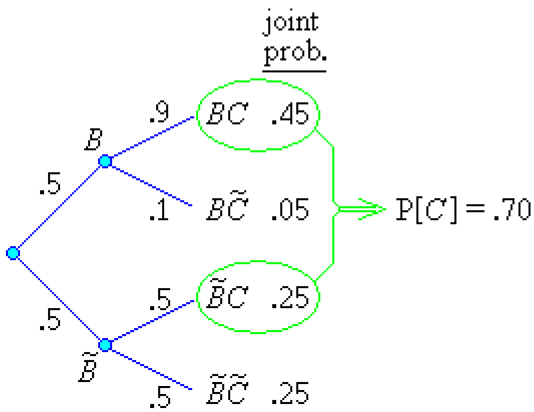
By the total probability law, $\mathrm{P}[C] = \mathrm{P}[B \cap C] + \mathrm{P}[\tilde{B} \cap C] = .45 + .25 = .70$,

(which is the answer to part (c)!)

By the general addition law of probability,

$\mathrm{P}[B \cup C] = \mathrm{P}[B] + \mathrm{P}[C] - \mathrm{P}[B \cap C] = .5 + .7 - .45 = .75$

OR



Employ a probability calculation tree:

The answer to part (c) is shown.

For part (a),

The first three of the four terminal branches together constitute $B \cup C$. Therefore

$\mathrm{P}[B \cup C] = .45 + .05 + .25 = .75$

---

4 (b) P[system works] $= \mathrm{P}[A \cap (B \cup C)] = \mathrm{P}[A] \cdot \mathrm{P}[B \cup C]$ (independent events)

$= .80 \times .75 = \boxed{\mathbf{.60}}$ or $\frac{8}{10} \times \frac{3}{4} = \frac{3}{5}$

(c) By the total probability law, $\mathrm{P}[C] = \mathrm{P}[C \cap B] + \mathrm{P}[C \cap \tilde{B}]$

By the general multiplication law, $\mathrm{P}[C] = \mathrm{P}[C \mid B] \cdot \mathrm{P}[B] + \mathrm{P}[C \mid \tilde{B}] \cdot \mathrm{P}[\tilde{B}]$

$\Rightarrow \quad \mathrm{P}[C] = .90 \times .50 + .50 \times .50 = .45 + .25 \quad \Rightarrow$

$$\boxed{\mathrm{P}[C] = .70}$$

---

5. The performances of eight machines of varying capacities in a standard test are measured before (x) and after (y) the application of a proposed new cleaning device. It is known that both populations are normally distributed with a common variance. The manager will invest in the new cleaning device only if there is evidence, at a level of significance of 5%, that the device has improved the average performance score by more than 1.0.

(a) Which of the two sample *t*-tests (paired or unpaired) should be conducted? State the reason for your selection. [3]
(b) Write down the appropriate null and alternative hypotheses. [3]
(c) Use the appropriate set of Minitab output below, to conduct the appropriate hypothesis test, at a level of significance of 5%. [4]
(d) What should the manager's decision be? [2]

Two-Sample T-Test and CI: After, Before

```
        N   Mean  StDev  SE Mean
After   8  154.0   49.9       18
Before  8  150.0   49.0       17

Difference = mu (After) - mu (Before)
Estimate for difference:  4.0
95% lower bound for difference:  -39.5
T-Test of difference = 1 (vs >): T-Value = 0.121  P-Value = 0.453  DF = 14
Both use Pooled StDev = 49.4397
```

Paired T-Test and CI: After, Before

```
            N   Mean  StDev  SE Mean
After       8  154.0   49.9     17.6
Before      8  150.0   49.0     17.3
Difference  8   4.00   4.47     1.58

95% lower bound for mean difference: 1.004
T-Test of mean difference = 1 (vs > 1): T-Value = 1.897  P-Value = 0.0498
```

5 (a) The same eight machines are in both samples. Therefore used a

paired test

(b) The burden of proof is on showing that $\mu_Y - \mu_X > 1$. Let $D = Y - X$. Therefore test

$$\mathcal{H}_o : \mu_D = 1 \quad \text{vs.} \quad \mathcal{H}_A : \mu_D > 1$$

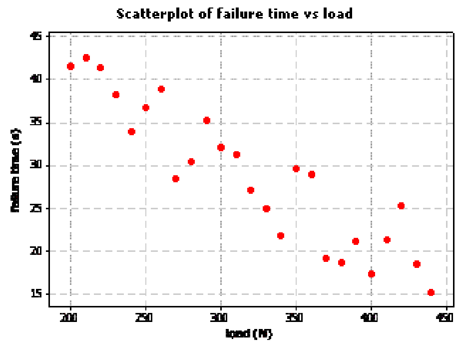
(c) Using method 3, the *p*-value for the paired *t*-test is reported by Minitab to be .0498 . $p < \alpha$ (just barely). Therefore, at a 5% level of significance,

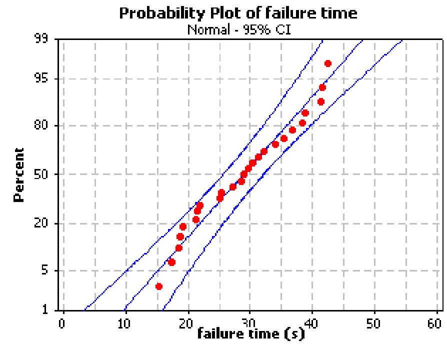
reject $\mathcal{H}_o$ in favour of $\mathcal{H}_A : \mu_Y - \mu_X > 1$

(d) We have sufficient evidence for $\mathcal{H}_A$. Therefore

invest in the new cleaning device

6. The time (y) to failure of a device is measured for each of 25 values of a load (x). A plot of the observed values of (x, y) and a normal probability plot of the residuals from simple linear regression are shown here.





(a) State two reasons why the simple linear regression model for Y as a function of x is appropriate. [2]

(a) [Any two of]

- the points (x, y) follow a linear trend.
- the distribution of the residuals is consistent with a normal distribution.
- the variance of Y seems to be independent of x.

6 (continued)

Use the following summary statistics to answer the remaining parts of this question.

$$n = 25 \qquad \sum x = 8\,000 \qquad \sum y = 719.81$$
$$\sum x^2 = 2\,690\,000 \qquad \sum xy = 216\,777.00 \qquad \sum y^2 = 22\,391.88$$
$$n\,S_{xx} = 3\,250\,000 \qquad n\,S_{xy} = -339\,055.00 \qquad n\,S_{yy} = 41670.5639$$

(b) Find the equation of the regression line $\left(\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x\right)$. [3]

(c) Construct the ANOVA table for this simple linear regression model. [5]

(d) To the nearest 1%, how much of the total variation in Y is explained by the simple linear regression model? [3]

(e) Conduct an appropriate hypothesis test to determine whether or not there is a significant linear association between Y and x. [3]

(f) Construct a 95% prediction interval for a future value of Y when $x = 320$. [4]

(b) $$\hat{\beta}_1 = \frac{n\,S_{xy}}{n\,S_{xx}} = \frac{-339\,055.00}{3\,250\,000} = -0.104\,324\,615\ldots$$

$$\hat{\beta}_0 = \frac{\sum y - \hat{\beta}_1 \sum x}{n} = \frac{719.81 - (-0.104\ldots)\times 8\,000}{25} = 62.176\,276\,923\ldots$$

Therefore, correct to 3 s.f. in the coefficients, the regression line is

$$\boxed{y = 62.2 - 0.104x}$$

(c) $$SST = S_{yy} = \frac{n\,S_{yy}}{n} = \frac{41670.5639}{25} = 1666.822\,556$$

$$SSR = \frac{\left(n\,S_{xy}\right)^2}{n\left(n\,S_{xx}\right)} = \frac{(-339\,055.00)^2}{25\times 3\,250\,000} = 1414.871\,298\ldots$$

$$SSE = SST - SSR = 1666.8\ldots - 1414.8\ldots = 251.951\,257\,230\ldots$$

$$MSR = \frac{SSR}{1} = 1414.8\ldots \qquad MSE = \frac{SSE}{n-2} = \frac{251.9\ldots}{23} = 10.954\,402\,488\ldots$$

$$f = t^2 = \frac{MSR}{MSE} = \frac{1414.8\ldots}{10.9\ldots} = 129.160\ldots$$

Therefore the ANOVA table is [next page]

6 (c) (continued)

Source	d.f.	SS	MS	f
R	1	1 414.871...	1 414.871...	129.160...
E	23	251.951...	10.954...	
T	24	1 666.822...		

(d) Required is the coefficient of determination, $r^2 = \frac{SSR}{SST} = \frac{1414.8\ldots}{1666.8\ldots} = .848\ldots \Rightarrow$

$$\boxed{r^2 = 85\%}$$

(e) Test $\mathcal{H}_o: \beta_1 = 0$ vs. $\mathcal{H}_A: \beta_1 \neq 0$

From the ANOVA table, $t^2 = f = 129.160\ldots \Rightarrow t = -11.364\ldots$

[Note: $t < 0$ because $\hat{\beta}_1 < 0$]

But $t_{.005,\,23} = 2.80734$. $|t_{obs}| \gg t_{.005,\,23}$

Therefore reject $\mathcal{H}_o$ in favour of $\mathcal{H}_A$ at any reasonable α.

There is a significant linear association between Y and x.

(f) $x_o = 320$. $\bar{x} = \frac{\sum x}{n} = \frac{8000}{25} = 320 \Rightarrow x_o = \bar{x}$

This simplifies the calculation of the prediction interval,

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,\,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{n\left(x_o - \bar{x}\right)^2}{\left(n S_{xx}\right)}}, \quad \text{where} \quad s = \sqrt{MSE}$$

$$\hat{\beta}_0 + \hat{\beta}_1 x_o = 62.176\ldots - 0.104\ldots \times 320 = 28.792\,400$$

OR note that $x_o = \bar{x} \Rightarrow y_o = \bar{y} = \frac{\sum y}{n} = \frac{719.81}{25} = 28.7924$

$$t_{\alpha/2,\,(n-2)} = t_{.025,\,23} = 2.06866$$

$$t_{\alpha/2,\,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{n\left(x_o - \bar{x}\right)^2}{\left(n S_{xx}\right)}} = 2.06866\sqrt{10.954\ldots\left(1 + \frac{1}{25} + 0\right)} = 6.982\,326\ldots$$

The 95% PI is therefore $28.792\,400 \pm 6.982\,326\ldots \Rightarrow$

$$\boxed{(21.81,\ 35.77)}$$

7. A partially completed decision tree is shown here.

G = the item is good
D = the item is defective
$D = \tilde{G}$

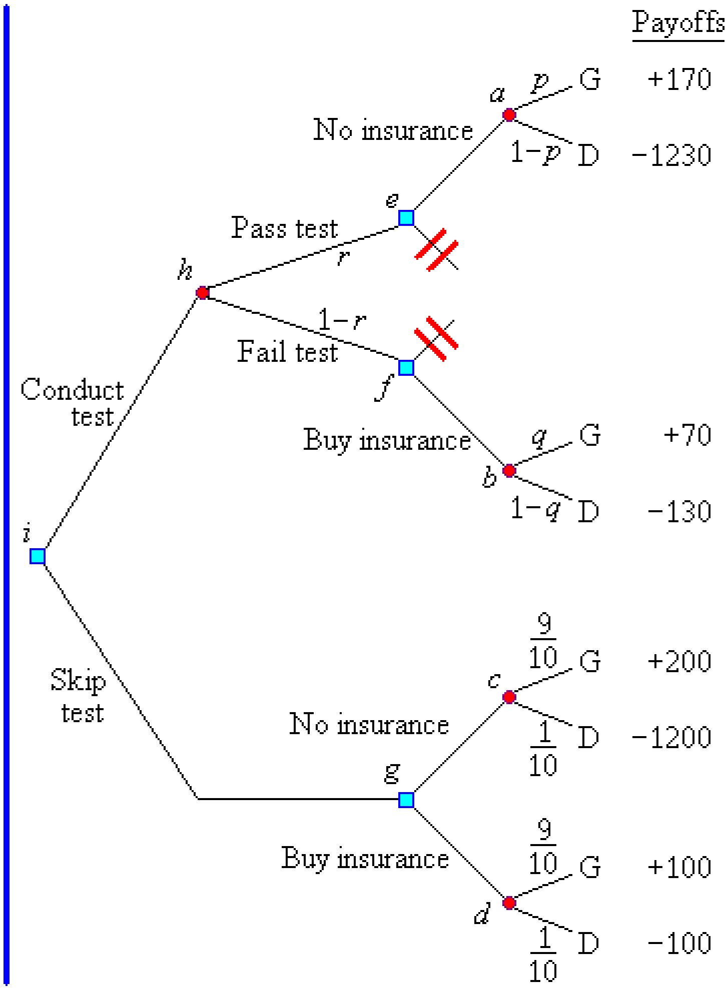
The payoffs for items are as shown at the right edge of the diagram (in units of dollars). Insurance protects against the loss due to a defective item.

The test costs $30.
Insurance, costing $100, is purchased if the item fails the test and is not purchased if the item passes the test.

It is known that 90% of all items are good.

If an item is good then it will pass the test 80% of the time.

If an item is defective, then it will pass the test 20% of the time.



(a) Verify that the probability labelled **(*p*)** is exactly $\frac{36}{37}$. [3]

(b) Verify that the probability labelled **(*q*)** is exactly $\frac{9}{13}$. [2]

(c) Find the probability labelled **(*r*)**. [2]

(d) Verify that the expected value labelled **(*a*)** is exactly $\frac{4890}{37}$. [1]

(e) Find the other expected values labelled **(*b*)**, **(*c*)**, **(*d*)**, **(*e*)**, **(*f*)**, **(*g*)**, **(*h*)**, **(*i*)** and hence determine the optimum strategy that maximizes the expected payoff. [10]

(f) By how much can the test cost be increased before the optimum strategy changes? [2]

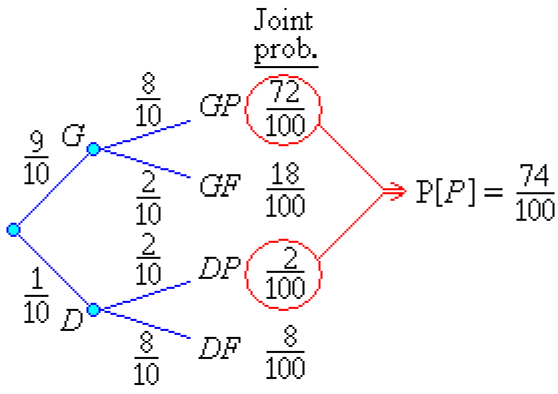
7 (a) Let P = item passes test and $F = \tilde{P}$ = item fails test.

$$p = \text{P}[G \mid P] = \frac{\text{P}[GP]}{\text{P}[P]}$$

We know that $\text{P}[G] = \frac{9}{10}$,

$\text{P}[P \mid G] = \frac{8}{10}$ and $\text{P}[P \mid D] = \frac{2}{10}$

From the secondary tree diagram,



$$p = \frac{72}{100} \div \frac{74}{100} = \frac{72}{74} = \frac{36}{37}$$

(b) $$q = \text{P}[G \mid F] = \frac{\text{P}[GF]}{\text{P}[F]}$$

$\text{P}[F] = 1 - \text{P}[P] = \frac{100-74}{100} = \frac{26}{100} \quad \Rightarrow \quad q = \frac{18}{100} \times \frac{100}{26} = \frac{9}{13}$

(c) From the diagram in part (a), $r = \text{P}[P] = \boxed{\mathbf{\frac{74}{100}}} = .74$

(d) $$a = +170p + (-1230)(1-p) = \frac{170 \times 36 - 1230 \times 1}{37} = \frac{6120-1230}{37} = \frac{4890}{37}$$

(e) $$b = +70q + (-130)(1-q) = \frac{70 \times 9 - 130 \times 4}{13} = \frac{630-520}{13} = \boxed{\mathbf{\frac{110}{13}}}$$

$$c = +200 \times \frac{9}{10} + (-1200) \times \frac{1}{10} = \frac{1800-1200}{10} = \boxed{\mathbf{60}}$$

$$d = +100 \times \frac{9}{10} + (-100) \times \frac{1}{10} = \frac{900-100}{10} = \boxed{\mathbf{80}}$$

$e = a = \boxed{\mathbf{\frac{4890}{37}}}$, $f = b = \boxed{\mathbf{\frac{110}{13}}}$, $g = \max(c,d) = d = \boxed{\mathbf{80}}$

$$h = e \cdot r + f \cdot (1-r) = \frac{4890}{37} \times \frac{74}{100} + \frac{110}{13} \times \frac{26}{100} = \frac{9780+220}{100} = \frac{10\,000}{100} = \boxed{\mathbf{100}}$$

$$i = \max(g,h) = h = \boxed{\mathbf{100}}$$

---

7 (e) (continued)

The optimum strategy is therefore

> **invest in the test**; and
> buy insurance if and only if the item fails the test,
> for a maximum expected gain of **$100**

Additional Note (not required for a complete solution in the examination):

The actual gain is one of

- +$ 170 if the item is good and passes the test (72% of the time);
- +$ 70 if the item is good but fails the test (18% of the time);
- –$ 130 if the item is defective and fails the test (8% of the time); or
- –$1,230 if the item is defective but passes the test (2% of the time)

(f) The amount by which the test cost can be increased without changing the optimum strategy is $h - g = 100 - 80 =$

> $20

BONUS QUESTION

8. A two-sample hypothesis test is designed to determine whether or not the true mean efficiency rating μ_{new} of a prototype machine is higher than the true mean efficiency rating μ_{old} of an existing machine. Random samples of equal size n are taken from each type of machine and their efficiencies are recorded. It is known that the efficiencies are normally distributed with the same known population standard deviation $\sigma = 10$ in both cases. [+5]

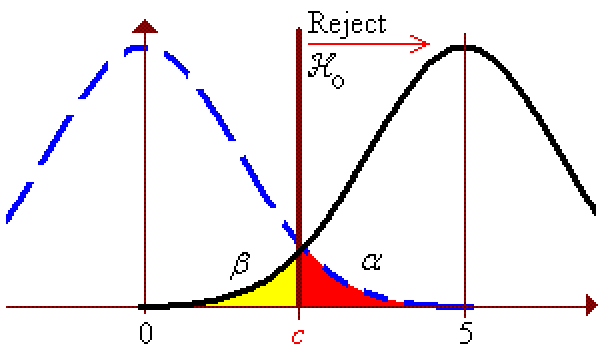
Find the minimum sample size $n_{\min}$ needed to ensure that the probabilities α (of type I error) and $\beta(5)$ (type II error when $\mu_{\text{new}} - \mu_{\text{old}} = 5$) are both 5%.
[If you quote a formula for $n_{\min}$, then derive that formula.]

Test $\mathcal{H}_o : \mu_{\text{new}} - \mu_{\text{old}} = 0$ vs. $\mathcal{H}_A : \mu_{\text{new}} - \mu_{\text{old}} > 0$ at $\alpha = .05$

Under $\mathcal{H}_o$, $\mathrm{E}\left[\overline{X}_{\text{new}} - \overline{X}_{\text{old}}\right] = 0$

$$\mathrm{V}\left[\overline{X}_{\text{new}} - \overline{X}_{\text{old}}\right] = \frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \frac{2\sigma^2}{n} = \frac{200}{n}$$

$$\Rightarrow \quad \overline{X}_{\text{new}} - \overline{X}_{\text{old}} \sim \mathrm{N}\left(0, \frac{200}{n}\right)$$



The boundary of the rejection region is at

$$c = \Delta_o + z_\alpha (s.e.) = 0 + 1.64\ldots\sqrt{\frac{200}{n}}$$

$$\beta(5) = \mathrm{P}\left[\overline{X}_{\text{new}} - \overline{X}_{\text{old}} < c \,\middle|\, \mu_{\text{new}} - \mu_{\text{old}} = 5\right]$$

$$= \mathrm{P}\left[Z < \frac{c-5}{\sqrt{\frac{200}{n}}}\right] = \Phi\left(\frac{c-5}{\sqrt{\frac{200}{n}}}\right) = \Phi\left(1.64\ldots - \sqrt{n}\frac{5}{\sqrt{200}}\right)$$

But we require $\beta(5) = .05 = \Phi(-1.64\ldots)$

$$\Rightarrow \quad 1.64\ldots - \sqrt{n}\frac{5}{\sqrt{200}} = -1.64\ldots \quad \Rightarrow \quad \sqrt{n} = 2\frac{\sqrt{200}}{5}\times 1.64\ldots$$

$$\Rightarrow \quad n = 200\left(\frac{2}{5}\times 1.64\ldots\right)^2 = 86.99\ldots \quad \Rightarrow$$

$$\boxed{n_{\min} = \mathbf{87}}$$

8. <u>Additional Note</u>:

The general case is:

Test $\mathcal{H}_o : \mu_B - \mu_A = \Delta_o$ vs. $\mathcal{H}_A : \mu_B - \mu_A > \Delta_o$ at some specified α .

For equal sample sizes n and equal population variances σ^2,

$$V\left[\bar{X}_B - \bar{X}_A\right] = \frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \frac{2\sigma^2}{n} \quad \Rightarrow \quad (s.e.) = \sigma\sqrt{\frac{2}{n}}$$

The boundary of the rejection region is at $c = \Delta_o + z_\alpha (s.e.)$

$$\beta(\Delta_1) = \Phi\left(-z_\beta\right) = \Phi\left(\frac{c - \Delta_1}{(s.e.)}\right) = \Phi\left(\frac{\Delta_o + z_\alpha (s.e.) - \Delta_1}{(s.e.)}\right) = \Phi\left(z_\alpha + \frac{\Delta_o - \Delta_1}{(s.e.)}\right)$$

$$\Rightarrow \quad -z_\beta = z_\alpha + \frac{\Delta_o - \Delta_1}{(s.e.)} \qquad \Rightarrow \quad \frac{\Delta_1 - \Delta_o}{(s.e.)} = z_\alpha + z_\beta$$

$$\Rightarrow \quad \frac{1}{\sigma}\sqrt{\frac{n}{2}}\left(\Delta_1 - \Delta_o\right) = z_\alpha + z_\beta \qquad \Rightarrow \quad \sqrt{n} = \frac{\left(z_\alpha + z_\beta\right)\sigma\sqrt{2}}{\Delta_1 - \Delta_o}$$

Therefore

$$\boxed{n_{\min} = \text{ceil}\left(2\left(\frac{\left(z_\alpha + z_\beta\right)\sigma}{\Delta_1 - \Delta_o}\right)^2\right)}$$

With $\Delta_o = 0$, $\Delta_1 = 5$, $\alpha = \beta = .05 \Rightarrow z_\alpha = z_\beta = 1.64485$ and $\sigma = 10$,

$$n_{\min} = \text{ceil}\left(2\left(\frac{2 \times 1.64485 \times 10}{5 - 0}\right)^2\right) = \text{ceil}\left(2\left(\frac{2 \times 1.64485 \times 10}{5 - 0}\right)^2\right)$$

$$= \text{ceil}(86.577\ldots) = 87$$

☜ [Back to the index of solutions]