1) A continuous random quantity X is known to be normally distributed with a population mean $\mu = 20.4$ and a population variance $\sigma^2 = 25.1$.

(a) Evaluate $\text{P}[X \le 15.0]$. [3]

(b) A random sample of size 4 is taken from this population. $\overline{X}$ is the sample mean. Evaluate $\text{P}[\overline{X} \le 15.0]$. [5]

Note: You do ***not*** need to use linear interpolation in this question.
Quote your answers correct to only two significant figures.

(a) $X \sim \text{N}(20.4, 25.1)$

$$\text{P}[X \le 15.0] = \text{P}\left[Z \le \frac{15.0 - 20.4}{\sqrt{25.1}}\right] \approx \Phi(-1.08) = .14007 \quad \text{(from the supplied tables)}$$

Therefore, correct to two significant figures,

$$\boxed{\text{P}[X \le 15.0] = .14}$$

(b) By the Central Limit Theorem, $\overline{X} \sim \text{N}\left(20.4, \frac{25.1}{4}\right) = \text{N}(20.4, 6.275)$

$$\text{P}[\overline{X} \le 15.0] = \text{P}\left[Z \le \frac{15.0 - 20.4}{\sqrt{6.275}}\right] \approx \Phi(-2.16) = .01539 \quad \text{(from the supplied tables)}$$

Therefore, correct to two significant figures,

$$\boxed{\text{P}[\overline{X} \le 15.0] = .015}$$

[although .016 is actually more accurate.]

continued ☞

2) The joint probability mass function $p(x, y)$ for random quantities X, Y is defined by the table:

	$p(x, y)$	Y: –1	0	1	
	–1	.20	.15	.15	
X	0	.15	.14	.11	
	1	.05	.01	.04	

(a) Find the covariance $\text{Cov}(X, Y)$. [8]

(b) Are the random quantities X, Y independent? Why or why not? [4]

(a) Extending the table,

	$p(x, y)$	Y: –1	0	1	$p_X(x)$
	–1	.20	.15	.15	**.50**
X	0	.15	.14	.11	**.40**
	1	.05	.01	.04	**.10**
	$p_Y(y)$	**.40**	**.30**	**.30**	**1**

$$\text{E}[X] = -1\times.5+0\times.4+1\times.1 = -.5+0+.1 = -0.4$$

$$\text{E}[Y] = -1\times.4+0\times.3+1\times.3 = -.4+0+.3 = -0.1$$

$$\text{E}[XY] = -1\times-1\times.20+0+-1\times1\times.15 \;+\; 0+0+0 \;+\; 1\times-1\times.05+0+1\times1\times.15$$

$$= .20-.15-.05+.04 = 0.04$$

$$\text{Cov}[X,Y] = \text{E}[XY] - \text{E}[X]\text{E}[Y] = 0.04 - (-0.4)\times(-0.1) = 0.04-0.04 \Rightarrow$$

$$\boxed{\text{Cov}[X,Y] = 0}$$

(b) Zero covariance does ***not*** imply independence.

One must test $p(x,y) = p_X(x)\cdot p_Y(y) \quad \forall(x,y)$.

The test actually passes for the entire top row { (–1, –1), (–1, 0), (–1, 1) } !

However it fails for the remaining six entries, for example

$p_X(0)\cdot p_Y(-1) = .4\times.4 = .16 \neq p(0,1) \Rightarrow$

NO : X, Y are not independent.

3) A box contains twelve (12) gear wheels, of which three (3) are protected with a rust-proofing treatment and the other nine (9) are not protected. A random sample of two (2) gear wheels is drawn, both at once, from the box. Let the random quantity X represent the number of gear wheels in the random sample that are protected.

(a) Show why the probability mass function (p.m.f.) for X is ***not*** binomial. [2]

(b) Find $\text{P}[X = 3]$. [2]

(c) Find the exact probability mass function $p(x)$ for X. [10]

(d) If the sample were drawn with replacement, then would the p.m.f. for X be binomial? Why or why not? [2]

(a) Let S_n = success (protected gear wheel) in trial n.

$\text{P}[S_1] = \text{P}[S_2] = \frac{3}{12} = \frac{1}{4} = 0.25$

$\text{P}[S_2 \mid S_1] = \frac{2}{11} = 0.\dot{1}\dot{8} \neq \text{P}[S_2]$

Therefore the third condition for a binomial p.m.f., (independence of trials) is false. The dependence is strong enough that the binomial p.m.f. is not even a good approximation to the exact p.m.f. (which is hypergeometric).

OR

The independence condition fails because the sample is drawn without replacement from a small population.

(b) Sample size $n = 2 \Rightarrow X > 2$ is absolutely impossible. Therefore

$$\boxed{\text{P}[X = 3] = 0}$$

(c) In order to obtain x successes in the sample, we must draw x successes from the 3 successes available in the population **and** we must draw $(2 - x)$ failures from the 9 failures available in the population. Therefore the exact p.m.f. is

$$\boxed{\text{P}[X = x] = \frac{{}^{3}C_{x} \cdot {}^{9}C_{2-x}}{{}^{12}C_{2}}}$$

The individual values are

$$\text{P}[X = 0] = \frac{{}^{3}C_{0} \cdot {}^{9}C_{2}}{{}^{12}C_{2}} = 1 \times \frac{9 \times 8}{2 \times 1} \times \frac{2 \times 1}{12 \times 11} = \underline{\underline{\mathbf{\frac{6}{11}}}}$$

$$\text{P}[X = 1] = \frac{{}^{3}C_{1} \cdot {}^{9}C_{1}}{{}^{12}C_{2}} = 3 \times 9 \times \frac{2 \times 1}{12 \times 11} = \underline{\underline{\mathbf{\frac{9}{22}}}}$$

$$\text{P}[X = 2] = \frac{{}^{3}C_{2} \cdot {}^{9}C_{0}}{{}^{12}C_{2}} = 3 \times 1 \times \frac{2 \times 1}{12 \times 11} = \underline{\underline{\mathbf{\frac{1}{22}}}}$$

3 (d) If the sampling were with replacement, then

continued ☞

$\text{P}\left[S_2 \mid S_1\right] = \frac{3}{12} = \text{P}\left[S_2\right]$ so that trials would be independent.
The other three conditions for a binomial p.m.f. also hold:
each trial consists of a pair of complementary outcomes (protected or not);
probability of success is constant across trials, $\text{P}\left[S_1\right] = \text{P}\left[S_2\right] = \frac{3}{12}$;
the number of trials is fixed, $n = 2$.
Therefore **YES** the p.m.f. would be binomial (exactly).

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4) A function $f(x)$ of a continuous variable x is defined by

$$f(x) = \begin{cases} 105\left(x^4 - 2x^5 + x^6\right) & (0 < x < 1) \\ 0 & (\text{otherwise}) \end{cases}$$

(a) Show that $f(x)$ is a well-defined probability density function (p.d.f.). [2]
(b) Find the cumulative distribution function (c.d.f.) $F(x)$ for this p.d.f. [8]
(c) Hence evaluate $\text{P}\left[X > \frac{1}{2}\right]$ exactly. Leave your answer as a fraction. [4]

BONUS QUESTION:
(d) Find the population mean μ as a fraction reduced to its lowest terms. [+3]

(a) $105\left(x^4 - 2x^5 + x^6\right) = 105x^4(1-x)^2 > 0$ for $0 < x < 1$.

$f(x) = 0$ everywhere else. Therefore $f(x) \geq 0$ for all x.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 105\left(x^4 - 2x^5 + x^6\right)dx = \left[105\left(\frac{x^5}{5} - \frac{x^6}{3} + \frac{x^7}{7}\right)\right]_0^1$$

$$= 105\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} - 0\right) = 21 - 35 + 15 = 1$$

Therefore $f(x)$ is a well-defined probability density function.

(b) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Clearly $F(x) = 0$ for all $x < 0$ and $F(x) = 1$ for all $x > 1$. For $0 \leq x \leq 1$,

$$F(x) = \int_0^x 105\left(t^4 - 2t^5 + t^6\right)dt = \left[105\left(\frac{t^5}{5} - \frac{t^6}{3} + \frac{t^7}{7}\right)\right]_0^x$$

$$= 105\left(\frac{x^5}{5} - \frac{x^6}{3} + \frac{x^7}{7} - 0\right) = x^5\left(21 - 35x + 15x^2\right)$$

continued ☞

4 (b) (continued)

Therefore the c.d.f. is

$$F(x) = \begin{cases} 0 & (x<0) \\ x^5\left(21-35x+15x^2\right) & (0 \le x \le 1) \\ 1 & (x>1) \end{cases}$$

(c) $\mathrm{P}\left[X > \tfrac{1}{2}\right] = 1 - \mathrm{P}\left[X \le \tfrac{1}{2}\right] = 1 - F\left(\tfrac{1}{2}\right)$

$$= 1 - \left(\frac{1}{2}\right)^5 \left(21 - 35\left(\frac{1}{2}\right) + 15\left(\frac{1}{2}\right)^2\right) = \left(\frac{1}{2}\right)^7 \left(128 - \left(21\times 4 - 35\times 2 + 15\right)\right)$$

$$= \frac{128 - (84-70+15)}{128} = \frac{128-29}{128} \Rightarrow$$

$$\mathrm{P}\left[X > \tfrac{1}{2}\right] = \frac{99}{128}$$

(d) $\mu = \mathrm{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 105\left(x^5 - 2x^6 + x^7\right) dx = \left[105\left(\frac{x^6}{6} - \frac{2x^7}{7} + \frac{x^8}{8}\right)\right]_0^1$

$$= \left[\frac{35x^6}{2} - 30x^7 + \frac{105x^8}{8}\right]_0^1 = \frac{140 - 240 + 105 - 0}{8} \Rightarrow$$

$$\mu = \frac{5}{8}$$

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