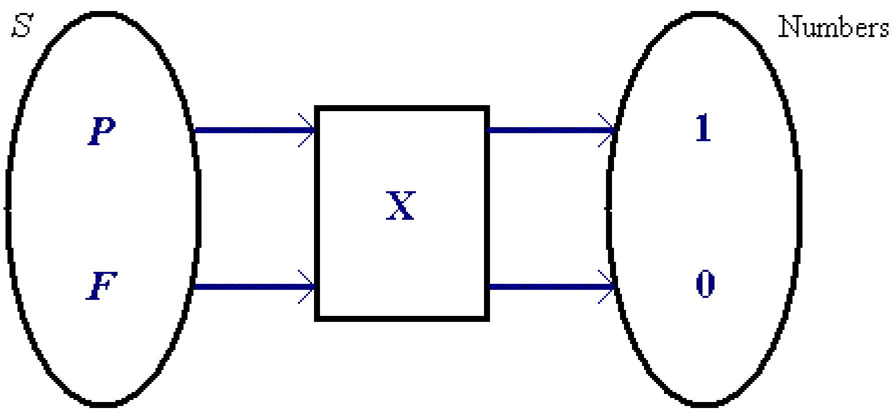
A **random quantity** [r.q.] maps an outcome to a number.

Example 6.01:

P = "A student passes ENGI 3423"
F = "That student fails ENGI 3423"

The sample space is $S = \{ P, F \}$



Define $X(P) = 1$, $X(F) = 0$, then
X is a random quantity.

Definition: A **Bernoulli random quantity** has only two possible values: 0 and 1.

Example 6.02

Let Y = the sum of the scores on two fair six-sided dice.

$Y(i, j) = i + j$

The possible values of Y are: **2, 3, 4, ... , 12**

Example 6.03

Let N = the number of components tested when one fails.

The possible values of N are: **1, 2, 3, ...**

A set D is **discrete** if

$n(D)$ is finite

OR

$n(D)$ is "countably infinite" (consecutive values can be found)

Examples:

6.03. Set $\mathbb{N}$ = (the set of all natural numbers) is **discrete (countably infinite)**

6.04. $A = \{ x : 1 \le x \le 2$ and x is real $\}$ is **not discrete (it is continuous)**

['1' is the smallest value, but what is the second-smallest value?]

A random quantity is **discrete** if its set of possible values is a discrete set.

Each value of a random quantity has some probability of occurring. The set of probabilities for all values of the random quantity defines a function $p(x)$, known as the

Probability Mass Function
(or probability function)
(*p.m.f.*):

$$p(x) = \text{P}[X = x]$$

Note: X is a random quantity, but x is a particular value of that random quantity.

All probability mass functions satisfy both of these conditions:

$$p(x) \ge 0 \quad \forall x$$

and

$$\sum_{\text{all } x} p(x) = 1$$

[Note that these two conditions together ensure that $p(x) \le 1 \;\; \forall x$.]

Example 6.05

$$f(x) = \begin{cases} cx^2 & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \quad \leftarrow \text{[NOTE: may omit this branch]}$$

[$f(x) = 0$ is assumed for all x not mentioned in the definition of $f(x)$.]

$f(x)$ is a probability mass function. Find the value of the constant c .

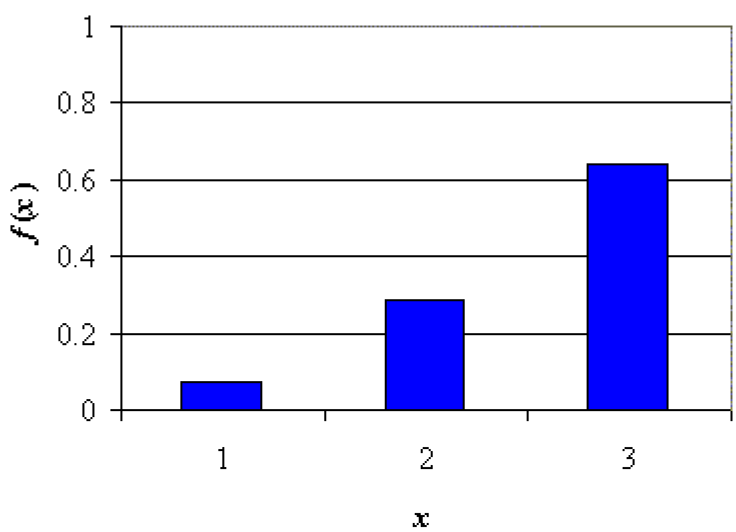
$p(x) \geq 0 \quad \forall x \quad \Rightarrow \quad c \geq 0$

$\sum p(x) = 1 \quad \Rightarrow \quad c(1)^2 + c(2)^2 + c(3)^2 = 1$

Therefore

$$c = \frac{1}{14}$$

Bar Chart:



Example 6.06

Find the p.m.f. for X = (the number of heads when two fair coins are tossed).

Let H_i = head on coin i and T_i = tail on coin i.

The possible values of X are X = **0** **(T_1T_2)**
1 **(H_1T_2 or T_1H_2)**
or **2** **(H_1H_2)**

$$
\begin{aligned}
P[X=0] &= P[T_1T_2] \\
&= P[T_1]\,P[T_2|T_1] \\
&= \tfrac{1}{2}\times\tfrac{1}{2} \qquad \text{(independent events)} \\
&= \tfrac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
P[X=1] &= P[H_1T_2] + P[T_1H_2] \qquad \text{(mutually exclusive events)} \\
&= P[H_1]\,P[T_2] + P[T_1]\,P[H_2] \qquad \text{(independent events)} \\
&= \tfrac{1}{2}\times\tfrac{1}{2} + \tfrac{1}{2}\times\tfrac{1}{2} \\
&= \tfrac{1}{4} + \tfrac{1}{4} \\
&= \tfrac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
P[X=2] &= P[H_1H_2] \\
&= \tfrac{1}{2}\times\tfrac{1}{2} \\
&= \tfrac{1}{4}
\end{aligned}
$$

Therefore the p.m.f. is

$$
f(x) = \begin{cases} \frac{1}{4} & (x=0,2) \\ \frac{1}{2} & (x=1) \\ 0 & (\text{otherwise}) \end{cases}
$$

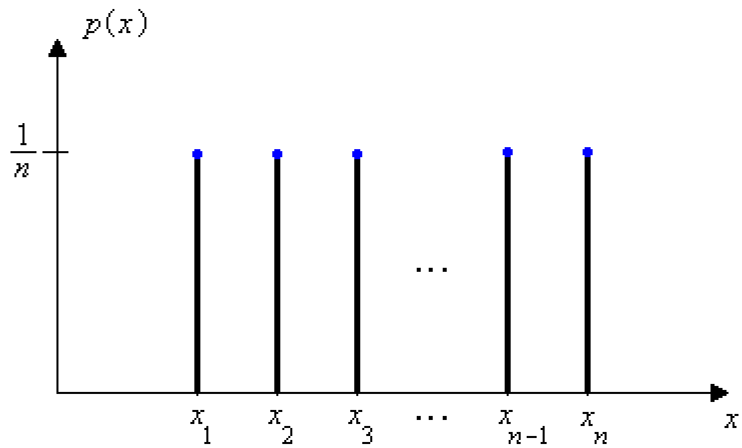
The Discrete Uniform Probability Distribution

A random quantity X , whose n possible values $\{ x_1, x_2, x_3, \ldots , x_n \}$ are all equally likely, possesses a discrete uniform probability distribution.

$$P[X = x_i] = \frac{1}{n} \qquad (i = 1,\ 2,\ \ldots,\ n)$$

An example is X = (the score on a fair standard six-sided die), for which $n = 6$ and $x_i = i$.

Line graph:



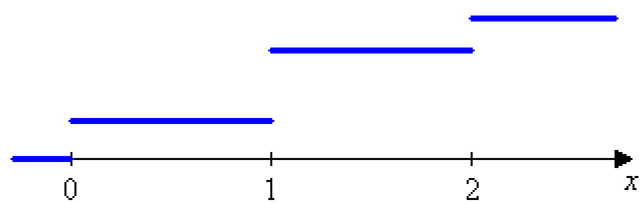
Cumulative Distribution Function (c.d.f.)

$$F(x) = \text{P}[X \le x] = \sum_{y:\, y \le x} p(y)$$

Example 6.07

Find the cumulative distribution function for
X = (the number of heads when two fair coins are tossed).

The possible values of X are 0, 1 and 2.



$F(0) = \text{P}[X \le 0] = p(0) = 1/4$.

$F(1) = \text{P}[X \le 1] = \text{P}[X < 1] + \text{P}[X = 1]$

$= F(0) + p(1)$

$= \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$

$F(2) = \text{P}[X \le 2]$

$= F(1) + p(2)$

$= \tfrac{3}{4} + \tfrac{1}{4} = \mathbf{1}$

When $x < 0$, $F(x) = \text{P}[X \le x] \le \text{P}[X < 0] = 0 \Rightarrow F(x) = 0$

When $x > 2$, $F(x) = \text{P}[X \le x] = F(2) + \text{P}[2 < X \le x] = \mathbf{1 + 0 = 1}$

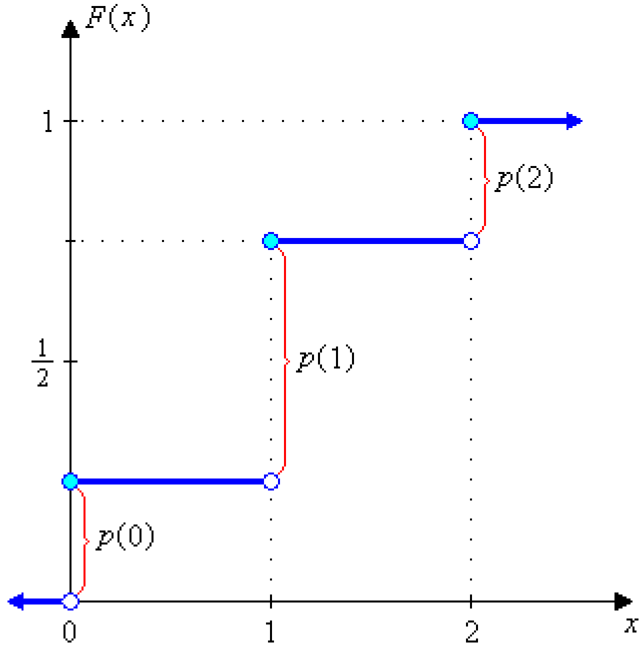
When $1 < x < 2$, $\text{P}[X \le x] = F(1) + \text{P}[1 < X \le x] = \tfrac{3}{4} + \mathbf{0} = \tfrac{3}{4}$

etc.

Thus

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1/4 & \text{if} \quad 0 \le x < 1 \\ 3/4 & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases}$$

The graph of the c.d.f. is:



In general, the graph of a discrete c.d.f. :

- is always non-decreasing
- is level between consecutive possible values (staircase appearance)
- has a finite discontinuity at each possible value (step height = $p(x)$)
- rises in steps from $F(x) = 0$ to $F(x) = 1$.

Drawing convention:

● filled circle = point included in interval
○ open circle = point excluded from interval

Example 6.08 (the inverse of the preceding problem):

Find the probability mass function $p(x)$ given the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1/4 & \text{if} \quad 0 \le x < 1 \\ 3/4 & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases}$$

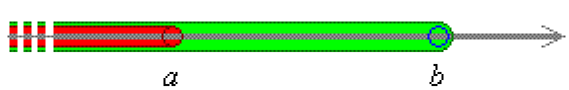
Steps (= possible values) are at $x = 0, 1, 2$ only.

$$p(0) = F(0) = \tfrac{1}{4}$$

$$p(1) = F(1) - F(0) = \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2}$$

$$p(2) = F(2) - F(1) = 1 - \tfrac{3}{4} = \tfrac{1}{4}$$

and we recover the original p.m.f.



In general,

last kept − last excluded

$$\boxed{P[a < X \le b] = F(b) - F(a)}$$

If a, b and **all** possible values are integers, then

$$\boxed{P[a \le X \le b] = F(b) - F(a-1)} \quad \text{and} \quad \boxed{p(a) = P[X = a] = F(a) - F(a-1)}$$

Example 6.09

Find and sketch the *c.d.f.* for X = (the score upon rolling a fair standard die once).

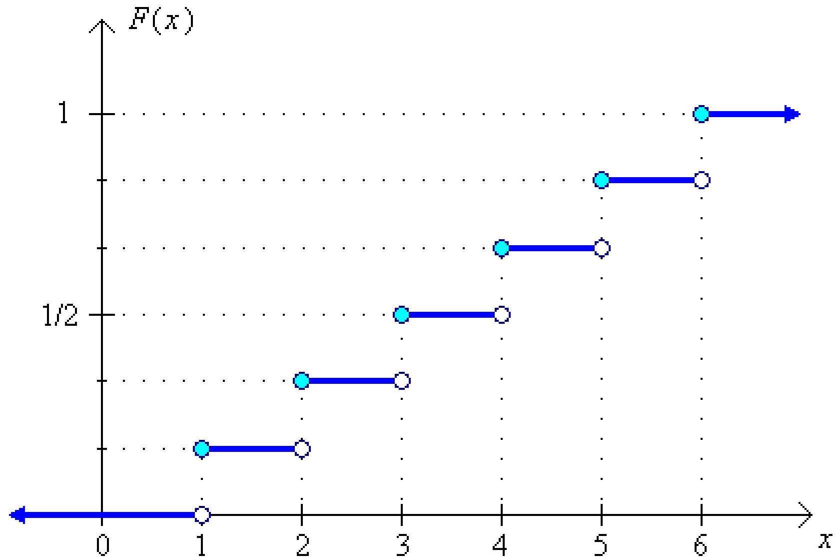
The *p.m.f.* is a uniform distribution

$$p(x) = \frac{1}{6} \qquad (x = 1,2,3,4,5,6)$$

Thus $F(x)$ increases from 0 to 1/6 at $x = 1$ and increases by steps of 1/6 at each subsequent integer value until $x = 6$. It follows easily that

$$F(x) = \begin{cases} 0 & (x < 1) \\ 1 & (x \ge 6) \\ \text{INT}(x)/6 & (\text{otherwise}) \end{cases}$$

The graph of $F(x)$ has the classic staircase appearance of the cumulative distribution function of a discrete random quantity.



Expected value of a random quantity

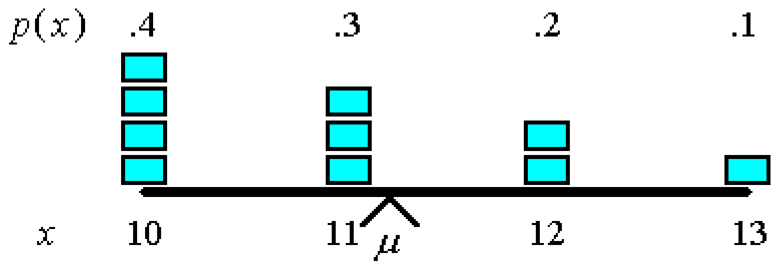
Example 6.10:

The random quantity X is known to have the p.m.f.

x	10	11	12	13
$p(x)$	.4	.3	.2	.1

If we measure values for X many times, what value do we expect to see on average?

Treat the values of $p(x)$ as point masses of probability:



The **expected value** E[X] (= **population mean** μ) is at the fulcrum (balance point) of the beam.

Taking moments about $x = 10$:

$\sum p(x)(x-10) = .4\times0 + .3\times1 + .2\times2 + .1\times3 = 1.0$

The fulcrum is at $x = 10 + 1$

Therefore $\mu = \text{E}[X] = \underline{\underline{11}}$

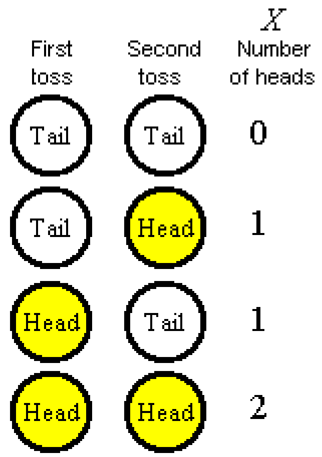
In general, for any random quantity X with a discrete probability mass function $p(x)$ and a set of possible values D, the population mean μ of X (and the expected value of X) is

$$\text{E}[X] = \mu_X = \sum_{x \in D} x \cdot p(x)$$

Shortcut: If X is symmetric about $x = a$, then $\text{E}[X] = a$

Example 6.11:
Let X = the number of heads when a coin has been tossed twice. Find E[X].

Solution:
List the all the possible combinations.
→ the probability mass function of the distribution of X.



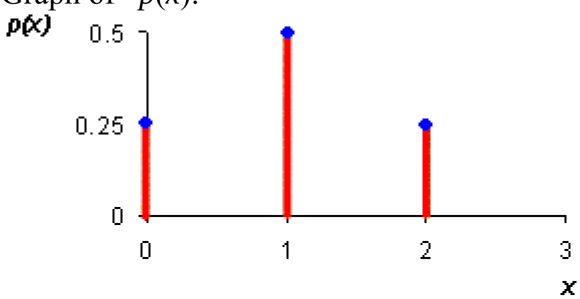
$$p(x) = \begin{cases} 1/4 & (x=0) \\ 1/2 & (x=1) \\ 1/4 & (x=2) \end{cases}$$

$$\text{E}[X] = 0\times\tfrac{1}{4} + 1\times\tfrac{1}{2} + 2\times\tfrac{1}{4}$$

$$= 0 + \tfrac{1}{2} + \tfrac{1}{2}$$

Therefore
μ = **1**

Alternative solution:
Graph of $p(x)$:



$p(x)$ is symmetric about x = **1** .

Therefore, E[X] = **1**

The expected value of a function

Definition:
If the random quantity X has set of possible values D and p.m.f. $p(x)$, then the expected value of any function $h(X)$, denoted by E[$h(X)$], is computed by

$$\text{E}\left[h(X)\right] = \sum_{\text{all } x} h(x)\cdot p(x)$$

E[$h(X)$] is computed in the same way that E[X] itself is, except that $h(x)$ is substituted in place of x.

Special case:

$h(x) = ax + b \quad \Rightarrow \qquad \boxed{\mathrm{E}\left[aX + b\right] = a\,\mathrm{E}\left[X\right] + b}$

Proof:

$$\mathbf{E}[aX + b] = \sum (ax + b)\, p(x)$$

$$= \sum (ax)\, p(x) + \sum b\, p(x)$$

$$= a \sum x\, p(x) + b \sum p(x)$$

$$= a\,\mathbf{E}[X] + b$$

Example 6.12:
C = tomorrow's temperature high in °C
F = tomorrow's temperature high in °F
Given $E[C] = 10$, find $E[F]$.

$$F = \tfrac{9}{5}C + 32$$

$$\Rightarrow \qquad \mathrm{E}\left[F\right] = \tfrac{9}{5}\mathrm{E}\left[C\right] + 32$$

$$= \tfrac{9}{5}\times 10 + 32$$

$$\therefore \qquad \mathrm{E}\left[F\right] = \underline{\underline{\mathbf{50}}}$$

The variance of X

The quantity usually employed to measure the spread in the values of a random quantity X is the **population variance** $\mathrm{V}[X] = \sigma^2 = \frac{1}{N}\sum_x (x-\mu)^2$

Definition:

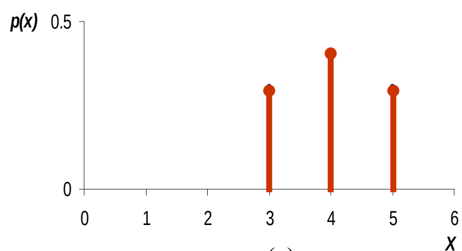
Let X have probability mass function $p(x)$ and expected value μ. Then

$$\mathrm{V}[X] = \sum_x (x-\mu)^2 \, p(x) = \mathrm{E}\left[(X-\mu)^2\right]$$

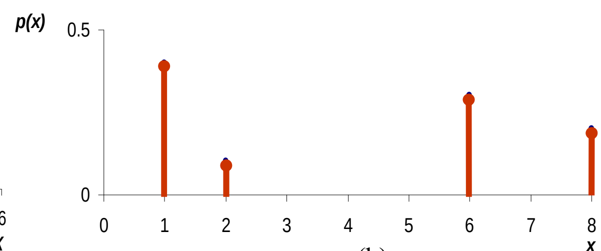
The **standard deviation** of X is $\sigma = \sqrt{\mathrm{V}[X]}$

Example 6.13:

Two different probability distributions [below] share the same mean $\mu = 4$



(a)



(b)

If X has p.m.f. as shown in Figure (a)

x	3	4	5
$p(x)$	.3	.4	.3

$\mu =$ **4 (by symmetry)**

$\mathrm{V}[X] =$ **$(3-4)^2\times.3 + (4-4)^2\times.4 + (5-4)^2\times.3 = .3 + 0 + .3 =$ $\underline{0.6}$**

and $\sigma = \sqrt{0.6} \approx 0.7746$

If X has p.m.f as shown in Figure (b)

x	1	2	6	8
$p(x)$	.4	.1	.3	.2

$\mu =$ **$\mathrm{E}[X] = 1\times.4 + 2\times.1 + 6\times.3 + 8\times.2 = .4 + .2 + 1.8 + 1.6 =$ $\underline{4}$**

$\mathrm{V}[X] =$ **$(1-4)^2\times.4 + (2-4)^2\times.1 + (6-4)^2\times.3 + (8-4)^2\times.2 = 3.6 + 0.4 + 1.2 + 3.2$**
$=$ $\underline{8.4}$

and $\sigma = \sqrt{8.4} \approx 2.898$ [Higher variance $\leftrightarrow$ greater spread]

Example 6.14:
Let X = number of heads when a coin has been tossed twice. Find V[X].
$\text{V}[X] = \text{E}[(X-\mu)^2]$

$$= \sum (x-\mu)^2 \, p(x)$$

$$= (0-1)^2 \times \tfrac{1}{4} + (1-1)^2 \times \tfrac{1}{2} + (2-1)^2 \times \tfrac{1}{4}$$

$$= \tfrac{1}{4} + 0 + \tfrac{1}{4} = \underline{0.5}$$

A shortcut formula for variance

$$\boxed{\text{V}[X] = \text{E}\left[X^2\right] - \left(\text{E}[X]\right)^2}$$

Proof: $\sigma^2 = \text{E}[X^2] - \mu^2$

$$\text{V}[X] = \text{E}[(X-\mu)^2] = \text{E}[X^2 - 2\mu X + \mu^2]$$

$$= \text{E}[X^2] - 2\mu \text{E}[X] + \mu^2 \text{E}[1] = \text{E}[X^2] - 2\mu\mu + \mu^2$$

$$= \text{E}[X^2] - \mu^2$$

Note: $\text{E}\left[f(X)\right] \neq f\left(\text{E}[X]\right)$ unless $f(x)$ is linear and/or X is constant.

Example 6.14 (continued):
Let X = number of heads when a coin has been tossed twice. Find V[X] using the shortcut formula.

$$\text{E}[X^2] = \sum x^2 \, p(x)$$

$$= 0^2 \times \tfrac{1}{4} + 1^2 \times \tfrac{1}{2} + 2^2 \times \tfrac{1}{4}$$

$$= 0 + \tfrac{1}{2} + 1 = 1.5$$

$$\text{V}[X] = \text{E}[X^2] - (\text{E}[X])^2$$

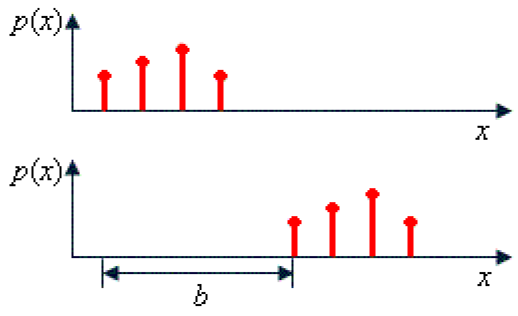
$$= 1.5 - 1^2$$

$$\therefore \quad \sigma^2 = \underline{0.5}$$

The shortcut is more convenient when μ is not an integer.

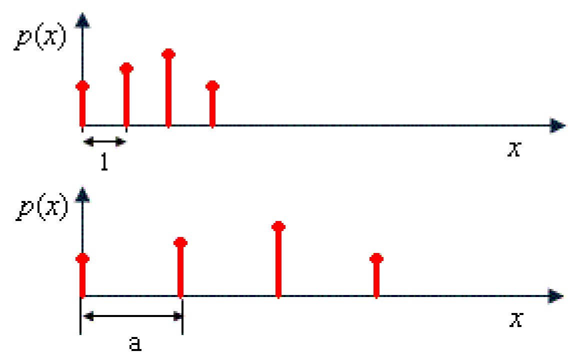
Rules of variance

Example 6.15:
Do the distributions in the following two figures have the same variance or not?



YES

Example 6.16:
Do the distributions in the following two figures have the same variance or not?



NO

$$\boxed{\mathrm{V}[aX+b] = a^2\mathrm{V}[X]}$$

Proof:

$$\begin{aligned}
\mathrm{V}[aX+b] &= \mathrm{E}[(\,(aX+b) - \mathrm{E}[aX+b]\,)^2] \\
&= \mathrm{E}[(\,(aX+b) - (a\mu+b)\,)^2] \\
&= \mathrm{E}[(aX - a\mu)^2] \\
&= \mathrm{E}[a^2\,(X-\mu)^2] \\
&= a^2\,\mathrm{E}[(X-\mu)^2] \\
&= a^2\,\mathrm{V}[X]
\end{aligned}$$

The addition of the constant b does not affect the variance, because the addition of b changes the location (and therefore mean value) but not the spread of values.