ENGI 4421 Probability and Statistics
Faculty of Engineering and Applied Science

Problem Set 4

Conditional probability, Bayes' theorem

---

1. One urn contains five red balls and five green balls. A second urn contains eight red balls and one green ball. A ball is chosen randomly from the first urn and placed in the second urn. Then a ball is chosen randomly from the second urn and placed in the first urn.

   (a) What is the probability that a red ball is selected from the first urn **and** a red ball is selected from the second urn?
   (b) What is the [unconditional] probability that a red ball is selected from the second urn?
   (c) At the conclusion of the selection process, what is the probability that the numbers of red and green balls in each urn are identical to the initial numbers?

Now consider the generalization to the case where there are m balls initially in urn 1, a of which are red balls, and n balls initially in urn 2, b of which are red balls, together with $m > 1$, $n > 1$, $0 < a < m$ and $0 < b < n$ (so that each urn initially contains at least one ball of each colour).

   (d) Show that the [unconditional] probability that a red ball is selected from the second urn during this process is, in general, different from the probability of drawing a red ball directly from the second urn [when the first urn is left alone].
   (e) Find a necessary and sufficient condition on the values of a and b for the two unconditional probabilities in part (d) above to be equal.

---

2. A test for unacceptable metal fatigue in a girder is not perfect. The test is positive, (that is, the test suggests that unacceptable metal fatigue is present), 95% of the time when unacceptable metal fatigue really is present. The test is positive 10% of the time when there really isn't any unacceptable metal fatigue. It is known that 5% of all girders at a particular site have developed unacceptable metal fatigue. If the next test result is positive, then what are the odds that that girder really does have unacceptable metal fatigue?

---

3. A factory has the following information about the quality control process on its production line:
   If an item is defective, then there is a 99.9% chance that the quality control process will reject it [which, with the benefit of hindsight, is the correct decision].
   If an item is good, then there is a 2% chance that the quality control process will reject it [a "false positive", which, with the benefit of hindsight, is not the correct decision].
   It is known that 0.1% of all items on the production line are defective.
   Given that the quality control process has just rejected an item, find the probability that that decision is correct [i.e. that the item really is defective].

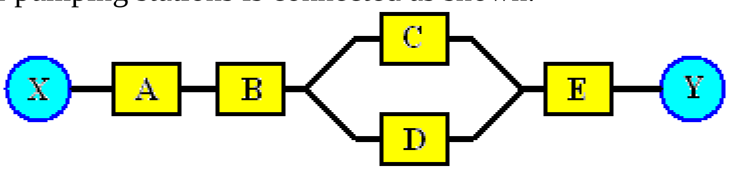
---

4. Alice, Bob and Carol take part in a game where each of them, in turn, rolls a fair die until one of them rolls a six and wins the game. Alice rolls first. If she doesn't roll a six, then Bob rolls next. If he doesn't roll a six, then Carol has her first chance to win. If none of them roll a six on the first attempt, then Alice gets a second roll and so on.

(a) Find the probability that Carol wins the game in the first round.
(b) Find the probability that the game lasts for more than two complete rounds.
(c) Find the probability that Alice wins.
(d) Find the probability that Carol wins, given that Alice doesn't roll a six on her first roll.

---

5. A network of pumping stations is connected as shown.



The reliability of each station is independent of the others, except for stations C and D, for which $\mathrm{P}\left[D \mid C\right] = \mathrm{P}\left[C \mid D\right] = .9$ and $\mathrm{P}\left[D \mid \tilde{C}\right] = \mathrm{P}\left[C \mid \tilde{D}\right] = .5$.

Water passes through each of the other stations 90% of the time.
Find the probability that water flows through the network from point X to point Y.

---

6. In a criminal trial, a prosecutor states that only one in every ten million innocent men have a DNA profile that matches the known DNA profile of the guilty person. The defendent's DNA profile matches that of the guilty person. The prosecutor goes on to state that the chance that the defendent is innocent is therefore only one in ten million, so unlikely that the jury must find the defendent guilty.

What is the major flaw in this argument?

---

7. This is a continuation of Problem Set 2 Question 7. Use the labels shown there for events and decisions. The following exact probabilities are known:

$P[D] = .05$

$P[N \mid D] = .99$

$P[N \mid G] = .10$

Use a separate tree diagram (**or** use Bayes' theorem) to calculate

(a) $P[D \mid N]$

(b) $P[D \mid Y]$

(c) $P[N]$

(d) Place the exact values of the following probabilities, (expressed as fractions in their lowest terms), at the appropriate locations on the original decision tree:

$P[D \mid N]$

$P[G \mid N]$

$P[D \mid Y]$

$P[G \mid Y]$

$P[D]$

$P[G]$

$P[N]$

$P[Y]$

☜ Back to the index of questions On to the solutions to this problem set ☞