Problem Set 6

Joint Probability Distributions; Propagation of Error; Decision Tree

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1. The joint probability mass function $p(x, y)$ for a pair of discrete random quantities X, Y is defined by the following table.

p		$y = -1$	$y = 0$	$y = 1$
x	-1	.10	.20	.20
	0	.06	.08	.16
	1	.04	.12	.04

(a) Find the correlation coefficient ρ for X and Y, correct to three decimal places.
(b) Find the probability that Y is non-negative, given that X is positive.
(c) Are X and Y stochastically independent?

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2. [Navidi, Exercises 2.6, Question 21] (Parts (a), (b) and (e) are bonus material.)

The lifetime of a certain component, in years, has the probability density function

$$f(x) = \begin{cases} e^{-x} & (x > 0) \\ 0 & (x \le 0) \end{cases}$$

Two such components, whose lifetimes are independent, are available. As soon as the first component fails, it is replaced with the second component. Let X denote the lifetime of the first component and let Y denote the lifetime of the second component.

(a) Find the joint probability density function of X and Y.
(b) Find $\mathrm{P}[X \le 1 \text{ and } Y > 1]$.
(c) Find μ_X.
(d) Find μ_{X+Y}.
(e) Find $\mathrm{P}[X + Y \le 2]$. (Hint: Sketch the region of the plane where $x + y \le 2$ and then integrate the joint probability density function over that region).

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3. [Navidi, Exercises 3.3, Question 4]

The velocity v of sound in air at temperature T is given by $v = 20.04\sqrt{T}$, where T is measured in kelvins (K) and v is in m/s. Assume that $T = 300 \pm 0.4\,\mathrm{K}$. Estimate v and find the uncertainty in the estimate.

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4. [Navidi, Exercises 3.2, Question 17]

A certain chemical process is run ten times at a temperature of 65ºC and ten times at a temperature of 80ºC. The yield at each run was measured as a percent of a theoretical maximum. The data are presented in this table.

65ºC	71.3	69.1	70.3	69.9	71.1	70.7	69.8	68.5	70.9	69.8
80ºC	90.3	90.8	91.2	90.7	89.0	89.7	91.3	91.2	89.7	91.1

(a) For each temperature, estimate the mean yield and find the uncertainty in the estimate.
(b) Estimate the difference between the mean yields at the two temperatures and find the uncertainty in the estimate.

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5. Ten independent measurements of the mass of some object produce an estimate $16.21 \pm \dfrac{0.05}{\sqrt{10}}\,\mathrm{kg}$. Five independent measurements of the mass of same object, made with a more precise device, produce an estimate $16.18 \pm \dfrac{0.02}{\sqrt{5}}\,\mathrm{kg}$.

Combine these estimates into a weighted average with minimum uncertainty and quote your answer to three decimal places.

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6. [Navidi, Supplementary Exercises 3, Question 11]

A laminated item is made up of six layers. The two outer layers each have a thickness of $1.25 \pm 0.10\,\mathrm{mm}$ and the four inner layers each have a thickness of $0.80 \pm 0.05\,\mathrm{mm}$. Assume that the thicknesses of the layers are independent. Estimate the thickness of the item and find the uncertainty in the estimate.

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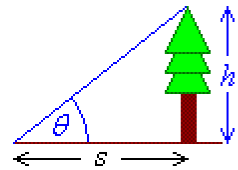
7. [Navidi, Exercises 3.4, Question 3]

From a fixed point on the ground, the distance to a certain tree is measured to be $s = 55.2 \pm 0.1\,\text{m}$ and the angle from the point to the top of the tree is measured to be $\theta = 0.50 \pm 0.02\,\text{radians}$. The height of the tree is given by $h = s \tan\theta$.



(a) Estimate h and find the uncertainty in the estimate.
(b) Which would provide a greater reduction in the uncertainty in h: halving the uncertainty in s to 0.05 m, or halving the uncertainty in θ to 0.01 radians?

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8. Refer back to question 7 on Problem Set 4 and use the labels and probabilities shown there for events and decisions.
The manager has the following additional information:
If the manager chooses to accept the item, then
 if it is good, then the profit is \$10.
 if it is defective, then the loss is \$100 (or, equivalently, the profit is –\$100).
If the manager chooses to reject the item, then the profit is zero.
If the manager chooses to invest in the quality control system, then the cost of that investment is equivalent to a testing fee of \$2.50 per item.
The prior probability that an item is defective is known to be $P[D] = .05$
The reliability of the quality control system is known from the conditional probabilities
 $P[N \mid D] = .99$
 $P[N \mid G] = .10$
and in Problem Set 4 the following values were found:
 $P[D \mid N] = 99/289$
 $P[D \mid Y] = 1/1711$
 $P[N] = 289/2000$
Determine the manager's optimum strategy, in order to maximize the expected profit.
[You may assume that the manager has no aversion to the risk of a loss.]
Should the manager invest in the quality control system?
If so, what action (accept or reject) should follow each verdict (Y or N) from the quality control system?
If not, what action (accept or reject) should be taken?
What is the best expected profit per item?
A similar example will be explored during a tutorial in June.

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9. ***Bonus question*:**

The general expression for the variance of the difference of two random quantities X, Y is

$$\mathrm{V}\left[X - Y\right] = \sum_{i=1}^{2}\sum_{j=1}^{2} a_i a_j \operatorname{Cov}\left[X_i, X_j\right] = \mathrm{V}\left[X\right] - 2\operatorname{Cov}\left[X, Y\right] + \mathrm{V}\left[Y\right]$$

Consider the case where the two random quantities have the same variance σ^2 .

(a) Express $\mathrm{V}[X - Y]$ in terms of σ^2 and the correlation coefficient ρ.

(b) In terms of σ^2, what are the least and greatest possible values of $\mathrm{V}[X - Y]$?

(c) Write down the value of $\mathrm{V}[X - Y]$ when X, Y are independent of each other.

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10. ***Bonus question*:**

A random quantity A is said to be an unbiased estimator of the parameter θ if and only if $\mathrm{E}[A] = \theta$.

The sample variance S^2 is an unbiased estimator of the population variance σ^2.

Show that the sample standard deviation S is a **biased** estimator of the population standard deviation σ.

[*Hint*: Make use of the shortcut formula $\mathrm{V}\left[X\right] = \mathrm{E}\left[X^2\right] - \left(\mathrm{E}\left[X\right]\right)^2$.]

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11. When combining two independent estimates $x \pm \sigma_X$ and $y \pm \sigma_Y$ into a weighted average $w \pm \sigma$ with minimum uncertainty, show that σ^2 is the harmonic mean of σ_X^2 and σ_Y^2, (in other words, that $\dfrac{1}{\sigma^2} = \dfrac{1}{\sigma_X^2} + \dfrac{1}{\sigma_Y^2}$).

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