Example 3.01 [Navidi Section 2.2; Devore Section 2.3]

Four cards, labelled **A** , **B** , **C** and **D** , are in an urn.
In how many ways can *three* cards be drawn
(a) with replacement?
(b) without replacement?
(c) without replacement (if the order of selection doesn't matter)?

(a) "With replacement" means that, each time a card is drawn, it is put back into the urn before the next card is drawn. Therefore the same card can be withdrawn more than once from the urn. The complete sample space is listed here.

AAA	**AAB**	**AAC**	**AAD**	**ABA**	**ABB**	**ABC**	**ABD**
ACA	**ACB**	**ACC**	**ACD**	**ADA**	**ADB**	**ADC**	**ADD**
BAA	**BAB**	**BAC**	**BAD**	**BBA**	**BBB**	**BBC**	**BBD**
BCA	**BCB**	**BCC**	**BCD**	**BDA**	**BDB**	**BDC**	**BDD**
CAA	**CAB**	**CAC**	**CAD**	**CBA**	**CBB**	**CBC**	**CBD**
CCA	**CCB**	**CCC**	**CCD**	**CDA**	**CDB**	**CDC**	**CDD**
DAA	**DAB**	**DAC**	**DAD**	**DBA**	**DBB**	**DBC**	**DBD**
DCA	**DCB**	**DCC**	**DCD**	**DDA**	**DDB**	**DDC**	**DDD**

In general, # ways to choose r objects from n with replacement is

(b) "Without replacement" means that each time a card is drawn, it stays out of the urn for all subsequent drawings of cards. The same card cannot be drawn more than once. Identify the reduced sample space from part (a):

AAA	**AAB**	**AAC**	**AAD**	**ABA**	**ABB**	**ABC**	**ABD**
ACA	**ACB**	**ACC**	**ACD**	**ADA**	**ADB**	**ADC**	**ADD**
BAA	**BAB**	**BAC**	**BAD**	**BBA**	**BBB**	**BBC**	**BBD**
BCA	**BCB**	**BCC**	**BCD**	**BDA**	**BDB**	**BDC**	**BDD**
CAA	**CAB**	**CAC**	**CAD**	**CBA**	**CBB**	**CBC**	**CBD**
CCA	**CCB**	**CCC**	**CCD**	**CDA**	**CDB**	**CDC**	**CDD**
DAA	**DAB**	**DAC**	**DAD**	**DBA**	**DBB**	**DBC**	**DBD**
DCA	**DCB**	**DCC**	**DCD**	**DDA**	**DDB**	**DDC**	**DDD**

In general, the number of ways in which r objects can be drawn from n objects without replacement (with the order of selection being important) is the number of **permutations**:

$${}^{n}\mathrm{P}_{r} \;=$$

<u>Definition</u> – the factorial function, for all positive integers n, is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Therefore the number of permutations of r objects from n objects is also

$${}^{n}\mathrm{P}_{r} =$$

Alternative symbols for permutations include $\mathrm{P}_{r,n} = (n)_r = {}^{n}\mathrm{P}_{r}$

(c) If only the identities of the cards drawn from the urn matters, not the order in which they were drawn, then we are seeking the number of **combinations** of 3 cards from 4.

The permutations **ABC**, **ACB**, **BAC**, are all the same combination, because all of these permutations contain the same three cards { **A**, **B**, **C** }.

However the permutation **ABD** is a different combination from the permutation **BAC**, because one card is different (**D** instead of **C**).

The single combination { **A**, **B**, **C** } contains a number of permutations equal to the number of ways in which the three letters can be re-arranged among themselves:

More generally, a single combination of r objects from n objects contains permutations.

Therefore the number of combinations of r objects from n objects (the number of ways of drawing them without replacement and with the order of selection being irrelevant) is

Note that this definition is consistent only if $0! = 1$.

Example 3.02

Four cards, labelled **A** , **B** , **C** and **D** , are in an urn.
In how many ways can *two* cards be drawn
(a) with replacement?
(b) without replacement?
(c) without replacement (if the order of selection doesn't matter)?

For part (a) of this question, the complete sample space is listed below.

AA	**AB**	**AC**	**AD**	**BA**	**BB**	**BC**	**BD**
CA	**CB**	**CC**	**CD**	**DA**	**DB**	**DC**	**DD**

(a)

(b)

(c)

The combinations are

AB **AC** **AD** **BC** **BD** **CD**

The permutations are

AB **BA** **AC** **CA** **AD** **DA** **BC** **CB** **BD** **DB** **CD** **DC**

Example 3.03

Evaluate $\binom{11}{6}$

Example 3.04

Evaluate $P_{2,9}$

Example 3.05

Simplify $\binom{n}{n-r}$

Example 3.06

Simplify $\binom{n}{n}$

Also note the identities

$$^{n}\mathrm{P}_{0} = \mathrm{P}_{0,n} = {}^{n}\mathrm{C}_{0} = \binom{n}{0} = \binom{n}{n} = {}^{n}\mathrm{C}_{n} = \mathbf{1}$$

$$^{n}\mathrm{P}_{n} = \mathrm{P}_{n,n} = n!, \qquad {}^{n}\mathrm{P}_{n-1} = \mathrm{P}_{n-1,n} = n!$$

and $$^{n}\mathrm{P}_{1} = \mathrm{P}_{1,n} = {}^{n}\mathrm{C}_{1} = \binom{n}{1} = \binom{n}{n-1} = {}^{n}\mathrm{C}_{n-1} = \boldsymbol{n}$$

<u>Summary</u>:

The number of ways to draw r objects from n distinguishable objects is:

[with replacement (ordered):] $= n^{r}$

[without replacement (ordered):] $= {}^{n}\mathrm{P}_{r}$

[without replacement (unordered):] $= {}^{n}\mathrm{C}_{r}$

The case "with replacement (unordered)" seldom arises in practice, but the number of ways of drawing r objects from n distinguishable objects in this case can be shown to be ${}^{n+r-1}\mathrm{C}_{r}$. See `www.engr.mun.ca/~ggeorge/4421/demos/Counts.xls` for an illustration of these values for some choices of r and n.

<u>Example 3.07</u>

(a) In how many ways can a team of three men and three women be chosen from a group of five men and six women?

(b) In how many ways can a team of three men and three women be chosen from a group of five men and six women when the team has one leader, one other member and a reserve for the men and likewise for the women?

(a)

Example 3.07 (continued)

(b)

Also note the identity ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ which leads to Pascal's triangle:

```
                1
             1     1
          1     2     1
       1     3     3     1
    1     4     6     4     1
 1     5    10    10     5     1
⋰                                ⋱
```

The number of combinations also appears in the binomial expansion $(a+b)^n$, where n is a natural number:

$$(a+b)^n = a^n + {}^{n}C_1 a^{n-1}b^1 + {}^{n}C_2 a^{n-2}b^2 + {}^{n}C_3 a^{n-3}b^3 + \ldots + {}^{n}C_{n-1} a^1 b^{n-1} + b^n$$

or

$$\boxed{(a+b)^n = \sum_{k=0}^{n} {}^{n}C_k \, a^{n-k} b^k}$$

<u>Example 3.08</u>

$z^3 = (x + jy)^3 =$

<u>Example 3.09</u>

$(1 + x)^4 =$

[End of Chapter 3]