Example 4.01 [Navidi Section 2.3; Devore Sections 2.4-2.5]

Given that rolling two fair dice has produced a total of at least 10, find the probability that exactly one die is a ‘6’.

Let A = "total ≥ 10"
and B = "exactly one '6' "

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

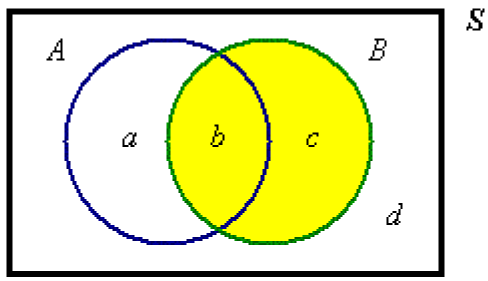
This can be re-written as the general multiplication law of probability:

Because intersection is commutative,

$$\mathrm{P}[BA] = \mathrm{P}[AB] \quad \Rightarrow \quad \mathrm{P}[B \mid A] \cdot \mathrm{P}[A] = \mathrm{P}[A \mid B] \cdot \mathrm{P}[B]$$

Events A, B are **independent** if $\mathrm{P}[B \mid A] = \mathrm{P}[B]$ (and are **dependent** otherwise).
If a set of events is independent, then knowledge that one or more of them occurred does not affect the probability of the other events in the set at all.

This concept of a reduced sample space can be generalized to any pair of possible events, as illustrated in these Venn diagrams. The sample space S is partitioned into four regions, whose probabilities are a, b, c and d.



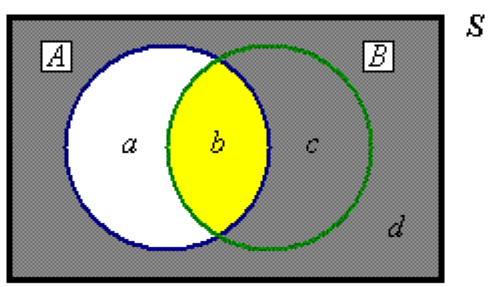
Because these four mutually exclusive regions, between them, cover the entire sample space, we must have

$$a + b + c + d =$$

The unconditional probability of event B (when we have no information on whether or not event A occurred) is

$$\mathrm{P}[B] =$$

If we now know that event A occurred, then our sample space is reduced to those sample points that are inside event A only:



Now the probability of event B, conditional on event A, is the proportion of the given event that is also inside the desired event B:

$$\mathrm{P}[B \mid A] =$$

Example 4.02

Show that, upon rolling two fair dice, the events
C = "the first fair die is a '6' " and D = "the second fair die is a '6' "
are independent.

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

Events E_1, E_2 are stochastically independent (or just **independent**) if $\mathrm{P}\left[E_1 \mid E_2\right] = \mathrm{P}\left[E_1\right]$.

Equivalently, the events are independent iff (if and only if) $\mathrm{P}\left[E_1 E_2\right] = \mathrm{P}\left[E_1\right] \cdot \mathrm{P}\left[E_2\right]$.
Compare this with the general multiplication law of probability:

$$\mathrm{P}\left[E_1 E_2\right] = \mathrm{P}\left[E_1 \mid E_2\right] \cdot \mathrm{P}\left[E_2\right]$$

<u>Example 4.03</u>

A bag contains two red, three blue and four yellow marbles. Three marbles are taken at random from the bag,

(a) without replacement;

(b) with replacement.

In each case, find the probability that the colours of the three marbles are all different.

Let "E" represent the desired event and "RBY" represent the event "red marble first and blue marble second and yellow marble third" and so on. Then

E =

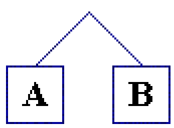
Example 4.03 (continued)

Independent vs. Mutually Exclusive Events:

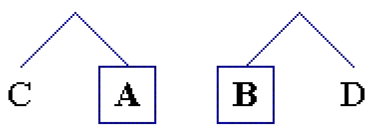
Two possible events E_1, E_2 are **independent** if and only if $\mathrm{P}[E_1 E_2] = \mathrm{P}[E_1] \cdot \mathrm{P}[E_2]$.
Two possible events E_1, E_2 are **mutually exclusive** if and only if $\mathrm{P}[E_1 E_2] = 0$.

No pair of possible events can satisfy both conditions, because $\mathrm{P}[E_1] \cdot \mathrm{P}[E_2] \neq 0$.
A pair of independent possible events cannot be mutually exclusive.
A pair of mutually exclusive possible events cannot be independent.
An example with two teams in the playoffs of some sport will illustrate this point.

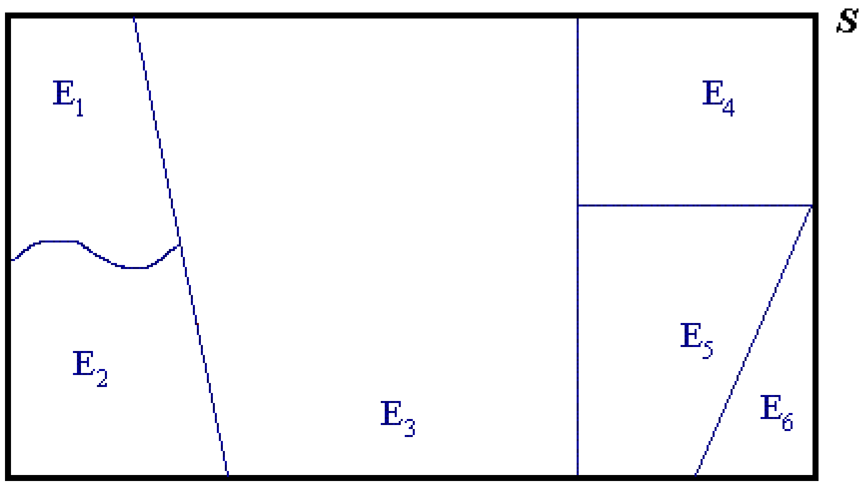
Example 4.04
Teams play each other: Outcomes for teams **A** and **B** are:



Teams play other teams: Outcomes for teams **A** and **B** are:



A set of mutually exclusive and collectively exhaustive events is a **partition** of ***S***.



A set of mutually exclusive events $E_1, E_2, \ldots, E_n$ is **collectively exhaustive** if

Bayes' Theorem:

When an event A can be partitioned into a set of n mutually exclusive and collectively exhaustive events E_i, then

$$\boxed{P\left[E_k \mid A\right] = \frac{P\left[A \mid E_k\right] \cdot P\left[E_k\right]}{\sum_{i=1}^{n} P\left[A \mid E_i\right] \cdot P\left[E_i\right]}}$$

<u>Example 4.05</u>

The stock of a warehouse consists of boxes of high, medium and low quality lamps in the respective proportions 1:2:2. The probabilities of lamps of these three types being unsatisfactory are 0, 0.1 and 0.4 respectively. If a box is chosen at random and one lamp in the box is tested and found to be satisfactory, what is the probability that the box contains

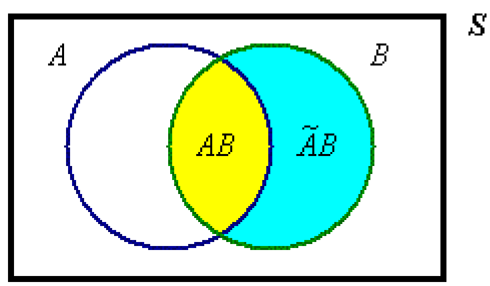
(a) high quality lamps;
(b) medium quality lamps;
(c) low quality lamps?

Example 4.05 (continued)

Another version of the total probability law:

$$\underline{P[AB]} + \underline{P[\tilde{A}B]} = \underline{P[B]}$$

$\Rightarrow$



Example 4.06

A test for a rare disease has a reliability of 99%, that is, if a person has the disease then the test will be positive 99% of the time, while if a person is free of the disease then the test will be negative 99% of the time. The disease is present in only 1% of the population.

Given that the test result for one individual is positive, find the probability that that person has the disease.

Let D = "person has the disease"
and A = "test result is positive".

First note that the required probability is $\text{P}[D \mid A]$, not $\text{P}[A \mid D]$. The order matters!

From the question,

$\text{P}\left[A \mid D\right] =$ $\text{P}\left[A \mid \overline{D}\right] =$

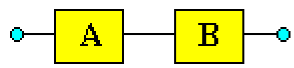
and
$\text{P}\left[D\right] =$

Example 4.06 (continued)

Reliability Analysis

If a subsystem consists of two components whose failures are independent of each other, then the probability that the subsystem works can be found in terms of the probabilities for each component to work. Two cases arise:

Connection in series:

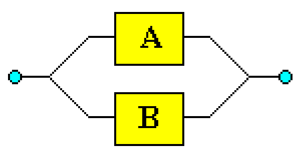


The subsystem works if and only if both components work.
Let A = "component A works", B = "component B works" and
E = "subsystem works".

$$\mathrm{P}[E] =$$

or, if the components are not independent, $\mathrm{P}[E] =$

Connection in parallel:



The subsystem ***fails*** if and only if both components fail.

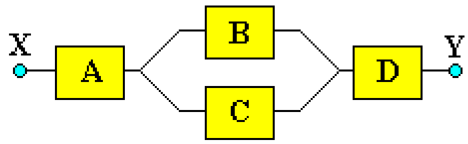
$$\mathrm{P}[E] =$$

or

or, if the components are not independent,

<u>Example 4.07</u>

Water flows through a network of four pumping stations, as shown.



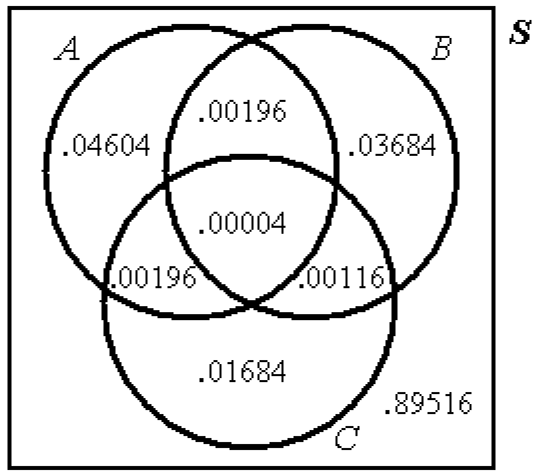
The reliabilities of the stations are independent of each other.
The probabilities for each station to pump water through it are
$\mathrm{P}[A] = .90$, $\mathrm{P}[B] = .50$, $\mathrm{P}[C] = .75$ and $\mathrm{P}[D] = .80$.
Find the probability that water flows through the system from point X to point Y.

Let E = "water flows through the subsystem formed by stations B and C"
and W = "water flows through the system from X to Y"
then

Some Additional Tutorial Examples

<u>Example 4.08</u> (This example is also in Problem Set 2)

Events A, B, C are such that the probabilities are as shown in this Venn diagram.



Are the three events independent?

P[A] = .05 , P[B] = .04 , P[C] = .02 ,

P[AB] = .002 , P[BC] = .0012 ,
P[CA] = .002 ,

P[ABC] = .00004

$\Rightarrow$

Three events { A, B, C } are mutually independent if and only if

P[$A\ B\ C$] = P[A] × P[B] × P[C]
and
all three pairs of events { A, B }, { B, C }, { C, A } are independent

[Reference: George, G.H., Mathematical Gazette, vol. **88**, #513, 85-86, Note 88.76 "*Testing for the Independence of Three Events*", 2004 November]

Example 4.09

Three women and three men sit at random in a row of six seats.
Find the probability that the men and women sit in alternate seats.

Example 4.10

A hiring committee with seven members has narrowed its choice for a new employee to either candidate *A* or candidate *B*. Each member has voted on a slip of paper for one of the candidates. Suppose that there are actually four votes for *A* and three for *B*. If the slips are selected for tallying in random order, what is the probability that *A* remains ahead of *B* throughout the vote count? (For example, this event occurs if the selected ordering is *AABABAB* but not for *ABBAAAB*).

[End of Chapter 4]

[Space for additional notes]