Propagation of Error [Navidi Chapter 3; not in Devore]

Any realistic measurement procedure contains error.
Any calculations based on that measurement will therefore also contain an error.
The errors are **propagated** from measurements to the calculated value.

From Chapter 1, for any constants a, b and any random quantity X we know that

$$\mathrm{E}\left[aX+b\right] = a\,\mathrm{E}\left[X\right]+b$$

and

$$\mathrm{V}\left[aX+b\right] = a^2\mathrm{V}\left[X\right] \quad\Rightarrow\quad \sigma_{aX+b} = \left|a\right|\sigma_X$$

From Chapter 7, for any constants $\left\{ c_1, c_2, \ldots, c_n \right\}$ and any set of *independent* random quantities $\left\{ X_1, X_2, \ldots, X_n \right\}$, we know that

$$\mathrm{E}\left[c_1X_1+c_2X_2+\ldots+c_nX_n\right] = c_1\mathrm{E}\left[X_1\right]+c_2\mathrm{E}\left[X_2\right]+\ldots+c_n\mathrm{E}\left[X_n\right]$$

and

$$\mathrm{V}\left[c_1X_1+c_2X_2+\ldots+c_nX_n\right] = c_1^2\mathrm{V}\left[X_1\right]+c_2^2\mathrm{V}\left[X_2\right]+\ldots+c_n^2\mathrm{V}\left[X_n\right]$$

A random sample of n observations drawn from a population of mean μ and variance σ^2 has a sample mean $\overline{X}$ for which

$$\mathrm{E}\left[\overline{X}\right] = \mu \quad\text{and}\quad \mathrm{V}\left[\overline{X}\right] = \frac{\sigma^2}{n} \quad\Rightarrow\quad \sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}}$$

Example 8.01

The diameter of a circle is measured to be 6.0 cm, with the uncertainty represented by the standard deviation of 0.1 cm. Find the uncertainty in the circumference of the circle.

Let D = measured diameter of the circle
and C = resulting calculated circumference of the circle
then

<u>Example 8.02</u>

A weigh scale is used to estimate masses of objects up to 100 kg with an uncertainty of 0.5 kg. The error in each measurement is independent of the errors in all other measurements.
Four components of a machine are measured and masses of (8.7 ± 0.5) kg, (10.2 ± 0.5) kg, (12.7 ± 0.5) kg and (15.1 ± 0.5) kg are reported.

What is the total mass of the four components?

Let M = total mass and M_i = mass of the i^{th} component, then

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Note on quoted uncertainty:

The uncertainty b in the expression $(a \pm b)$ is, by default, the appropriate standard deviation. However, b is sometimes taken to be the maximum possible error.
On other occasions it is some multiple of the standard deviation (often 2σ or 3σ).
In chapters 11 and 12 we will express confidence intervals in the form $(a \pm b)$.

Be sure which of these interpretations is intended when you encounter $(a \pm b)$.

Example 8.03

The mass of a component is measured six times on a weigh scale whose uncertainty is unknown. The six measurements (in kg) are 5.09, 5.16, 5.08, 5.10, 5.14 and 5.12. Estimate the mass of the component and find the uncertainty in this estimate.

Let $\overline{M}$ = sample mean of the measured masses and S = sample standard deviation.
The true uncertainty σ is unknown, but it may be estimated by the observed s.

$$\overline{m} = \frac{30.69}{6} = 5.115$$

and

Repeated Measurements with Different Uncertainties

Example 8.04

Two independent estimates of the duration of a chemical process, using different clocks, are (4.0 ± 0.2) s and (4.1 ± 0.1) s. Find the uncertainty in the average time of 4.05 s.

Let $\overline{T}$ = average of the times on the two clocks. $\overline{t} = \dfrac{4.0 + 4.1}{2} = 4.05$

Example 8.04 (continued)

However, one clock is more precise than the other. It makes sense that greater weight should be placed on the value provided by the more precise clock. Let us use a weighted average T_W in place of the simple average $\overline{T}$.

$$T_W = cT_1 + (1-c)T_2 \quad \Rightarrow \quad \sigma^2 = \mathrm{V}[T_W] = c^2 \mathrm{V}[T_1] + (1-c)^2 \mathrm{V}[T_2]$$

where c is a weight in the interval [0, 1].

We need the value of c that minimises σ^2 (and therefore generates the best possible precision).

In general, when two independent estimates for the same value are reported as $(x_1 \pm \sigma_1)$ and $(x_2 \pm \sigma_2)$, then the most precise average value will be $(x \pm \sigma)$, where

$$x = c_1 x_1 + c_2 x_2 , \quad \sigma^2 = c_1^{\,2} \sigma_1^{\,2} + c_2^{\,2} \sigma_2^{\,2} , \quad c_1 = \frac{\sigma_2^{\,2}}{\sigma_1^{\,2} + \sigma_2^{\,2}} \quad \text{and} \quad c_2 = \frac{\sigma_1^{\,2}}{\sigma_1^{\,2} + \sigma_2^{\,2}}$$

Linear Combination of Dependent Measurements

The general expression for the variance of a linear combination of random quantities
$Y = \sum_{i=1}^{n} a_i X_i$ is

$$\mathrm{V}[Y] = \mathrm{V}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}\left[X_i, X_j\right]$$

This can be re-written as

$$\mathrm{V}[Y] = \sum_{i=1}^{n} a_i^2 \mathrm{V}\left[X_i\right] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} a_i a_j \mathrm{Cov}\left[X_i, X_j\right]$$

$$= \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} a_i a_j \rho_{ij}\sigma_i\sigma_j$$

where ρ_{ij} is the correlation coefficient between X_i and X_j and $\sigma_i^2 = \mathrm{V}\left[X_i\right]$.

The maximum possible value of $\mathrm{V}[Y]$ occurs when $\rho_{ij} = \mathrm{sgn}\left(a_i a_j\right) \quad \forall (i, j)$.

We then obtain $\mathrm{V}[Y] = \left(\sum_{i=1}^{n} |a_i| \sigma_i\right)^2 \quad \Rightarrow \quad \sigma_Y = |a_1|\sigma_1 + |a_2|\sigma_2 + \ldots + |a_n|\sigma_n$

In all other cases, a conservative estimate of the uncertainty in $Y = \sum_{i=1}^{n} a_i X_i$ is

$$\boxed{\sigma_Y \le |a_1|\sigma_1 + |a_2|\sigma_2 + \ldots + |a_n|\sigma_n}$$

Example 8.05

Suppose that we are not sure whether or not the two estimates, using different clocks, of the duration of a chemical process in Example 8.04 are independent. The estimates are (4.0 ± 0.2) s and (4.1 ± 0.1) s. Find a conservative overall estimate.

As before, $\overline{T} = \dfrac{T_1 + T_2}{2} = \dfrac{1}{2}\left(T_1 + T_2\right) \quad \Rightarrow \quad \overline{t} = \dfrac{4.0 + 4.1}{2} = 4.05$

Non-Linear Functions of One Measurement

Suppose that $U(X)$ is a non-linear function of a random quantity X whose true mean is μ.

If X is an unbiased estimator of μ, then, in general, $U(X)$ will be a biased estimator of $U(\mu)$.

If X is close to μ, then the Taylor series approximation for $U(X)$,

$$U(X) = U(\mu) + \left.\frac{dU}{dX}\right|_{X=\mu}(X-\mu) + \frac{1}{2!}\left.\frac{d^2U}{dX^2}\right|_{X=\mu}(X-\mu)^2 + \ldots$$

may be truncated to just the first order,

$$U(X) \approx U(\mu) + \left.\frac{dU}{dX}\right|_{X=\mu}(X-\mu) = \left(U(\mu) - \mu\left.\frac{dU}{dX}\right|_{X=\mu}\right) + \left(\left.\frac{dU}{dX}\right|_{X=\mu}\right)X$$

This is a linear function of X, whose standard deviation follows from the general formula $\sigma_{aX+b} = |a|\sigma_X$:

$$\sigma_U \approx \left|\left(\left.\frac{dU}{dX}\right|_{X=\mu}\right)\right|\sigma_X$$

This is the formula for propagation of error. In practice we do not know the true value of μ, so we evaluate the derivative at the observed value of X instead.

<u>Example 8.06</u>

In Example 8.01, the diameter D of the circle was measured to be (6.0 ± 0.1) cm.
Estimate the area A of the circle.

Relative Uncertainty [**for bonus questions only**]

If U is an unbiased measurement whose true value is μ_U and whose absolute uncertainty is σ_U, then the **relative uncertainty** in U is the dimensionless quantity

$$\frac{\sigma_U}{\mu_U}$$

(also known as the **coefficient of variation**). In practice, the relative uncertainty is calculated as

$$\frac{\sigma_U}{U}$$

Note that $\sigma_{\ln U} \approx \left| \left(\frac{d}{dU} (\ln U) \Big|_U \right) \right| \sigma_U = \frac{\sigma_U}{U}$.

Therefore the absolute uncertainty in (ln U) is also the relative uncertainty in U.

Example 8.06 (again) [**for bonus questions only**]

Find the relative uncertainty in the area of the circle of diameter (6.0 ± 0.1) cm and hence find the absolute uncertainty in the area of the circle

$$A = \pi \left(\frac{D}{2} \right)^2 = \frac{\pi}{4} D^2 \qquad \Rightarrow \quad \ln A = \ln \left(\frac{\pi}{4} D^2 \right) = \ln \left(\frac{\pi}{4} \right) + \ln \left(D^2 \right) = \ln \left(\frac{\pi}{4} \right) + 2 \ln (D)$$

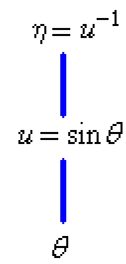
$$\Rightarrow \quad \frac{d}{dD} (\ln A) = 0 + \frac{2}{D} = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sigma_A}{A} \approx \frac{1}{3} \sigma_D = \frac{1}{3} \times 0.1 = \frac{\mathbf{1}}{\mathbf{30}} = 3.\dot{3}\%$$

$$\Rightarrow \quad \sigma_A \approx \frac{1}{30} A = \frac{1}{30} \times \pi \left(\frac{6}{2} \right)^2 = \frac{9\pi}{30} = \frac{3\pi}{10} \approx \mathbf{0.94} \text{ cm}^2$$

Example 8.07 (Navidi textbook, exercises 3.3, page 185, question 8) [**examinable**]

The refractive index η of a piece of glass is related to the critical angle θ by $\eta = \dfrac{1}{\sin\theta}$.
The critical angle is reported to be (0.70 ± 0.02) rad. Estimate the refractive index and find the uncertainty in the estimate.



Non-Linear Functions of Several Measurements [**for bonus questions only**]

If $\{ X_1, X_2, \ldots, X_n \}$ are independent measurements whose uncertainties $\{ \sigma_1, \sigma_2, \ldots, \sigma_n \}$ are small and if $U = U(X_1, X_2, \ldots, X_n)$ is a non-linear function of $\{ X_1, X_2, \ldots, X_n \}$, then

$$\sigma_U \approx \sqrt{\left(\frac{\partial U}{\partial X_1}\right)^2 \sigma_1^{\,2} + \left(\frac{\partial U}{\partial X_2}\right)^2 \sigma_2^{\,2} + \ldots + \left(\frac{\partial U}{\partial X_n}\right)^2 \sigma_n^{\,2}}$$

where the partial derivatives are evaluated at the observed value of $(X_1, X_2, \ldots, X_n)$.

This is the **multivariate propagation of error formula**.

If the $\{ X_1, X_2, \ldots, X_n \}$ are not independent, then a conservative estimate is

$$\sigma_U \le \left|\frac{\partial U}{\partial X_1}\right|\sigma_1 + \left|\frac{\partial U}{\partial X_2}\right|\sigma_2 + \ldots + \left|\frac{\partial U}{\partial X_n}\right|\sigma_n$$

Example 8.08 (Navidi textbook, exercises 3.4, page 193, question 8(b))
[**for bonus questions only**]

The pressure P (in kPa), temperature T (in K) and volume V (in litres) of one mole of an ideal gas are related by the equation

$$PV = 8.31T$$

Given that P = (242.52 ± 0.03) kPa and T = (290.11 ± 0.02) K , estimate V .

$$V = \frac{8.31T}{P} = \frac{8.31 \times 290.11}{242.52} = 9.94068\ldots$$

$$V = 8.31 T P^{-1} \quad \Rightarrow \quad \frac{\partial V}{\partial T} = 8.31 P^{-1} = \frac{8.31}{242.52} = 0.0342652\ldots$$

and $$\frac{\partial V}{\partial P} = -8.31 T P^{-2} = \frac{-8.31 \times 290.11}{(242.52)^2} = -0.0409891$$

Assuming that the measurements of pressure and temperature are independent,

$$\sigma_V \approx \sqrt{\left(\frac{\partial V}{\partial T}\right)^2 \sigma_T{}^2 + \left(\frac{\partial V}{\partial P}\right)^2 \sigma_P{}^2} = \sqrt{(0.034\ldots \times 0.02)^2 + (-0.040\ldots \times 0.03)^2} \approx 0.001408$$

Therefore

$$V = (\mathbf{9.9407 \pm 0.0014}) \text{ litre}$$

If we cannot assume independence, then a conservative estimate is given by

$$\sigma_V \le \left|\frac{\partial V}{\partial T}\right|\sigma_T + \left|\frac{\partial V}{\partial P}\right|\sigma_P = 0.034\ldots \times 0.02 + 0.040\ldots \times 0.03 \approx 0.001915$$

Therefore

$$V = (\mathbf{9.9407 \pm 0.0019}) \text{ litre}$$

[The answer is given to the fifth significant figure in V, because P and T were both quoted to that level of precision. However, the constant is quoted to only 3 s.f., so the answer may be unreliable beyond the third significant figure.]

[End of Chapter 8]

[Space for Additional Notes]