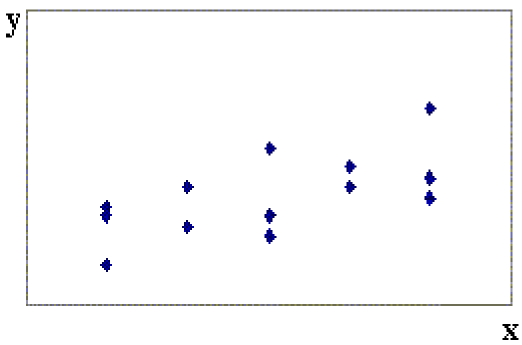
Simple Linear Regression [Navidi Sections 7.2-7.4; Devore Chapter 12]
[This topic is treated somewhat differently here from the approach in the textbooks.]
Sometimes an experiment is set up where the experimenter has control over the values of one or more variables X and measures the resulting values of another variable Y, producing a field of observations.

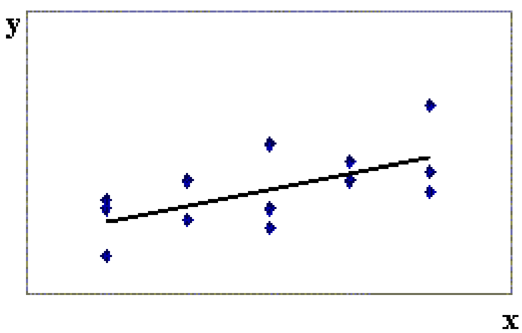


The question then arises: What is the best line (or curve) to draw through this field of points?

Values of X are controlled by the experimenter, so the non-random variable x is called the **controlled** variable or the **independent** variable or the **regressor.**

Values of Y are random, but are influenced by the value of x. Thus Y is called the **dependent** variable or the **response** variable.

We want a "line of best fit" so that, given a value of x, we can predict the value of Y for that value of x.



The **simple linear regression model** is that the **predicted value** of y is

$$y = \beta_0 + \beta_1 x$$

and that the **observed value** of Y is

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where ε_i is the **error**.

It is assumed that the errors are normally distributed as $\varepsilon_i \sim \mathrm{N}\left(0, \sigma^2\right)$, with a constant variance σ^2. The point estimates of the errors ε_i are the **residuals** $e_i = y_i - \hat{y}_i$.

With the assumptions

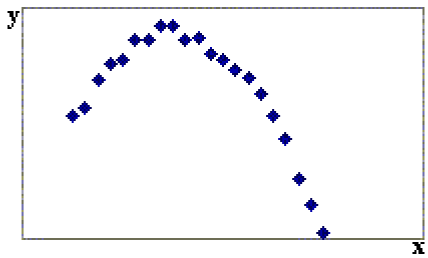
1) $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
2) $x = x_0 \Rightarrow Y \sim \mathrm{N}\left(\beta_0 + \beta_1 x_o, \sigma^2\right)$

in place, it then follows that $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators of the coefficients β_0 and β_1.

Methods for dealing with non-linear regression are available in the course text, but are beyond the scope of this course.

Examples illustrating violations of the assumptions in the simple linear regression model:

1.



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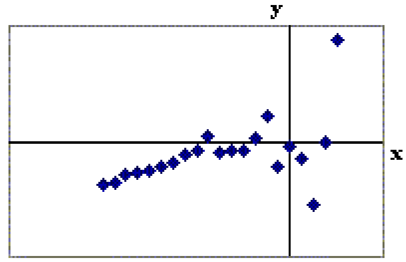

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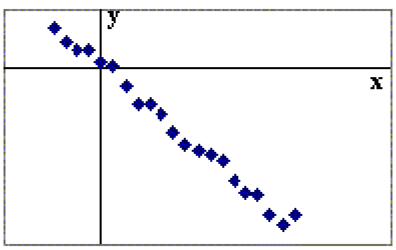

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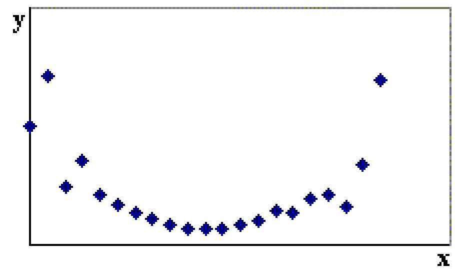
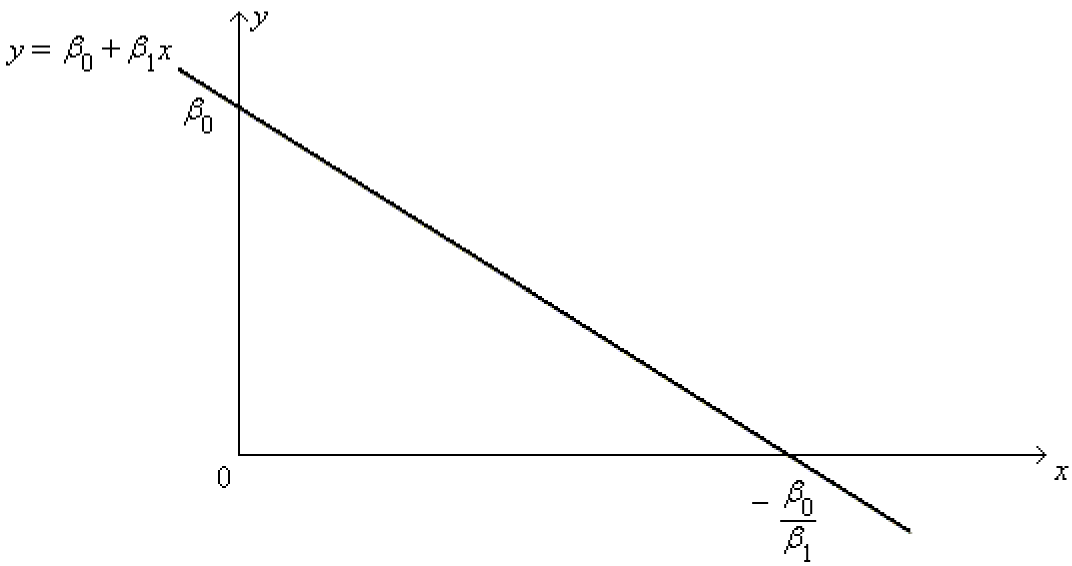


If the assumptions are true, then the probability distribution of $Y \mid x$ is $\mathrm{N}\left(\beta_0 + \beta_1 x_o, \sigma^2\right)$.



<u>Example 15.01</u>

Given that $Y_i = 10 - 0.5x_i + \varepsilon_i$, where $\varepsilon_i \sim \mathrm{N}(0, 2)$, find the probability that the observed value of y at $x = 8$ will exceed the observed value of y at $x = 7$.

$$Y_i \sim \mathrm{N}\left(10 - 0.5x_i,\ 2\right)$$

Let $\quad Y_7$ = the observed value of y at $x = 7$
and $\quad Y_8$ = the observed value of y at $x = 8$,
then

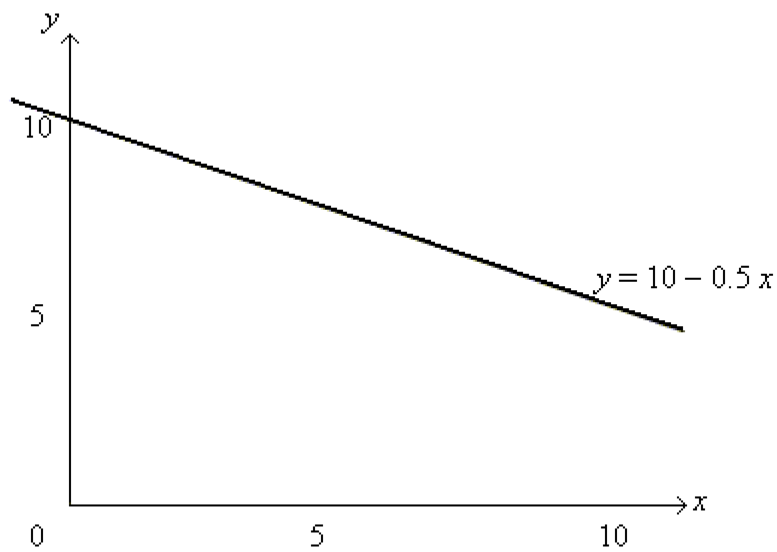
$Y_7 \sim \mathrm{N}($ $\qquad$ and $\qquad Y_8 \sim \mathrm{N}($

$\Rightarrow \qquad Y_8 - Y_7 \sim \mathrm{N}($

$\mu =$ $\qquad\qquad \sigma =$

$\mathrm{P}\left[Y_8 - Y_7 > 0\right] =$

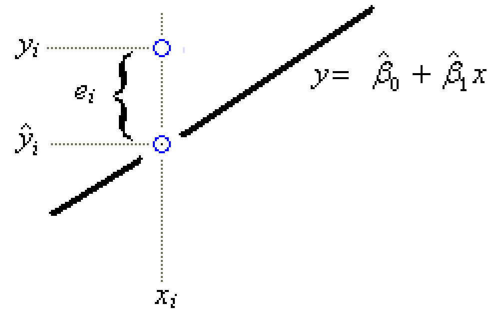
For any x_i in the range of the regression model, more than 95% of all Y_i will lie within $2\sigma \left(= 2\sqrt{2}\right)$ either side of the regression line.



Derivation of the coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ of the regression line $y = \hat{\beta}_0 + \hat{\beta}_1 x$:

We need to minimize the errors.

Each error is estimated by the observed residual $e_i = y_i - \hat{y}_i$.



Use the SSE (sum of squares due to errors)

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2 = f\left(\hat{\beta}_0, \hat{\beta}_1\right)$$

Find $\hat{\beta}_0$ and $\hat{\beta}_1$ such that $\dfrac{\partial S}{\partial \hat{\beta}_0} = \dfrac{\partial S}{\partial \hat{\beta}_1} = 0$.

$$\frac{\partial S}{\partial \hat{\beta}_0} = 2\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)(0 - 1 - 0) = 0 \quad \Rightarrow \qquad \textbf{(1)}$$

and

$$\frac{\partial S}{\partial \hat{\beta}_1} = 2\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)(0 - 0 - x_i) = 0 \quad \Rightarrow \qquad \textbf{(2)}$$

or, equivalently, $\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$ **(3)**

$$\Rightarrow \quad \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix} = \qquad \textbf{(4)}$$

The solution to the linear system of two **normal equations** **(1)** and **(2)** is, from the lower row of matrix equation **(4)**:

$$\boxed{\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}}, \quad \text{(where } \boxed{n S_{xy} = n\sum xy - \sum x \cdot \sum y}$$

$$\text{and} \quad n S_{xx} = n\sum x^2 - \left(\sum x\right)^2 \ )$$

or, equivalently, $\hat{\beta}_1 = \dfrac{\text{sample covariance of } (x, y)}{\text{sample variance of } x}$;

[Another alternative arises from $\rho = \dfrac{\text{Cov}[X,Y]}{\sigma_X \cdot \sigma_Y} \Rightarrow \text{Cov}[X,Y] = \sigma_X \cdot \sigma_Y \cdot \rho$

$$\Rightarrow \quad \hat{\beta}_1 = \frac{s_X \cdot s_Y \cdot r}{s_X^2} = r\frac{s_Y}{s_X} \ ]$$

From equation **(1)**: $\boxed{\hat{\beta}_0 = \frac{1}{n}\left(\sum y - \hat{\beta}_1 \sum x\right)}$

A form that is less susceptible to round-off errors (but less convenient for manual computations) is

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

The regression line of Y on x is $\boxed{y - \bar{y} = \hat{\beta}_1 (x - \bar{x})}$.

Equation **(1)** guarantees that all simple linear regression lines pass through the centroid $(\bar{x}, \bar{y})$ of the data.

It turns out that the simple linear regression method remains valid even if the values of the regressor x are also random.

However, note that interchanging x with y, (so that Y is the regressor and X is the response), results in a *different* regression line (unless X and Y are perfectly correlated).

Example 15.02
(the same data set as Example 12.05: paired two sample t test)

Nine volunteers are tested before and after a training programme. Find the line of best fit for the posterior (after training) scores as a function of the prior (before training) scores.

Volunteer:	1	2	3	4	5	6	7	8	9
After training:	75	66	69	45	54	85	58	91	62
Before training:	72	65	64	39	51	85	52	92	58

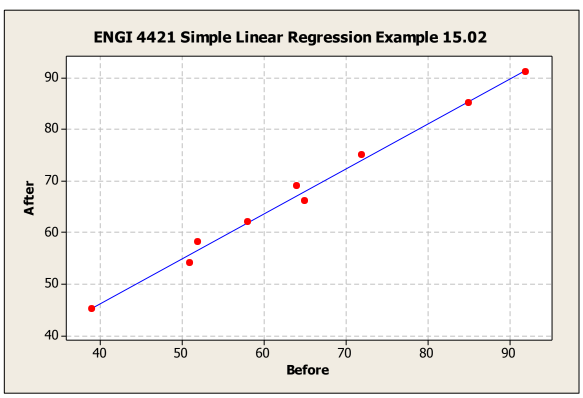
Let Y = score after training and X = score before training.

In order to use the simple linear regression model, the assumptions

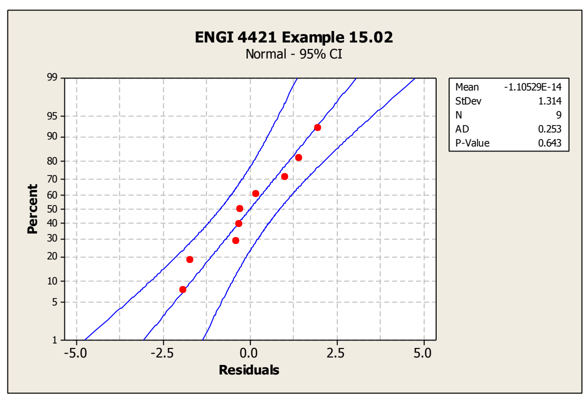
$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$
$$x = x_0 \quad \Rightarrow \quad Y \sim \mathrm{N}\left(\beta_0 + \beta_1 x_o, \sigma^2\right)$$

must hold.

From a scatter plot and a normal probability plot of the data
(in `http://www.engr.mun.ca/~ggeorge/4421/demos/regress2.xls`),
and `http://www.engr.mun.ca/~ggeorge/4421/demos/ex1202.mpj`),
one can see that the assumptions are reasonable.



Scatter plot



Normal probability plot of residuals

Calculations:

i	x_i	y_i	x_i^2	$x_i y_i$	y_i^2
1	72	75	5184	5400	5625
2	65	66	4225	4290	4356
3	64	69	4096	4416	4761
4	39	45	1521	1755	2025
5	51	54	2601	2754	2916
6	85	85	7225	7225	7225
7	52	58	2704	3016	3364
8	92	91	8464	8372	8281
9	58	62	3364	3596	3844
Sum:	**578**	**605**	**39384**	**40824**	**42397**

$$nS_{xy} = n\sum xy - \sum x \sum y = 9\times 40824 - 578\times 605 = \mathbf{17726}$$

$$nS_{xx} = n\sum x^2 - \left(\sum x\right)^2 = 9\times 39384 - 578^2 = \mathbf{20372}$$

$$\Rightarrow \quad \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{17726}{20372} = \underline{\underline{\mathbf{0.870116}}}$$

and $$\hat{\beta}_0 = \frac{1}{n}\left(\sum y - \hat{\beta}_1 \sum x\right) = \frac{1}{9}\left(605 - 0.807116\times 578\right) = \underline{\underline{\mathbf{11.34145}}}$$

Each predicted value $\hat{y}_i$ of Y is then estimated using $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \approx 11.34 + 0.87\,x$ and the point estimates of the unknown errors ε_i are the observed residuals $e_i = y_i - \hat{y}_i$.

A measure of the degree to which the regression line fails to explain the variation in Y is the sum of squares due to error,

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2$$

which is given in the adjoining table.

x_i	y_i	$\hat{y}_i$	e_i	e_i^2
72	75	73.98979	1.0102	1.0205
65	66	67.89898	–1.8990	3.6061
64	69	67.02886	1.9711	3.8854
39	45	45.27597	–0.2760	0.0762
51	54	55.71736	–1.7174	2.9493
85	85	85.30130	–0.3013	0.0908
52	58	56.58747	1.4125	1.9952
92	91	91.39211	–0.3921	0.1537
58	62	61.80817	0.1918	0.0368
			SSE =	**13.8141**

An Alternative Formula for *SSE*:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad \Rightarrow$$

$$SSE = \sum_{i=1}^{n} \left(y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x} \right) - \hat{\beta}_1 x_i \right)^2 = \sum_{i=1}^{n} \left(\left(y_i - \bar{y} \right) - \hat{\beta}_1 \left(x_i - \bar{x} \right) \right)^2$$

$$= \sum_{i=1}^{n} \left(y_i - \bar{y} \right)^2 - 2\hat{\beta}_1 \sum_{i=1}^{n} \left(x_i - \bar{x} \right)\left(y_i - \bar{y} \right) + \hat{\beta}_1^2 \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2$$

$$= S_{yy} - 2\hat{\beta}_1 S_{xy} + \hat{\beta}_1^2 S_{xx}$$

But $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}$

$\Rightarrow \quad SSE = S_{yy} - \hat{\beta}_1 S_{xy}$ or $SSE = \dfrac{S_{xx} S_{yy} - S_{xy}^{\ 2}}{S_{xx}}$ or

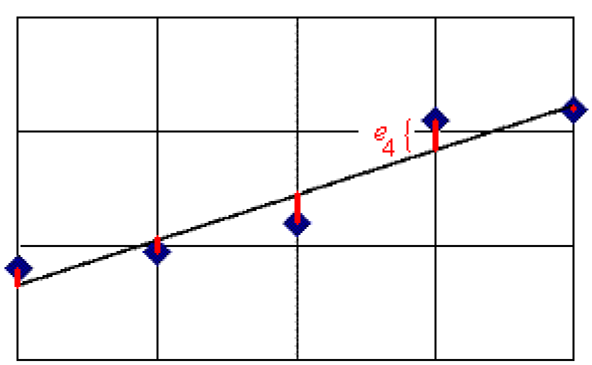
$$\boxed{SSE = \frac{\left(nS_{xx} \right)\left(nS_{yy} \right) - \left(nS_{xy} \right)^2}{n \times \left(nS_{xx} \right)}}$$

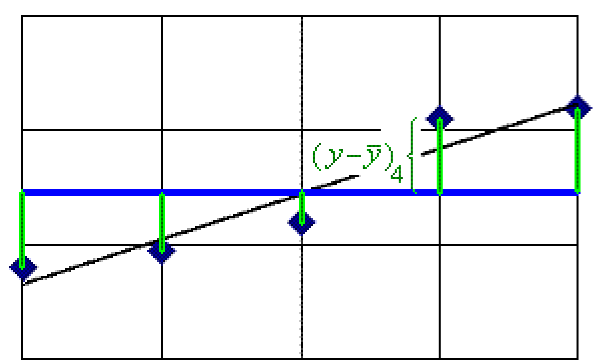
In this example,

$$SSE = \frac{20\,372 \times 15\,548 - 17\,726^2}{9 \times 20\,372} = 13.814\ldots$$

However, this formula is *very* sensitive to round-off errors:
If all terms are rounded off prematurely to three significant figures, then

$$SSE = \frac{20\,400 \times 15\,500 - 17\,700^2}{9 \times 20\,400} = 15.85 \quad (2 \text{ d.p.})$$





$SSE = \sum_{i=1}^{n} e_i^2 =$ $\qquad\qquad$ $SST =$

The total variation in Y is the SST (sum of squares - total):

$$SST = \frac{n\,S_{yy}}{n} = \sum\left(y_i - \bar{y}\right)^2$$ (which is $(n-1)\times$ the sample variance of y).

In this example, $SST = 15\,548 / 9 =$ **$\underline{1\,727.555...}$**

The total variation (SST) can be partitioned into the variation that can be explained by the regression line $\left(SSR = \sum\left(\hat{y}_i - \bar{y}\right)^2\right)$ and the variation that remains unexplained by the regression line (SSE). $SST = SSR + SSE$.

The proportion of the variation in Y that is explained by the regression line is known as the **coefficient of determination**

$$r^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

In this example, $r^2 = 1 - \dfrac{13.81\ldots}{1727.555\ldots} = .992\,004\ldots$

Therefore the regression model in this example explains 99.2% of the total variation in y.

Note:

$$SSR = \hat{\beta}_1 \cdot S_{xy} = \frac{{S_{xy}}^2}{S_{xx}}$$

and $SST = S_{yy}$

$\Rightarrow$

$$r^2 = \frac{{S_{xy}}^2}{S_{xx}\,S_{yy}}$$

The coefficient of determination is just the square of the sample correlation coefficient r. Thus $r = \sqrt{r^2} \approx .996$. It is no surprise that the two sets of test scores in this example are very strongly correlated. Most of the points on the graph are very close to the regression line $y = 0.87x + 11.34$.

A point estimate of the unknown population variance σ^2 of the errors ε is the sample variance or **mean square error** $s^2 = MSE = SSE \,/$ (number of degrees of freedom).

But the calculation of s^2 includes two parameters that are estimated from the data: $\hat{\beta}_0$ and $\hat{\beta}_1$. Therefore two degrees of freedom are lost and $\boxed{MSE = \dfrac{SSE}{n-2}}$.

In this example, $MSE \approx 1.973$.

A concise method of displaying some of this information is the **ANOVA table** (used in Chapters 10 and 11 of Devore for analysis of variance). The f value in the top right corner of the table is the square of a t value that can be used in an **hypothesis test** on the value of the slope coefficient β_1.

Source	Degrees of Freedom	Sums of Squares	Mean Squares	f
Regression	1	SSR = 1713.741...	MSR = SSR / 1 = 1713.741...	= MSR/MSE = 868.4...
Error	$n-2$ = 7	SSE = 13.81...	MSE = SSE / (n–2) = 1.973...	
Total	$n-1$ = 8	SST = 1727.555...		

To test $\mathcal{H}_0 : \beta_1 = 0$ (no useful linear association) against $\mathcal{H}_A : \beta_1 \neq 0$ (a useful linear association exists), we compare $|t| = \sqrt{f}$ to $t_{\alpha/2,\,(n-2)}$.

In this example, $|t| = \sqrt{868.4\ldots} \approx 29.4 \gg t_{.0005,7}$ (the p-value is $< 10^{-7}$) so we reject $\mathcal{H}_o$ in favour of $\mathcal{H}_A$ at any reasonable level of significance α .

The standard error s_b of $\hat{\beta}_1$ is $s_b = \dfrac{s}{\sqrt{S_{xx}}}$ so the t value is also equal to $\boxed{\dfrac{\hat{\beta}_1 - 0}{\sqrt{\dfrac{n\,MSE}{n\,S_{xx}}}}}$.

Yet another alternative test of the significance of the linear association is an hypothesis test on the population correlation coefficient ρ, $\left(\mathcal{H}_o : \rho = 0 \text{ vs. } \mathcal{H}_A : \rho \neq 0\right)$, using the test statistic $\boxed{t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}}}$, which is entirely equivalent to the other two t statistics above.

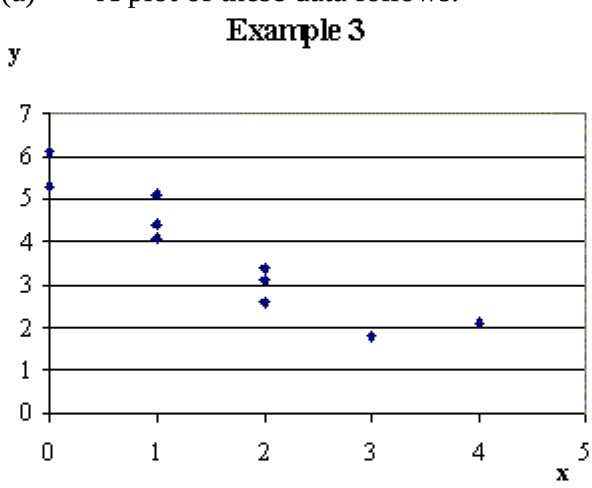
Example 15.03

(a) Find the line of best fit to the data

x	0	0	1	1	1	2	2	2	3	4
y	6.1	5.3	4.1	5.1	4.4	3.4	2.6	3.1	1.8	2.1

(b) Estimate the value of y when $x = 2$.
(c) Why can't the regression line be used to estimate y when $x = 10$?
(d) Find the sample correlation coefficient.
(e) Does a useful linear relationship between Y and x exist?

(a) A plot of these data follows.



The Excel spreadsheet file for these data can be found at

"`http://www.engr.mun.ca/~ggeorge/4421/demos/regress3.xls`".

The summary statistics are

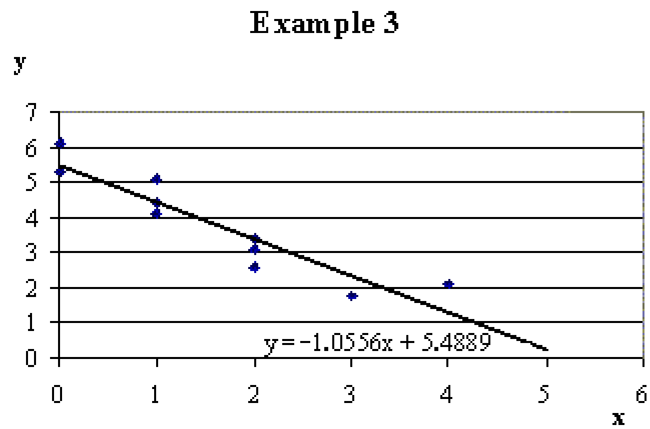
$\Sigma x = 16$ $\Sigma y = 38$ $n = 10$

$\Sigma x^2 = 40$ $\Sigma xy = 45.6$ $\Sigma y^2 = 163.06$

From which

$$n\,S_{xy} = n\,\Sigma\,xy - \Sigma\,x\,\Sigma\,y = -152$$

$$n\,S_{xx} = n\,\Sigma\,x^2 - (\Sigma\,x)^2 = 144 \qquad n\,S_{yy} = n\,\Sigma\,y^2 - (\Sigma\,y)^2 = 186.6$$



$\Rightarrow \quad \hat{\beta}_1 =$

and $\hat{\beta}_0 =$

So the regression line is

(b) $x = 2 \quad \Rightarrow \quad y =$

(c) $x = 10 \quad \Rightarrow \quad y =$

Problem:

(d) $$r = \frac{S_{xy}}{\sqrt{S_{xx}\,S_{yy}}} = \frac{-152}{\sqrt{144 \times 186.6}} = -.92727\ldots \approx \underline{\mathbf{-.93}}$$

(e) $$SSR = \frac{\left(n\,S_{xy}\right)^2}{n\left(n\,S_{xx}\right)} = \frac{(-152)^2}{10 \times 144} = 16.0\dot{4}$$

$$SST = S_{yy} = (186.6 / 10) = 18.66$$

and $SSE = SST - SSR = 18.66 - 16.04\ldots = 2.615\ldots$

The ANOVA table is then:

Source	d.f.	*SS*	*MS*	*f*
R		16.04444...		
E				
T		18.66000		

from which $t = -\sqrt{f} \approx$ But $t_{.0005,8} = 5.041\ldots$

Therefore reject $\mathcal{H}_o : \beta_1 = 0$ in favour of $\mathcal{H}_A : \beta_1 \neq 0$ at any reasonable level of significance α.

OR $\quad t \;=\; \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} \;=\; \dfrac{-.92727...\times\sqrt{8}}{\sqrt{1-.85983...}} \;\approx\; -7.005$

$\Rightarrow$ reject $\mathcal{H}_o : \rho = 0$ in favour of $\mathcal{H}_A : \rho \neq 0$ (a significant linear association exists).

Confidence and Prediction Intervals

The simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ leads to a line of best fit in the least squares sense, which provides an expected value of Y for each value for x:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = \mathrm{E}[Y \mid x] = \mu_{Y|x}.$$

The uncertainty in this expected value has two components:

- the square of the standard error of the scatter of the observed points about the regression line $\left(= \sigma^2 / n\right)$, and
- the uncertainty in the position of the regression line itself, which increases with the distance of the chosen x from the centroid of the data but decreases with increasing spread of the full set of x values: $\sigma^2 \left(\dfrac{(x-\bar{x})^2}{S_{xx}} \right)$.

The unknown variance σ^2 of individual points about the true regression line is estimated by the mean square error $s^2 = MSE$.

Thus a $100(1-\alpha)\%$ **confidence interval** for the expected value of Y at $x = x_o$ has endpoints at

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{\frac{1}{n} + \frac{(x_o - \bar{x})^2}{S_{xx}}}$$

The **prediction error** for a single point is the residual $E = Y - \hat{y}$, which can be treated as the difference of two independent random variables. The variance of the prediction error is then

$\mathrm{V}[E] =$

Thus a $100(1-\alpha)\%$ **prediction interval** for a single future observation of Y at $x = x_o$ has endpoints at

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{(x_o - \bar{x})^2}{S_{xx}}}$$

The prediction interval is always wider than the confidence interval.

Example 15.03 (continued)

(f) Find the 95% confidence interval for the expected value of Y at $x = 2$ and $x = 5$.
(g) Find the 95% prediction interval for a future value of Y at $x = 2$ and at $x = 5$.

(f) $\alpha = 5\% \;\Rightarrow\; \alpha/2 = .025$

Using the various values from parts (a) and (e):

$n = 10 \qquad t_{.025,8} = 2.306\ldots \qquad s = 0.57179\ldots \qquad \bar{x} = 1.6$

$S_{xx} = 14.4 \qquad \hat{\beta}_0 = 5.4888\ldots \qquad \hat{\beta}_1 = -1.0555\ldots$

$x_o = 2 \;\Rightarrow\;$ the 95% CI for $\mu_{Y|2}$ is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{\frac{1}{n} + \frac{(x_o - \bar{x})^2}{S_{xx}}} = 3.3777\ldots \pm 1.3185\ldots \times \sqrt{0.1111\ldots}$$

$= 3.3777\ldots \pm 0.4395\ldots \;\Rightarrow\;$ <u>$2.94 \le \mathrm{E}[\,Y \mid 2\,] < 3.82$</u> (to 3 s.f.)

Example 15.03 (continued)

$x_o = 5 \;\Rightarrow\;$ the 95% CI for $\mu_{Y|5}$ is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{\frac{1}{n} + \frac{\left(x_o - \bar{x}\right)^2}{S_{xx}}} = 0.2111\ldots \pm 1.3185 \times \sqrt{0.902777\ldots}$$

$= 0.2111... \pm 1.2528... \;\Rightarrow\; \underline{-1.04 \le \mathrm{E}[\,Y \mid 5\,] \le 1.46}$ (to 3 s.f.)

(g) $x_o = 2 \;\Rightarrow\;$ the 95% PI for Y is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{\left(x_o - \bar{x}\right)^2}{S_{xx}}} = 3.3777\ldots \pm 1.3185\ldots \times \sqrt{1.1111\ldots}$$

$= 3.3777... \pm 1.3898... \;\Rightarrow\; \underline{1.99 \le Y < 4.77}$ (to 3 s.f.) at $x = 2$

$x_o = 5 \;\Rightarrow\;$ the 95% PI for Y is

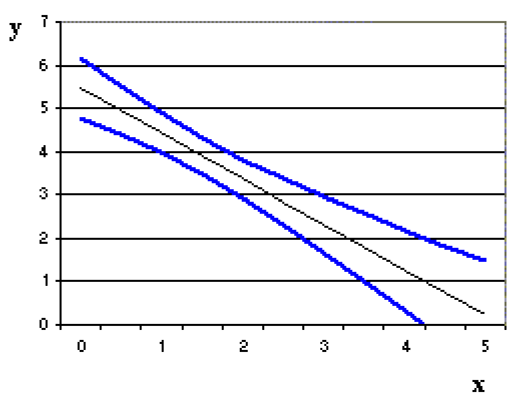
$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o\right) \pm t_{\alpha/2,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{\left(x_o - \bar{x}\right)^2}{S_{xx}}} = 0.2111\ldots \pm 1.3185 \times \sqrt{1.902777\ldots}$$

$= 0.2111... \pm 1.8188... \;\Rightarrow\; \underline{-1.61 < Y < 2.03}$ (to 3 s.f.) at $x = 5$

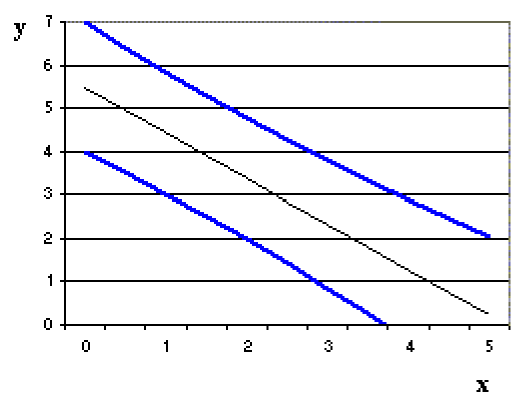
Note how the confidence and prediction intervals both become wider the further away from the centroid the value of x_o is. The two intervals at $x = 5$ are wide enough to cross the x-axis, which is an illustration of the dangers of **extrapolation** beyond the range of x for which data exist.

Sketch of confidence and prediction intervals for Example 3 (f) and (g):

(f) 95% Confidence Intervals



(g) 95% Prediction Intervals



Confidence Intervals on the Slope

It can be shown that

$$\mathrm{E}\left[\hat{\beta}_1\right] = \beta_1 \quad \text{and} \quad \mathrm{V}\left[\hat{\beta}_1\right] = \frac{MSE}{S_{xx}} = \frac{(nS_{xx})(nS_{yy}) - (nS_{xy})^2}{(n-2)(nS_{xx})^2}$$

Therefore a $100(1-\alpha)\%$ confidence interval on the true slope β_1 is

$$\boxed{\hat{\beta}_1 \pm t_{\alpha/2,\, n-2} \frac{s}{\sqrt{S_{xx}}}}$$

Example 15.02 (continued)

$n = 9$, $S_{xx} = \dfrac{20372}{9}$, $\hat{\beta}_1 = \dfrac{17726}{20372} \approx 0.870116$, $s = \sqrt{MSE} = \sqrt{13.814\ldots}$

$t_{.025,7} = 2.36462$

A 95% confidence interval on the slope is

$$0.870\ldots \pm 2.36\ldots\sqrt{\frac{9\times 13.814\ldots}{20372}} = 0.870\ldots \pm 0.184\ldots = (0.685, 1.055)$$

At this level of confidence, it is just plausible that a unit increase in "after" score may be associated with each unit increase in "before" score.

Example 15.03 (continued)

$n = 10$, $S_{xx} = 14.4$, $\hat{\beta}_1 = \dfrac{-152}{144} = -1.0\dot{5}$, $s = \sqrt{MSE} = \sqrt{0.3269\dot{4}}$

$t_{.005,8} = 3.35539$

A 99% confidence interval on the slope is

$$-1.0\dot{5} \pm 3.35\ldots\sqrt{\frac{0.3269\dot{4}}{14.4}} = -1.0\dot{5} \pm 0.50559\ldots = [-1.56, -0.55]$$

A unit decrease in Y for each unit increase in X is very consistent with this confidence interval.

Summary of Formulae for Simple Linear Regression:

First, check that the observations are consistent with $Y \sim \mathrm{N}\left(\beta_0 + \beta_1 x\,,\ \sigma^2\right)$, that is, a linear trend, a constant variance and residuals consistent with a normal distribution.

Calculate $n\,S_{xy} = n\sum xy - \sum x \cdot \sum y$ and similarly $n\,S_{xx}$, $n\,S_{yy}$.

Calculate $\hat{\beta}_1 = \dfrac{n\,S_{xy}}{n\,S_{xx}}$ and $\hat{\beta}_0 = \dfrac{\sum y - \hat{\beta}_1 \sum x}{n}$

The line of best fit to the data in the least squares sense is $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$.

Entries in the ANOVA table:

$$SST = S_{yy} = \frac{n\,S_{yy}}{n}, \qquad SSR = \frac{\left(n\,S_{xy}\right)^2}{n\left(n\,S_{xx}\right)}, \qquad SSE = SST - SSR$$

$$MSR = \frac{SSR}{1}, \qquad MSE = s^2 = \frac{SSE}{n-2}, \qquad f = t^2 = \frac{MSR}{MSE}$$

Coefficient of determination

$$r^2 = \frac{SSR}{SST} = \frac{\left(n\,S_{xy}\right)^2}{\left(n\,S_{xx}\right)\left(n\,S_{yy}\right)}$$

Sample correlation coefficient $= r = \operatorname{sign}\left(\hat{\beta}_1\right)\sqrt{r^2}$

To test $\mathcal{H}_o :\ \rho = 0$ vs. $\mathcal{H}_A :\ \rho \neq 0$ (or, equivalently, $\mathcal{H}_o :\ \beta_1 = 0$ vs. $\mathcal{H}_A :\ \beta_1 \neq 0$):
Use *any* of

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}, \qquad t = \frac{\hat{\beta}_1 - 0}{s_b}, \qquad t = \sqrt{\frac{MSR}{MSE}}, \qquad t = \sqrt{\frac{\left(n\,S_{xy}\right)^2 (n-2)}{\left(n\,S_{xx}\right)\left(n\,S_{yy}\right) - \left(n\,S_{xy}\right)^2}}$$

in a two-tailed single-sample t-test with $(n-2)$ degrees of freedom.

In the second formula, $s_b = \sqrt{\mathrm{V}\left[\hat{\beta}_1\right]} = \sqrt{\dfrac{MSE}{S_{xx}}} = \sqrt{\dfrac{\left(n\,S_{xx}\right)\left(n\,S_{yy}\right) - \left(n\,S_{xy}\right)^2}{(n-2)\left(n\,S_{xx}\right)^2}}$

To test $\mathcal{H}_o: \ \beta_1 = \beta_{1o}$ vs. $\mathcal{H}_A: \ \beta_1 > \beta_{1o}$ use

$$t = \frac{\hat{\beta}_1 - \beta_{1o}}{s_b} \qquad \text{or} \qquad t = \left(\left(n S_{xy} \right) - \beta_{1o} \left(n S_{xx} \right) \right) \sqrt{\frac{(n-2)}{\left(n S_{xx} \right)\left(n S_{yy} \right) - \left(n S_{xy} \right)^2}}$$

The $(1-\alpha)\times 100\%$ **confidence interval** estimate for $\mu = \mathrm{E}\left[Y \mid x = x_o \right]$ is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o \right) \pm t_{\alpha/2,\,(n-2)}\, s \sqrt{\frac{1}{n} + \frac{n\left(x_o - \bar{x} \right)^2}{\left(n S_{xx} \right)}}$$

The $(1-\alpha)\times 100\%$ **prediction interval** estimate for $Y \mid x = x_o$ is

$$\left(\hat{\beta}_0 + \hat{\beta}_1 x_o \right) \pm t_{\alpha/2,\,(n-2)}\, s \sqrt{1 + \frac{1}{n} + \frac{n\left(x_o - \bar{x} \right)^2}{\left(n S_{xx} \right)}}$$

[End of Chapter 15]

[End of ENGI 4421!]