Newton's Law of Cooling / Heating

A common application in a first course on ordinary differential equations is Newton's Law of Cooling (or Heating). A general case is presented here.

At time t = 0, an object at temperature $T(0) = T_0$ is immersed in a fluid at temperature T_f.

The temperature of the immersed object at subsequent times, $T(t)$, is governed by the first order linear ordinary differential equation

$$\frac{dT}{dt} = k\left(T - T_f\right)$$

where k is a constant of proportionality. To find k, we need additional information.

At some time $t_1 > 0$ the temperature of the immersed object is $T(t_1) = T_1$

(with $T_0 < T_1 < T_f$ for heating, $T_f < T_1 < T_0$ for cooling).

We can then find the temperature of the immersed object $T(t)$, in terms of T_0, T_f, t_1, T_1.

We can also find the time t_2 when $T(t_2) = T_2$.

$$\frac{dT}{dt} = k\left(T - T_f\right) \quad \Rightarrow \quad \frac{dT}{dt} - kT = -kT_f$$

This is a first order linear ODE $\dfrac{dy}{dx} + P(x)\,y = R(x)$,

with $y = T,\ x = t,\ P(x) = -k,\ R(x) = -kT_f$

$$P = -k \quad \Rightarrow \quad \int P\,dt = \int -k\,dt = -kt \quad \Rightarrow \quad e^h = e^{-kt}$$

$$\int e^h R\,dt = \int e^{-kt}\left(-kT_f\right)dt = T_f e^{-kt}$$

General solution: $\quad T(t) = e^{-h}\left(\int e^h R\,dt + C\right) = e^{kt}\left(T_f e^{-kt} + C\right) = T_f + C\,e^{kt}$

Initial condition: $\quad T(0) = T_0 \quad \Rightarrow \quad T_0 = T_f + C \quad \Rightarrow \quad C = T_0 - T_f$

Complete solution: $\quad T(t) = T_f + \left(T_0 - T_f\right)e^{kt}$

Additional information $T(t_1) = T_1$

$$\Rightarrow \quad T_1 = T_f + \left(T_0 - T_f\right)e^{kt_1} \quad \Rightarrow \quad e^{kt_1} = \frac{T_1 - T_f}{T_0 - T_f}$$

$$\Rightarrow \quad e^k = \left(\frac{T_1 - T_f}{T_0 - T_f}\right)^{1/t_1} \quad \Rightarrow \quad e^{kt} = \left(\frac{T_1 - T_f}{T_0 - T_f}\right)^{t/t_1}$$

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Therefore the temperature of the immersed object at any time $t > 0$ is

$$\boxed{T(t) = T_f + \left(T_0 - T_f\right)\left(\frac{T_1 - T_f}{T_0 - T_f}\right)^{t/t_1}}$$

To find the time t_2 when $T(t_2) = T_2$,

$$T_2 = T_f + \left(T_0 - T_f\right)\left(\frac{T_1 - T_f}{T_0 - T_f}\right)^{t_2/t_1} \quad \Rightarrow \quad \left(\frac{T_1 - T_f}{T_0 - T_f}\right)^{t_2/t_1} = \frac{T_2 - T_f}{T_0 - T_f}$$

$$\Rightarrow \quad \frac{t_2}{t_1} \ln\left(\frac{T_1 - T_f}{T_0 - T_f}\right) = \ln\left(\frac{T_2 - T_f}{T_0 - T_f}\right) \quad \Rightarrow$$

$$\boxed{t_2 = t_1 \ln\left(\frac{T_2 - T_f}{T_0 - T_f}\right) \Big/ \ln\left(\frac{T_1 - T_f}{T_0 - T_f}\right)}$$

The custom Excel spreadsheet file "Newton_Cool.xlsx" allows the user to enter valid values for T_0, T_f, t_1, T_1. It will return the time t_2 when $T(t_2) = T_2$ and the temperature $T_3 = T(t_3)$ at time t_3.
