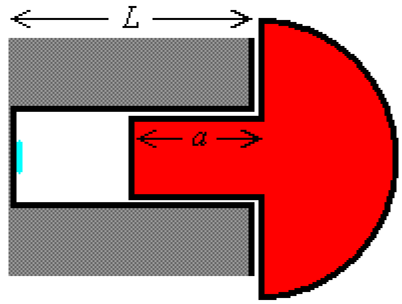
The Bug-Rivet Paradox

An idealised bug of negligible dimensions is hiding at the end of a hole of length L.

A rivet has a shaft length of $a < L$.

Clearly the bug is safe when the rivet head is flush to the [very resilient] surface.



Consider what happens when the rivet slams into the surface at a speed of $v = \beta c$ (where c is the speed of light and $0 < \beta < 1$).

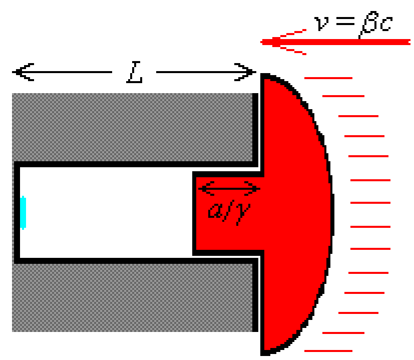
The special theory of relativity states that objects moving relative to our frame of reference are shortened in the direction of motion by a factor $\frac{1}{\gamma} = \sqrt{1-\beta^2}$ (where $\gamma =$ the Lorentz factor).

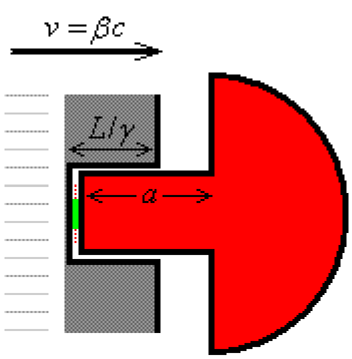
From the point of view (frame of reference) of the bug, the rivet shaft is even shorter and therefore the bug should continue to be safe, however fast the rivet is moving.

$$a_{\text{app}} = \frac{a}{\gamma} = a\sqrt{1-\beta^2} < a < L$$

This does assume that both objects are ideally rigid.

We shall return to this point of view later.





From the frame of reference of the rivet, the rivet is stationary and unchanged, but the hole is moving fast and is shortened by the Lorentz contraction to

$$L_{\text{app}} = \frac{L}{\gamma} = L\sqrt{1-\beta^2}$$

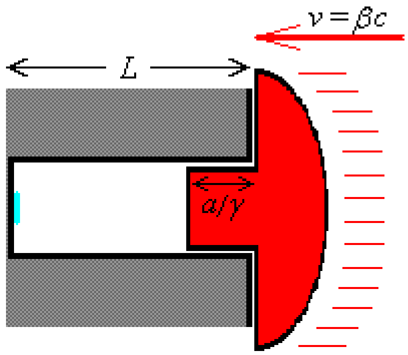
If the approach speed is fast enough, so that $L_{\text{app}} < a$, then the end of the hole slams into the tip of the rivet before the surface can reach the head of the rivet. The bug is squashed!

This is the paradox: is the bug squashed or not?

There are many sources for this paradox (a relative of the pole-barn paradox), such as "`http://en.wikipedia.org/wiki/Wikipedia:Reference_desk/Archives/Science/2006_October_19#Bug_Rivet_Paradox`"
and a nice animation at
" `http://math.ucr.edu/~jdp/Relativity/Bug_Rivet.html` ".

Resolution of the Paradox:

One of the consequences of special relativity is that two events that are simultaneous in one frame of reference are no longer simultaneous in other frames of reference. Perfectly rigid objects are impossible.



In the frame of reference of the bug, the entire rivet cannot come to a complete stop all at the same instant. Information cannot travel faster than the speed of light. It takes time for knowledge that the rivet head has slammed into the surface to travel down the shaft of the rivet. Until each part of the shaft receives the information that the rivet head has stopped, that part keeps going at speed $v = \beta c$.

The information proceeds down the shaft at speed c while the tip continues to move at speed $v = \beta c$.

The tip cannot stop until a time $t_1 = \frac{a}{\gamma} \div (c - \beta c) = \frac{a}{\gamma c(1-\beta)}$ after the head has stopped.

During that time the tip travels a distance vt_1.

The bug will be squashed if $vt_1 > L - \frac{a}{\gamma}$

$$\Rightarrow \frac{\beta c a}{\gamma c(1-\beta)} > L - \frac{a}{\gamma} \Rightarrow \frac{a}{\gamma}\left(\frac{\beta}{1-\beta}+1\right) > L \Rightarrow \frac{a}{\gamma}\left(\frac{\beta+1-\beta}{1-\beta}\right) > L$$

Also recall that $\frac{1}{\gamma} = \sqrt{1-\beta^2} \Rightarrow \frac{1}{\gamma(1-\beta)} = \frac{\sqrt{1-\beta^2}}{1-\beta} = \frac{\sqrt{(1-\beta)(1+\beta)}}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}}$

The bug will be squashed if $a\sqrt{\frac{1+\beta}{1-\beta}} > L \Rightarrow \frac{a}{L} > \sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \frac{1-\beta}{1+\beta} < \left(\frac{a}{L}\right)^2$

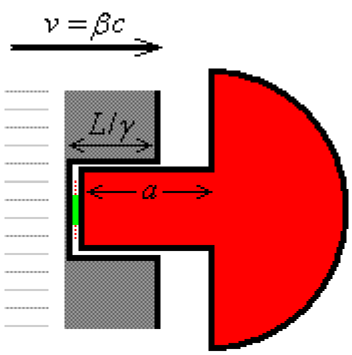
$$\Rightarrow 1-\beta < \left(\frac{a}{L}\right)^2 + \beta\left(\frac{a}{L}\right)^2 \Rightarrow \beta\left(1+\left(\frac{a}{L}\right)^2\right) > 1-\left(\frac{a}{L}\right)^2 \Rightarrow \beta > \frac{1-\left(\frac{a}{L}\right)^2}{1+\left(\frac{a}{L}\right)^2}$$

Therefore the bug will definitely be squashed if the speed exceeds $v_{\min} = \beta_{\min} c$, where

$$\boxed{\beta_{\min} = \frac{1-\left(\frac{a}{L}\right)^2}{1+\left(\frac{a}{L}\right)^2}}.$$

One can verify that $\lim_{a \to 0^+} \beta_{\min} = 1^-$ and that $\lim_{a \to L^-} \beta_{\min} = 0^+$.

Note that the impact of the rivet head always happens before the bug is squashed.



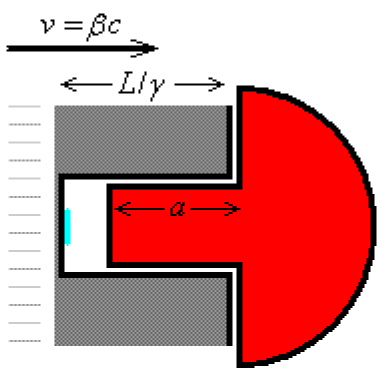
In the frame of reference of the rivet, the bug is definitely squashed if $\frac{L}{\gamma} < a \;\Rightarrow\; L\sqrt{1-\beta^2} < a \;\Rightarrow\; 1-\beta^2 < \left(\frac{a}{L}\right)^2$

$\Rightarrow\; 1-\left(\frac{a}{L}\right)^2 < \beta^2 \;\Rightarrow\; \beta > \sqrt{1-\left(\frac{a}{L}\right)^2}$.

The bug is squashed ***before*** the impact of the surface on the rivet head.

This is a higher speed than the $\beta_{\min} = \dfrac{1-\left(\frac{a}{L}\right)^2}{1+\left(\frac{a}{L}\right)^2}$ above.

The entire surface cannot come to an abrupt stop at the same instant. It takes time for the information about the impact of the rivet tip on the end of the hole to reach the surface that is rushing towards the rivet head. Let us now examine the case where the speed is not high enough for the Lorentz-contracted hole to be shorter than the rivet shaft in the frame of reference of the rivet. Now the observers agree that the impact of the rivet head happens first.



When the surface slams into contact with the head of the rivet, it takes time for information about that impact to travel down to the end of the hole. During this time the hole continues to move towards the tip of the rivet.

The time it takes for the propagating information to reach the tip of the stationary rivet is $t_2 = \frac{a}{c}$, during which time the bug moves a distance $vt_2 = \beta c\frac{a}{c} = \beta a$.

The bug is squashed if

$$vt_2 > \frac{L}{\gamma} - a \;\Rightarrow\; \beta a > \frac{L}{\gamma} - a \;\Rightarrow\; (1+\beta)a > \frac{L}{\gamma} \;\Rightarrow\; \frac{a}{L} > \frac{1}{1+\beta}\sqrt{1-\beta^2}$$

$$\Rightarrow\; \frac{1}{1+\beta}\sqrt{(1+\beta)(1-\beta)} < \frac{a}{L} \;\Rightarrow\; \sqrt{\frac{(1+\beta)(1-\beta)}{(1+\beta)^2}} < \frac{a}{L} \;\Rightarrow\; \sqrt{\frac{1-\beta}{1+\beta}} < \frac{a}{L}$$

$$\Rightarrow\; \frac{1-\beta}{1+\beta} < \left(\frac{a}{L}\right)^2 \;\Rightarrow\; 1-\beta < \left(\frac{a}{L}\right)^2 + \beta\left(\frac{a}{L}\right)^2 \;\Rightarrow\; \beta\left(1+\left(\frac{a}{L}\right)^2\right) > 1-\left(\frac{a}{L}\right)^2$$

This leads to the same minimum speed that guarantees the squashing of the bug as was the case in the frame of reference of the bug:

$$\boxed{\beta_{\min} = \left(1-\left(\frac{a}{L}\right)^2\right)\Big/\left(1+\left(\frac{a}{L}\right)^2\right)}.$$

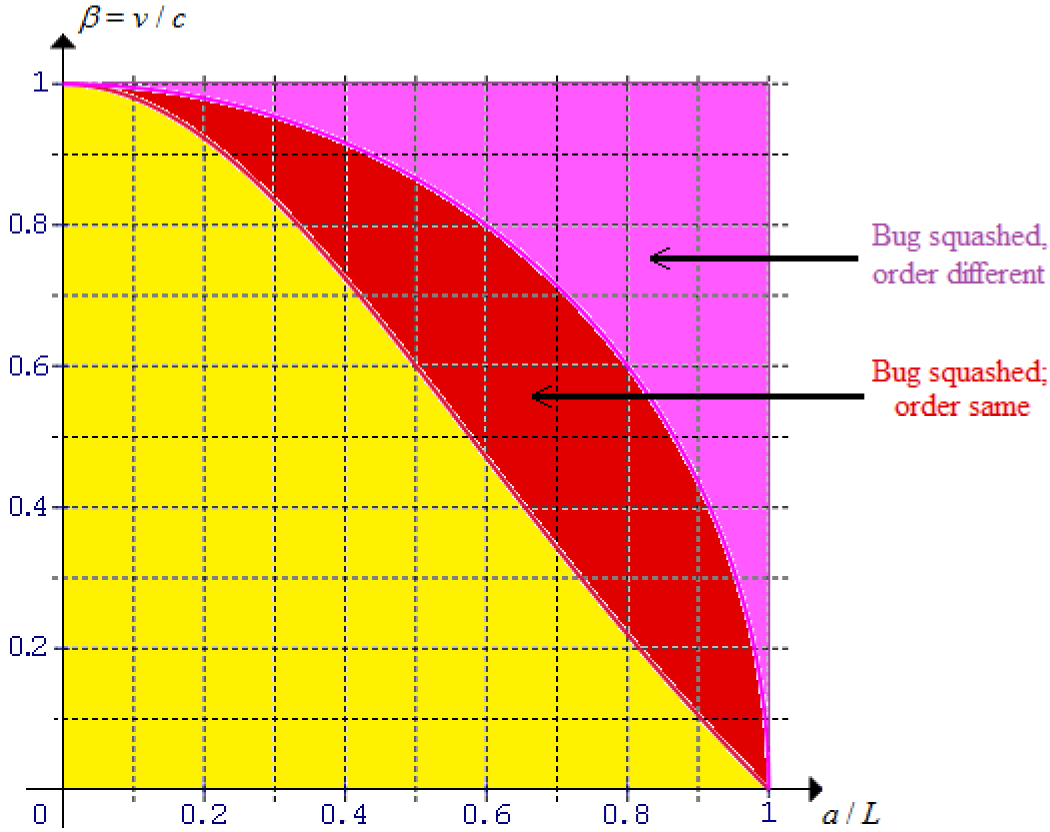
Note that observers travelling with each of the two frames of reference (bug and rivet) agree that the bug is squashed if $\beta > \beta_{\min}$, which resolves the paradox. They also agree that the impact of rivet head on surface happens before the bug is squashed, provided that

$$\frac{1-\left(\frac{a}{L}\right)^2}{1+\left(\frac{a}{L}\right)^2} < \beta < \sqrt{1-\left(\frac{a}{L}\right)^2}$$

(the red middle region in the graph below, of speed $\beta = v/c$ against shaft length a/L).

However, they disagree on which event happens first if $\beta > \sqrt{1-\left(\frac{a}{L}\right)^2}$ (the purple upper-right region of the graph). For speeds this high, the observer in the bug's frame of reference still deduces that the rivet-head impact happens first, but the other observer deduces that the bug is squashed first.

At the critical speed $\beta_c = \sqrt{1-\left(\frac{a}{L}\right)^2}$ the two events are simultaneous in the frame of the rivet, (the rivet fits perfectly in the shortened hole), but they are not simultaneous in the other frame of reference.



Dr. Glyn George
2012 May; graph added 2015 July
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