

FLOWS IN STREAM TUBES

CONSERVATION LAWS IN INTEGRAL FORM

Conservation of Mass states that the time rate of change of mass of a specific group of fluid particles in a flow is zero. Conservation of Momentum states that the time rate of change of momentum of a specific group must balance with the net load acting on it. Conservation of Energy states that the time rate of change of energy of a specific group must balance with heat and work interactions of the group with its surroundings. Mathematically one can write:

Conservation of Mass

$$D/Dt \int_V \rho \, dV = \int_V \partial \rho / \partial t \, dV + \int_S \rho \, \mathbf{v} \cdot \mathbf{n} \, dS = 0$$

Conservation of Momentum

$$\begin{aligned} D/Dt \int_V [\rho \mathbf{v}] \, dV &= \int_V \partial [\rho \mathbf{v}] / \partial t \, dV + \int_S [\rho \mathbf{v}] \, \mathbf{v} \cdot \mathbf{n} \, dS \\ &= \int_S \boldsymbol{\sigma} \, dS + \int_V \rho \mathbf{b} \, dV \end{aligned}$$

Conservation of Energy

$$\begin{aligned} D/Dt \int_V [\rho e] \, dV &= \int_V \partial [\rho e] / \partial t \, dV + \int_S [\rho e] \, \mathbf{v} \cdot \mathbf{n} \, dS \\ &= - \int_S \mathbf{q} \cdot \mathbf{n} \, dS + \int_S \mathbf{v} \cdot \boldsymbol{\sigma} \, dS \end{aligned}$$

In these equations, V is fluid volume, S is fluid surface area, t is time, \mathbf{n} is outward unit normal on S , \mathbf{v} is velocity, ρ is density, $\boldsymbol{\sigma}$ denotes surface stresses such as pressure and viscous traction, \mathbf{b} denotes body forces such as gravity, e is energy density and \mathbf{q} denotes heat flux.

CONSERVATION LAWS IN STREAM TUBE FORM

Conservation of Mass for a stream tube is:

$$\sum [\rho CA]_{\text{OUT}} - \sum [\rho CA]_{\text{IN}} = 0$$

In this equation, ρ is density, C is flow speed and A is pipe area. Letting ρCA equal \dot{M} allows one to rewrite mass as

$$\sum \dot{M}_{\text{OUT}} - \sum \dot{M}_{\text{IN}} = 0 \quad \sum \dot{M}_{\text{OUT}} = \sum \dot{M}_{\text{IN}}$$

Conservation of Momentum for a stream tube is:

$$\begin{aligned} & \sum [\rho \mathbf{v} CA]_{\text{OUT}} - \sum [\rho \mathbf{v} CA]_{\text{IN}} \\ &= - \sum [P A \mathbf{n}]_{\text{OUT}} - \sum [P A \mathbf{n}]_{\text{IN}} + \mathbf{R} \end{aligned}$$

Expansion gives

$$\sum [\dot{M} U]_{\text{OUT}} - \sum [\dot{M} U]_{\text{IN}} = - \sum P A n_x + R_x$$

$$\sum [\dot{M} V]_{\text{OUT}} - \sum [\dot{M} V]_{\text{IN}} = - \sum P A n_y + R_y$$

$$\sum [\dot{M} W]_{\text{OUT}} - \sum [\dot{M} W]_{\text{IN}} = - \sum P A n_z + R_z$$

In these equations, P is pressure, U V W are velocity components and \mathbf{R} is the wall force on the fluid.

Conservation of Energy for a stream tube is

$$\begin{aligned} \sum [\dot{M} (C^2/2 + gz)]_{\text{OUT}} - \sum [\dot{M} (C^2/2 + gz)]_{\text{IN}} = \\ - \sum [P A C]_{\text{OUT}} + \sum [P A C]_{\text{IN}} + \sum \dot{T} - \sum \dot{L} \end{aligned}$$

Manipulation gives

$$\sum [\dot{M} gh]_{\text{OUT}} - \sum [\dot{M} gh]_{\text{IN}} = + \sum \dot{T} - \sum \dot{L}$$

where h is known as head and is given by

$$h = C^2/2g + P/\rho g + z$$

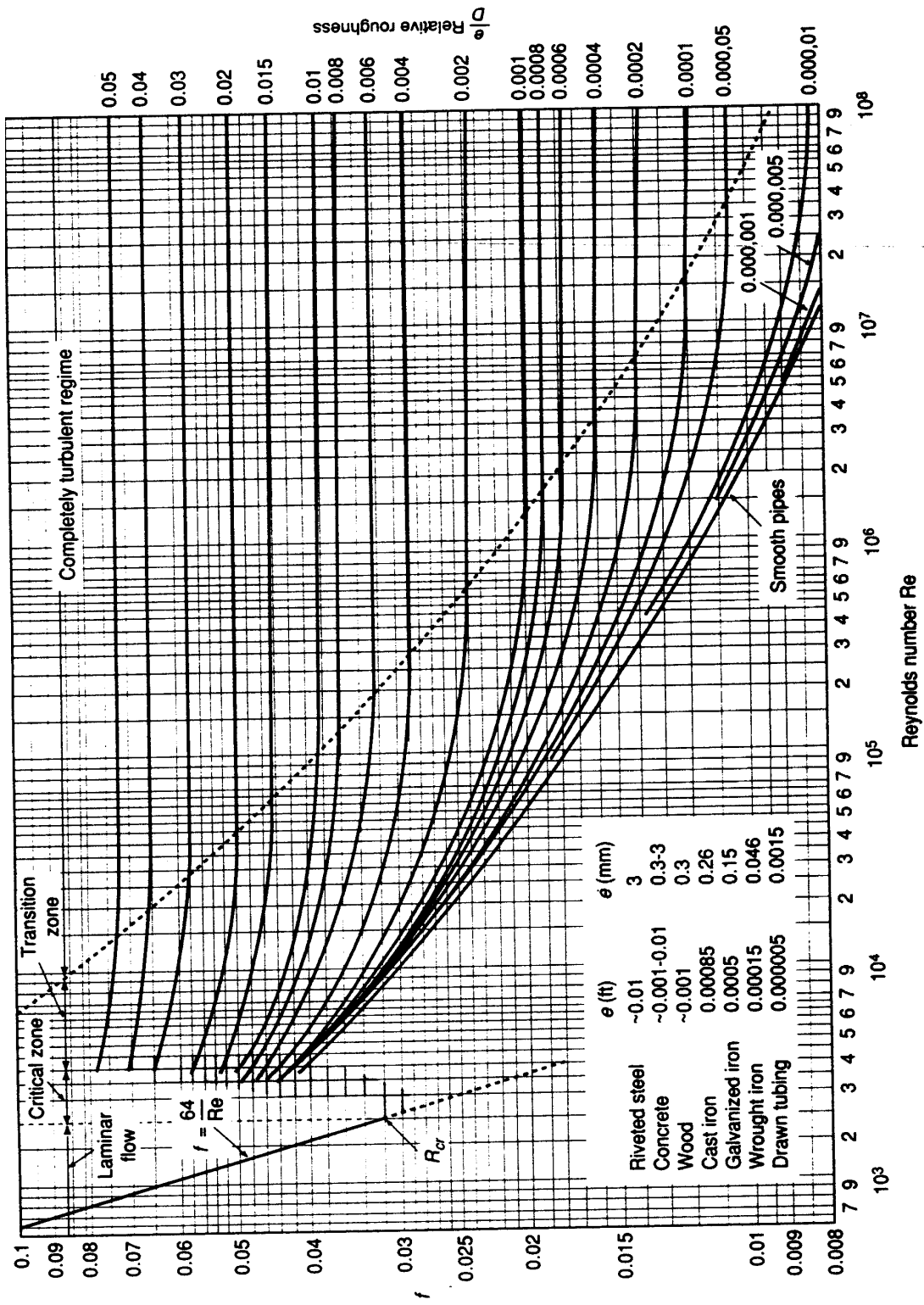
It represents each energy as an equivalent height of fluid. One can represent shaft power and lost power as

$$\dot{T} = \dot{M} gh_T \quad \dot{L} = \dot{M} gh_L$$

The head loss is given by

$$h_L = (fL/D + \sum K) C^2/2g$$

where f is pipe friction factor, L is pipe length, D is pipe diameter and K accounts for losses at constrictions such as bends. The Moody Diagram gives f as a function of Reynolds Number $Re = CD/\nu$ and pipe relative roughness $\epsilon = e/D$.



Moody diagram. (From L. F. Moody, *Trans. ASME*, Vol. 66, 1944.)

BERNOULLI EQUATION

When there is no shaft work and friction is insignificant, conservation of energy for a stream tube shows that h_{OUT} is equal to h_{IN} , which implies that h is constant:

$$C^2/2g + P/\rho g + z = K$$

This equation is known as the Bernoulli Equation. It can also be derived from conservation of momentum. For a short stream tube, a force balance gives:

$$\rho DC/Dt = \rho (\partial C/\partial t + C\partial C/\partial s) = -\partial P/\partial s - \rho g \partial z/\partial s$$

For steady flow this becomes

$$\rho C dC/ds = \rho d[C^2/2]/ds = -dP/ds - \rho g dz/ds$$

Integration of this gives the Bernoulli equation:

$$C^2/2 + P/\rho + gz = \kappa$$

This equation shows that, when pressure goes down in a flow, speed goes up and visa versa. From an energy perspective, flow work causes the speed changes. From a momentum perspective, it is due to pressure forces.

SYSTEM DEMAND

For a system where a pipe connects two reservoirs, the head H versus flow Q system demand equation has the form:

$$H = X + Y Q^2$$

$$X = \Delta [P/\rho g + z] \quad Y = [fL/D + \Sigma K]/[2gA^2]$$

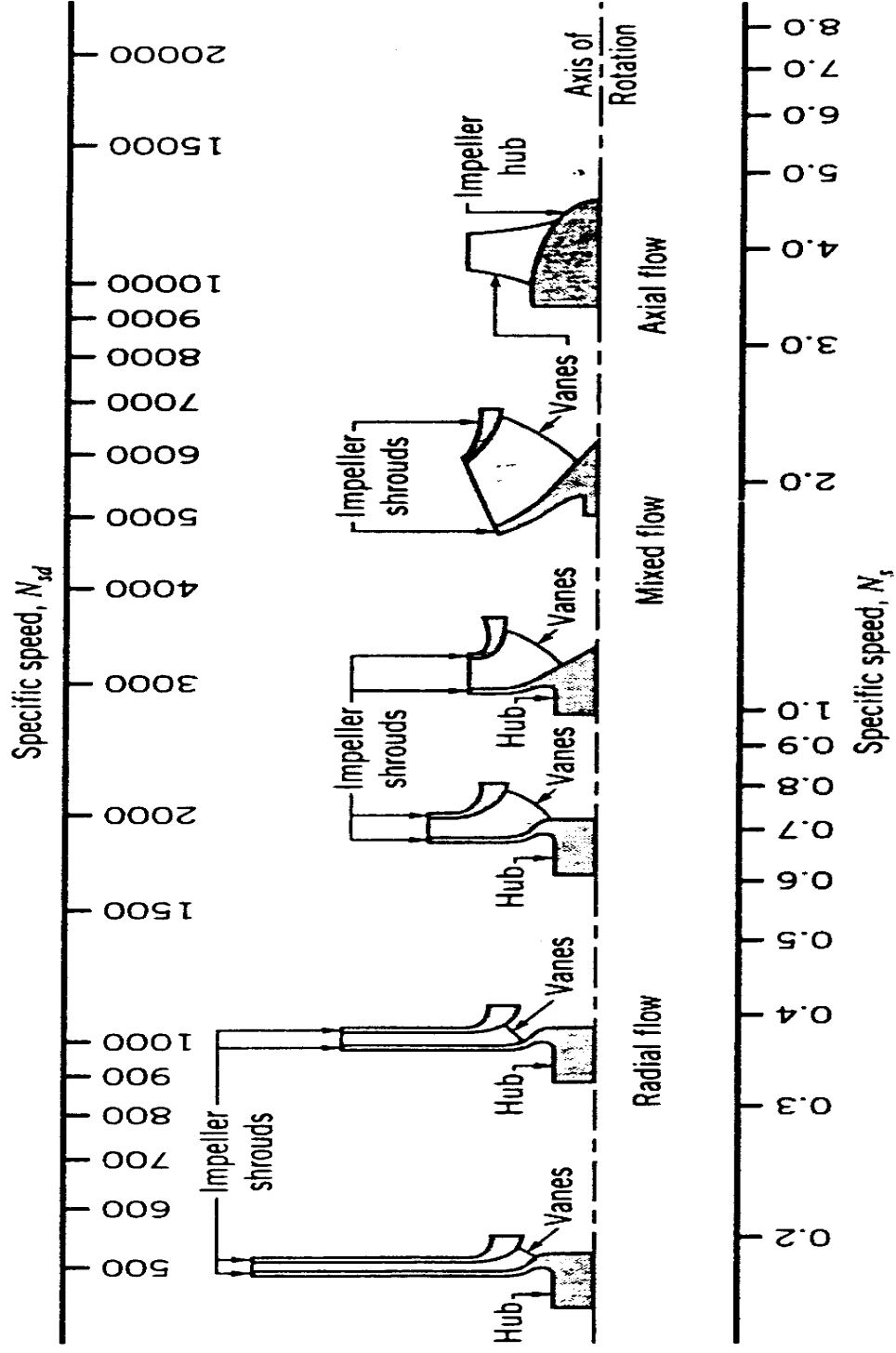
X accounts for pressure and height changes between the reservoirs and Y accounts for losses along the pipe.

PUMP SELECTION

To pick a pump, one first calculates the specific speed **N** based on the system operating point. This is a nondimensional number which does not have pump size in it:

$$\mathbf{N} = [N \sqrt{Q}]/[H^{3/4}]$$

This allows one to pick the appropriate type of pump. Axial pumps have high Q but low H which gives them high **N** . Radial pumps have lower Q but higher H which gives them lower **N** . Positive Displacement pumps have the lowest Q but highest H which gives them the lowest **N** . Next one scans pump catalogs of the type indicated by specific speed and picks the size of pump that will meet the system demand, while it is operating at its best efficiency point (BEP) or best operating point (BOP). Finally, to prevent cavitation, the pump is located in the system at a point where it has the Net Positive Suction Head or NPSH recommended by the manufacturer:



$$\text{NPSH} = P_s/\rho g + C_s C_s/2g - P_v/\rho g$$

In this equation, P_v is the absolute vapor pressure of the fluid being pumped, and P_s and C_s are the absolute pressure and speed at the pump inlet. For a system where a pipe connects a low reservoir to a high reservoir, conservation of energy from the low reservoir to the pump inlet gives:

$$P_o/\rho g - [P_s/\rho g + C_s C_s/2g + d] = h_L$$

where P_o is the absolute pressure of the air above the low reservoir and d is the height of the pump above the surface of the low reservoir. Manipulation gives

$$d = (P_o - P_v)/\rho g - h_L - \text{NPSH}$$

This shows that d might have to be negative.

ELECTRICAL ANALOGY

Electrical power **P** is $V I$ where V is volts and I is current. By analogy, fluid power **P** is $P Q$ where P is pressure and Q is volumetric flow rate. Note that power is force F times speed C . In a flow, force F is pressure P times area A . So power is P times A times C . Now volumetric flow rate Q is C times A . So power becomes P times Q . One can write pressure P in terms of head H as: $P = \rho g H$. Power becomes: **P** = $\rho g H Q$. Voltage drop along a wire is $\Delta V = RI$ where R is the resistance of the wire. By analogy, the pressure drop along a pipe due to losses is $\Delta P = RQ^2$ where R is the resistance of the pipe.

PRESSURE ITERATION METHOD FOR PIPE NETWORKS

In the pressure iteration method, one would first assume pressure at each node in the network where it is not known. Then for each node one would assume pressures at the surrounding nodes to be fixed. Next for each pipe connected to the node one balances head loss with pressure/gravity head: here pumps are treated as negative head losses while turbines are treated as positive head losses. This allows us to calculate the flow in each pipe and its direction. One then calculates the sum of the flows into the node treating flows in as positive and flows out as negative. If the $\Sigma Q > 0$ then the node acts like a sink and the pressure there is too low and must be increased a bit. If the $\Sigma Q < 0$ then the node acts like a source and the pressure there is too high and must be lowered a bit. Each node in the network is treated the same way. One sweeps through the network nodes again and again until the ΣQ for each node is approximately zero.

FLOW ITERATION METHOD FOR PIPE NETWORKS

In the flow iteration method, one assumes a distribution of flow which satisfies $\Sigma Q = 0$ at each node in the network. The flow iteration method modifies flows throughout the network in a way which maintains $\Sigma Q = 0$ at each node. In the method one identifies pipe loops in the network. Then for each loop one calculates the sum of the head losses as one moves around it in a clockwise sense. If flow in a pipe is clockwise head loss is taken to be positive whereas if flow is counterclockwise head loss is taken to be negative. For a

loop if the $\Sigma h_L > 0$ then there is too much clockwise flow: so flows must be reduced a bit in a clockwise sense. This decreases clockwise flows and increases counterclockwise flows. If the $\Sigma h_L < 0$ then there is not enough clockwise flow: so flows must be increased a bit in a clockwise sense. This increases clockwise flows and decreases counterclockwise flows. Each loop in the network is treated the same way. One sweeps through the network loops again and again until the Σh_L for each loop is approximately zero. Special pseudo loops are used to connect reservoirs.

TURBOMACHINES

Swirl is the only component of fluid velocity that has a moment arm around the axis of rotation or shaft of a turbomachine. Because of this, it is the only one that can contribute to shaft power. The shaft power equation is:

$$\mathbf{P} = \Delta [T \omega] = \Delta [\rho Q V_t R \omega]$$

The swirl or tangential component of fluid velocity is V_t . The symbol Δ indicates we are looking at changes from inlet to outlet. The tangential momentum at an inlet or an outlet is $\rho Q V_t$. Multiplying momentum by moment arm R gives the torque T . Multiplying torque by the speed ω gives the power \mathbf{P} . The power equation is good for pumps and turbines. Power is absorbed at an inlet and expelled at an outlet. If the outlet power is greater than the inlet power, then the machine is a pump. If the outlet power is less than the inlet power, then the machine is a turbine. Geometry can be used to connect V_t to the flow rate Q and the rotor speed ω .