SCALING LAWS

PREAMBLE

Scaling laws allow us to predict prototype behavior from model data. Generally the model and prototype must look the same. This is known as geometric similitude. The flow patterns at both scales must also look the same. This is known as kinematic or motion similitude. Finally, certain force ratios in the flow must be the same at both scales. This is known as kinetic or dynamic similitude. Sometimes getting all force ratios the same is impossible and one must use engineering judgement to resolve the issue.

The simplest way to derive scaling laws is to use common sense. If you need to develop a nondimensional power coefficient, you need to divide power by a reference power. The reference power could be based on things like the properties of the fluid and conditions imposed by the surroundings. One could also derive the scaling laws using a more formal procedure known as the Method of Indices. Most fluids texts call this the Buckingham π Theorem. For this, the variables and parameters of interest are divided into primary and secondary categories. When using the Buckingham π Theorem, each nondimensional coefficient is known as a π.

ILLUSTRATION: PUMPS

For a pump, it is customary to let N be the rotor RPM and D be the rotor diameter. All flow speeds U scale as ND and all areas A scale as D^2. Pressures are set by the dynamic pressure $\rho U^2/2$. Ignoring constants, one can define a reference pressure $[\rho N^2 D^2]$ and a reference flow $[ND^3]$. Since fluid power is just pressure times flow, one can also define a reference power $[\rho N^3 D^5]$. Dividing dimensional quantities by reference quantities gives the scaling laws:

Pressure Coefficient $C_P = P / [\rho N^2 D^2]$

Flow Coefficient $C_Q = Q / [ND^3]$

Power Coefficient $C_{\mathbf{P}} = \mathbf{P} / [\rho N^3 D^5]$

On the pressure versus flow characteristic of a pump, there is a best efficiency point (BEP) or best operating point (BOP). For geometrically similar pumps that have the same operating point on the C_P versus C_Q curve, the coefficients show that if D is doubled, P increases 4 fold, Q increases 8 fold and **P** increases 32 fold, whereas if N is doubled, P increases 4 fold, Q doubles and **P** increases 8 fold.

ILLUSTRATION: TURBINES

For a turbine, we are interested mainly in the power output of the device as a function of its rotational speed. The simplest way to develop a nondimensional power is to divide power **P** by something which has the units of power. The power in a flow is its dynamic pressure P times volumetric flow rate Q. For a flow, the dynamic pressure P is

$$P = \rho V^2/2$$

where ρ denotes the density of fluid and V is the speed of the flow. Volumetric flow Q is the speed of the flow V times its flow area A. So, a reference power is

$$\rho V^2/2 \quad VA$$

So, we can define a power coefficient $C_{\mathbf{P}}$

$$C_{\mathbf{P}} = \mathbf{P} \ \mathbf{/} \ [\rho \ V^3/2 \quad A]$$

To develop a nondimensional version of the rotational speed, we can divide the tip speed of the blades rω by the flow speed V. So, we can define a speed coefficient $C_{\mathbf{S}}$

$$C_{\mathbf{S}} = r\omega / V$$

One could derive the power and rotor speed coefficients using the Buckingham π Theorem. Power and speed would be primary variables. The flow speed and area and the density of the fluid would be secondary variables. For power, the goal is to find π_P where

$$\pi_P = \mathbf{P} \quad V^a \quad \rho^b \quad A^c$$

We need to find the a b c that make the right hand side dimensionless. In terms of the basic units of mass M and length L and time T, one can write

$$M^0 L^0 T^0 \quad = \quad M\ L/T^2 \quad L\ /T \quad [L/T]^a \quad [M/L^3]^b \quad [L^2]^c$$

Inspection shows that

$$a=-3 \qquad b=-1 \qquad c=-1$$

With this, π_P becomes

$$\pi_P = \mathbf{P} \ \boldsymbol{/} \ [\rho V^3 \ A]$$

Similarly, for rotor speed, the goal is to find π_S where

$$\pi_S = \omega \quad V^a \quad \rho^b \quad r^c$$

Manipulation shows that

$$a=-1 \qquad b=0 \qquad c=+1$$

With this, π_S becomes

$$\pi_S = r\omega \ / \ V$$

As can be seen, each π is basically the same as a C.

ILLUSTRATION : WAKE DRAG ON BODIES

For a body moving through a fluid, the wake drag on it can be represented nondimenionally as a drag coefficient:

$$C_D = \mathbf{D} / [[\rho U^2/2] A]$$

The reference drag is the dynamic pressure associated with the motion of the body times its profile area as seen from upstream. Usually C_D is a function of Reynolds Number:

$$Re = UD/\nu$$

This is inertia forces divided by viscous forces.

ILLUSTRATION : WAVE DRAG ON SHIPS

The drag on a ship due to wave generation can also be represented as a drag coefficient:

$$C_D = \mathbf{D} / [[\rho U^2/2] A]$$

In this case C_D is a function of Froude Number:

$$Fr = U/\sqrt{[gL]}$$

This is inertia forces divided by gravity forces.

ILLUSTRATION : OSCILLATORY MOTION

Sometimes flows are oscillatory. In this case we need to nondimensionalize the flow period **T** with a reference period T. For a body with characteristic dimension D in a flow with speed U, the reference period is the transit time:

$$T = D/U$$

So the nondimensional period is:

$$C_T = \mathbf{T}/T$$

Vortices are often shed from bodies in an asymmetric pattern. In this case, the transit time divided by the vortex shedding period gives the Strouhal Number:

$$St = T/\mathbf{T}$$

This is just the reciprocal of the period coefficient.