CONSERVATION LAWS FOR A STREAMTUBE

LAGRANGIAN VS EULERIAN FORMULATIONS

The Lagrangian Formulation focuses on a specific group of fluid particles in a flow. It is the most natural way to develop the governing equations but it not very practical from a mathematical point of view because there are just too many groups in a flow to follow. The Eulerian Formulation focuses on a specific region in space. Mathematically this control volume approach is much more practical. Here we start with the Lagrangian Formulation but use the Transport Theorem to switch to the Eulerian Formulation.

CONSERVATION OF MASS

Consider an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain mass ρdV where ρ is the fluid density. Integration over the volume gives the total mass of the group. According to Conservation of Mass, the time rate of change of the mass of the group is zero. Mathematically we can write

$$D/Dt \int_{V(t)} \rho \, dV = 0 \quad .$$

Using the Transport Theorem this can be rewritten as

$$\int_{V(t)} \partial\rho/\partial t \; dV \;+\; \int_{S(t)} \rho \; \mathbf{v.n} \; dS \;=\; 0$$

where **v** is the fluid velocity and **n** is the unit outward normal at points on S. For steady flow in a streamtube with multiple inlets and outlets conservation of mass reduces to

$$\Sigma \; (\rho CA)_{OUT} \;-\; \Sigma \; (\rho CA)_{IN} \;=\; \Sigma \; \dot{M}_{OUT} \;-\; \Sigma \; \dot{M}_{IN} \;=\; 0$$

where C is the flow speed and A is the tube area.

CONSERVATION OF MOMENTUM

Consider again an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain momentum $\rho dV\mathbf{v}$. Integration over V gives the total momentum of the group. According to Conservation of Momentum, the time rate of change of the momentum of the group is equal to the net force acting on it. The forces acting can be of two types: surface forces and body forces. Surface forces in turn can be of two types: pressure and viscous traction. Body forces are generally due only to gravity. Mathematically we can write

$$D/Dt \; \int_{V(t)} \rho\mathbf{v} \; dV \;=\; \int_{S(t)} \boldsymbol{\sigma} \; dS \;+\; \int_{V(t)} \rho\mathbf{b} \; dV$$

where $\boldsymbol{\sigma}$ is a vector representing the stress or force per unit area at any point on the surface S and **b** is a vector representing the body force per unit mass at any point within the volume V. Using the Transport Theorem the integral can be rewritten as

$$\int_{V(t)} \partial \rho \mathbf{v}/\partial t \; dV \; + \; \int_{S(t)} \rho \mathbf{v} \; \mathbf{v}.\mathbf{n} \; dS \; =$$

$$+ \; \int_{S(t)} \boldsymbol{\sigma} \; dS \; + \; \int_{V(t)} \rho \mathbf{b} \; dV \quad .$$

For short streamtubes friction and gravity are often insignificant. In this case for steady flow in a streamtube with multiple inlets and outlets conservation of momentum reduces to

$$\Sigma \; (\rho \mathbf{v} CA)_{OUT} \; - \; \Sigma \; (\rho \mathbf{v} CA)_{IN} \; = \; \Sigma \; (\dot{M} \, \mathbf{v})_{OUT} \; - \; \Sigma \; (\dot{M} \, \mathbf{v})_{IN}$$

$$=$$

$$- \; \Sigma \; (PA\mathbf{n})_{OUT} \; - \; \Sigma \; (PA\mathbf{n})_{IN} \; + \; \mathbf{R}$$

where **R** is the wall force on the fluid in the streamtube.

CONSERVATION OF ENERGY

Consider once more an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain energy edV where e is the fluid energy density. The energy density consists of internal energy and observable kinetic and potential energies:

$$e \; = \; u \; + \mathbf{v}.\mathbf{v}/2 \; + \; gz \quad .$$

Integration over the volume gives the total energy of the group. According to Conservation of Energy, the time rate of change of the energy of the group is equal to rate at which heat flows to the group from the surroundings plus the rate at which the surroundings does work on the group. Mathematically we can write

$$D/Dt \int_{V(t)} \rho e \, dV = - \int_{S(t)} \mathbf{q.n} \, dS + \int_{S(t)} \mathbf{v.\sigma} \, dS \quad .$$

A body force due to gravity work term is not present in this integral because it has already been accounted for as potential energy in energy density. Using the Transport Theorem the integral can be rewritten as

$$\int_{V(t)} \partial \rho e/\partial t \, dV + \int_{S(t)} \rho e \, \mathbf{v.n} \, dS =$$

$$- \int_{S(t)} \mathbf{q.n} \, dS + \int_{S(t)} \mathbf{v.\sigma} \, dS$$

For steady adiabatic isothermal flow in a streamtube with multiple inlets and outlets conservation of energy becomes

$$\Sigma \, [(\rho CA)(C^2/2+gz+P/\rho)]_{OUT} - \Sigma \, [(\rho CA)(C^2/2+gz+P/\rho)]_{IN}$$

$$= \Sigma \, (\dot{M} gh)_{OUT} - \Sigma \, (\dot{M} gh)_{IN} = \dot{T} - \dot{L}$$

where h is the flow head at inlets and outlets

$$h = C^2/2g + P/\rho g + z$$

and $\dot{L}$ accounts for losses and $\dot{T}$ accounts for shaft work.

REYNOLDS TRANSPORT THEOREM

Consider an arbitrary specific group of fluid particles anywhere in a flow and follow it for a short period of time Δt. Let α be any property of the fluid within the group. The Lagrangian rate of change of the integral of α over the volume V of the group is

$$D/Dt \int_{V(t)} \alpha(t)\, dV = \lim_{\Delta t \to 0} \left[\int_{V(t^*)} \alpha(t^*)\, dV - \int_{V(t)} \alpha(t)\, dV \right] / \Delta t$$

where $t^* = t + \Delta t$. Now adding and subtracting the integral of $\alpha(t^*)$ over V(t) inside the [] brackets allows us to rewrite the limit as

$$\lim_{\Delta t \to 0} \left[\int_{V(t)} \alpha(t^*)\, dV - \int_{V(t)} \alpha(t)\, dV \right] / \Delta t$$

$$+ \lim_{\Delta t \to 0} \left[\int_{V(t^*)} \alpha(t^*)\, dV - \int_{V(t)} \alpha(t^*)\, dV \right] / \Delta t \quad .$$

The first limit gives the Eulerian local derivative

$$\int_{V(t)} \partial\alpha/\partial t\, dV \quad .$$

Geometric considerations give $\Delta V = [\mathbf{v}\Delta t].[\mathbf{n}dS]$ where S(t) is the surface which encloses V(t). At any point on this surface $\mathbf{v}$ is the velocity of the fluid and $\mathbf{n}$ is the unit outward normal there. The ΔV equation allows us to replace the second limit with

$$\int_{S(t)} \alpha(t)\ \mathbf{v.n}\ dS\ .$$

So we can replace the original integral as follows

$$D/Dt \int_{V(t)} \alpha(t)\ dV = \int_{V(t)} \partial\alpha/\partial t\ dV + \int_{S(t)} \alpha(t)\ \mathbf{v.n}\ dS\ .$$

This is **Reynolds Transport Theorem.**