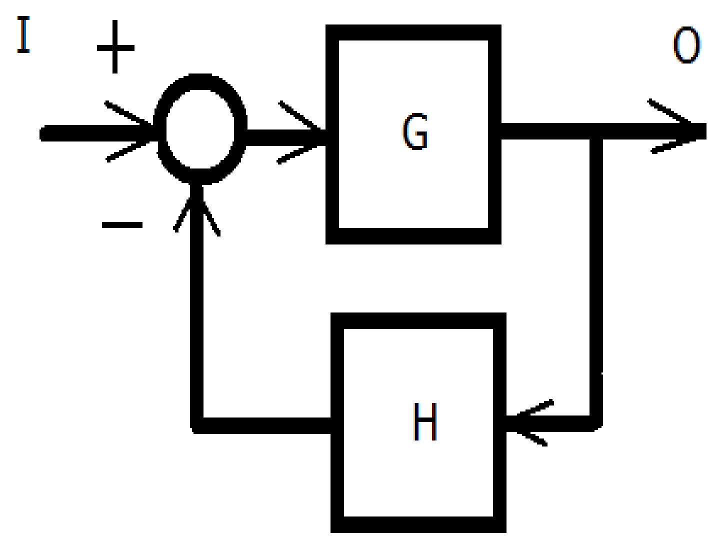
CONTROL SYSTEM STABILITY

The standard form block diagram is



This gives the transfer function

$$O / I = G / [1 + GH]$$

A unit impulse jars a system from a rest state and the motion thereafter is nonforced. It is a good input to test stability. For a unit impulse input I=1 and the response becomes

$$O = \mathbf{H} = G / [1 + GH] = \mathbf{N} / \mathbf{D}$$

where **N** and **D** are polynomials. The characteristic equation is

$$\mathbf{D} = 0$$

Partial Fraction Expansion (PFE) gives

$$O = \sum \Gamma / [S - \lambda]$$

where each λ is a root of the characteristic equation.

Inverse Laplace Transformation (ILT) gives

$$O = \sum \Gamma e^{+\lambda t}$$

The G and H transfer functions can be written as

$$G = A/B \qquad H=X/Y$$

$$GH = A/B\ X/Y = AX/BY = N/D$$

where A B X Y N D are polynomials. In this case

$$O = G / [1+GH] = A/B / [1+A/B\ X/Y]$$
$$= AY / [BY + AX] = \mathbf{N} / \mathbf{D}$$

This shows that

$$\mathbf{D} = N + D$$

The [1+GH] function is:

$$1 + GH = 1+ N/D = [N+D] / D = \mathbf{D} / D$$

Setting **D** equal to 0 gives the overall characteristic equation. Setting D equal to 0 gives the characteristic equation for the sub systems.

The [1+GH] function can be factored to give:

$$\Gamma\ \pi\ [S-Z] / \pi\ [S-P]$$

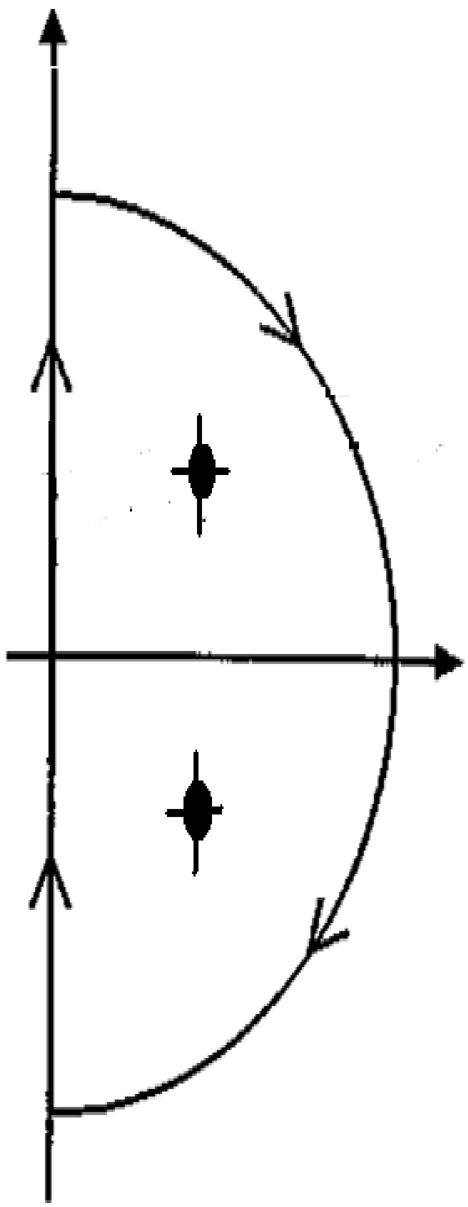
where the symbol π indicates product. Zeros Z are values of S which make [1+GH] zero. Poles P are values of S which make [1+GH] infinite. Note that each [S-Z] factor is basically a vector with its origin at Z. Similarly each [S-P] factor is basically a vector with its origin at P.

NYQUIST CONCEPT

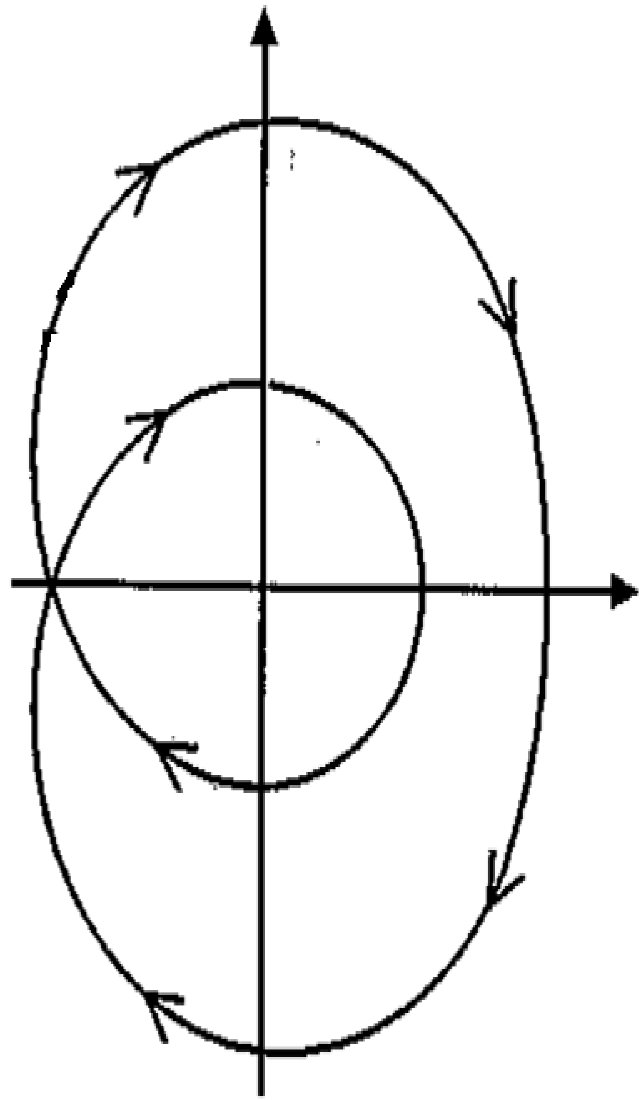
The Nyquist Concept starts by surrounding the entire unstable half of the S plane with a clockwise contour. The [1+GH] function is basically a vector made from zero and pole factors which are also vectors:

$$\Gamma \;\; \pi\,[S-Z] \,/\, \pi\,[S-P] \;=\; \frac{\Gamma\,(S-Z_1)\,(S-Z_2)\;::::: \;(S-Z_n)}{(S-P_1)\,(S-P_2)\;:::::\;(S-P_m)} \;=\; R\angle\Theta$$

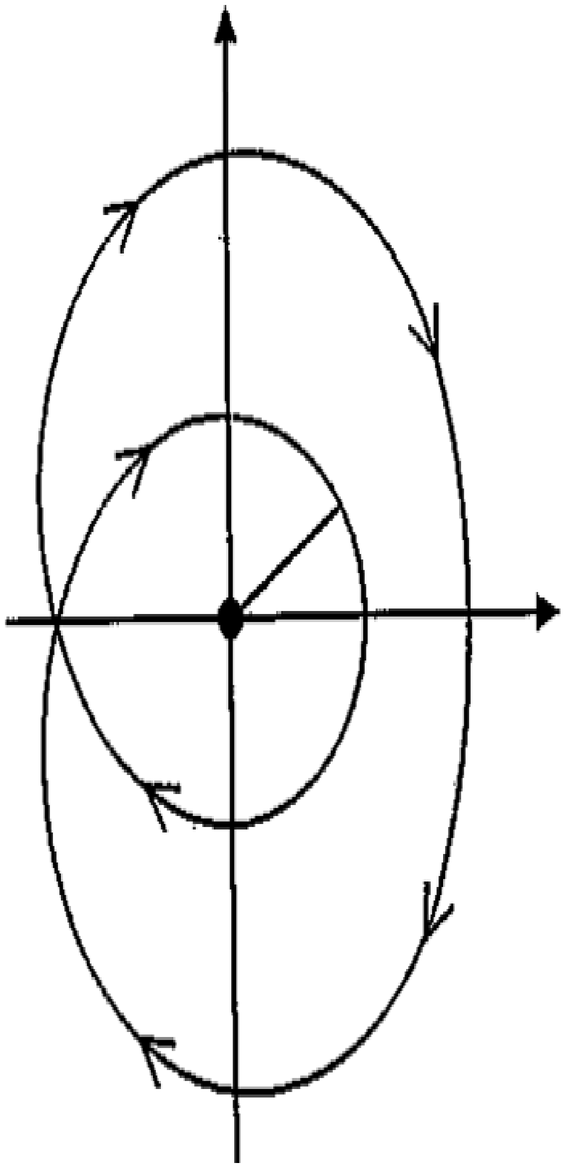
When the tip of the S vector moves clockwise around the Nyquist contour, zeros Z inside it cause clockwise rotations of [1+GH] while poles P inside it cause counter clockwise rotations. Only zeros and poles inside cause such rotations: zeros and poles outside only cause [1+GH] to nod up and down. The sketches on the next page show a complex conjugate pair of roots inside the Nyquist contour and the corresponding [1+GH] plot. As can be seen, the [1+GH] vector rotates twice clockwise as the tip of the S vector moves clockwise around the Nyquist contour. These clockwise rotations are caused by the two unstable zeros inside the contour. Subtracting one from [1+GH] and its origin gives GH and minus one. One can use a GH plot with a radius drawn from minus one to determine rotations.



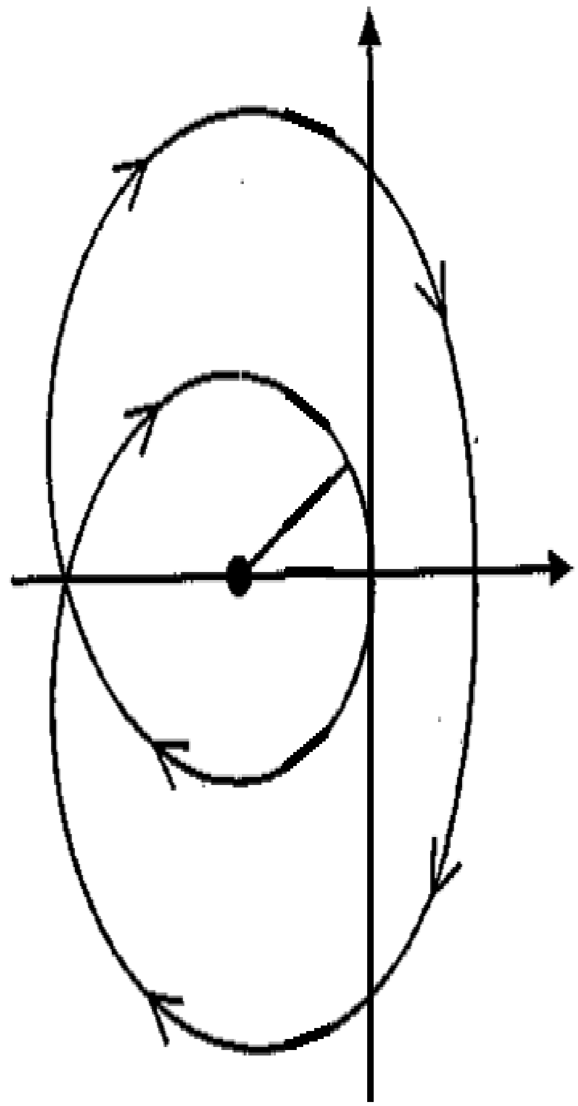
S PLANE



1+GH PLANE



1+GH PLOT



GH PLOT

ROOT LOCUS CONCEPT

When S is a Z or root of the overall characteristic equation, [1+GH] is equal to zero. This implies that GH is equal to minus unity: GH = -1. This means its magnitude is unity and its angle is plus or minus 180 degrees. So any S which satisfies these constraints is a root of the overall characteristic equation. To determine borderline proportional gain and period the angle constraint is used to determine the period and the magnitude constraint is used to determine the gain. Consider the GH function

$$GH = K X [(S-v) (S-w)] / [(S-a) (S-b) (S-c)]$$

Its poles and zeros are shown in the sketch. The location of the square point in the sketch is adjusted to satisfy the angle constraint:

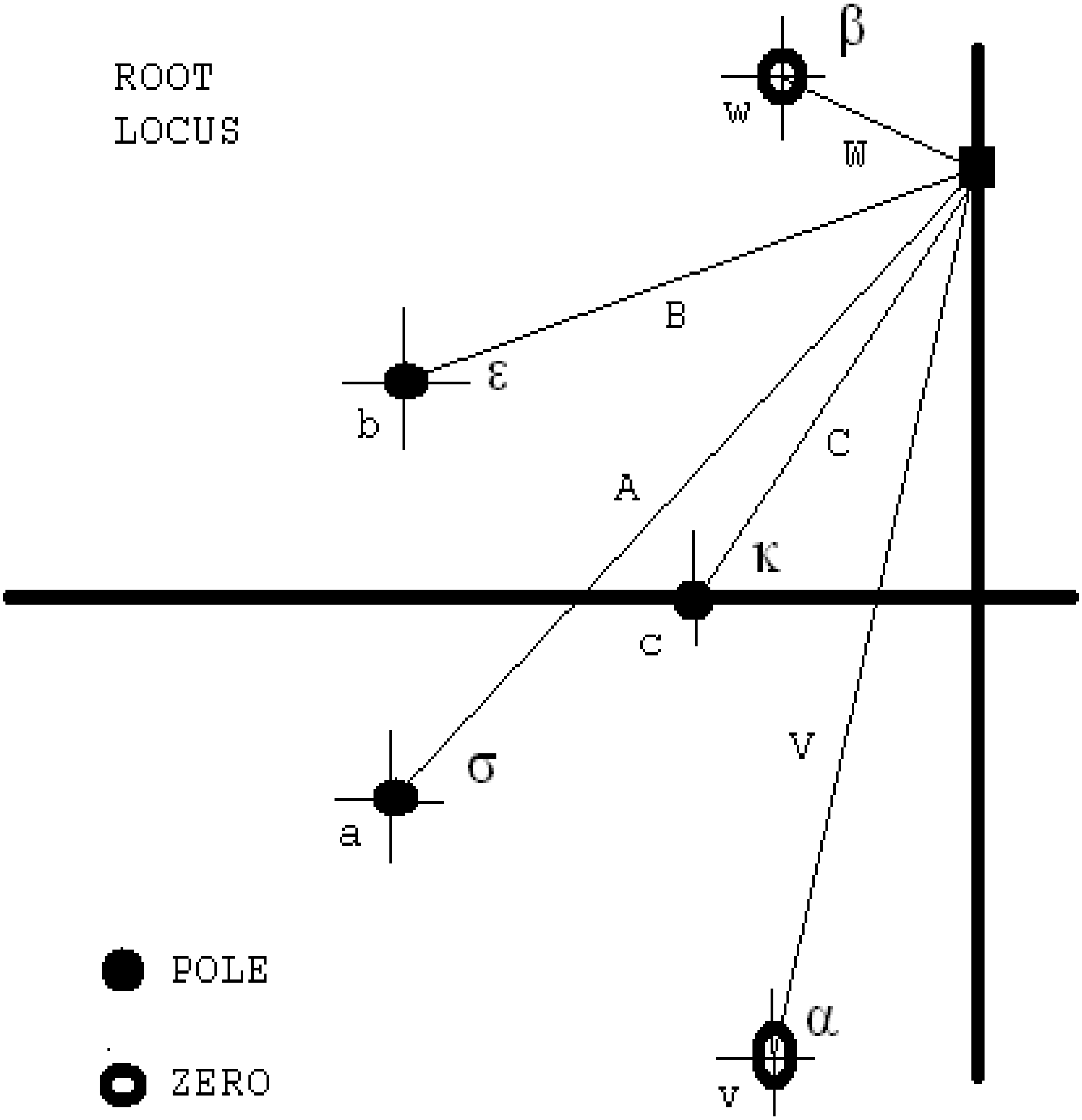
$$\alpha + \beta - \varepsilon - \kappa - \sigma = \pm 180$$

The magnitude constraint requires that

$$\mathbf{K} [X V W] / [A B C] = 1$$

where the lengths V W A B C can be measured. Manipulation gives

$$\mathbf{K} = [A B C] / [X V W]$$


ROOT
LOCUS
β
w
W
B
ε
b
C
A
K
c
V
σ
a
POLE
ZERO
α
v

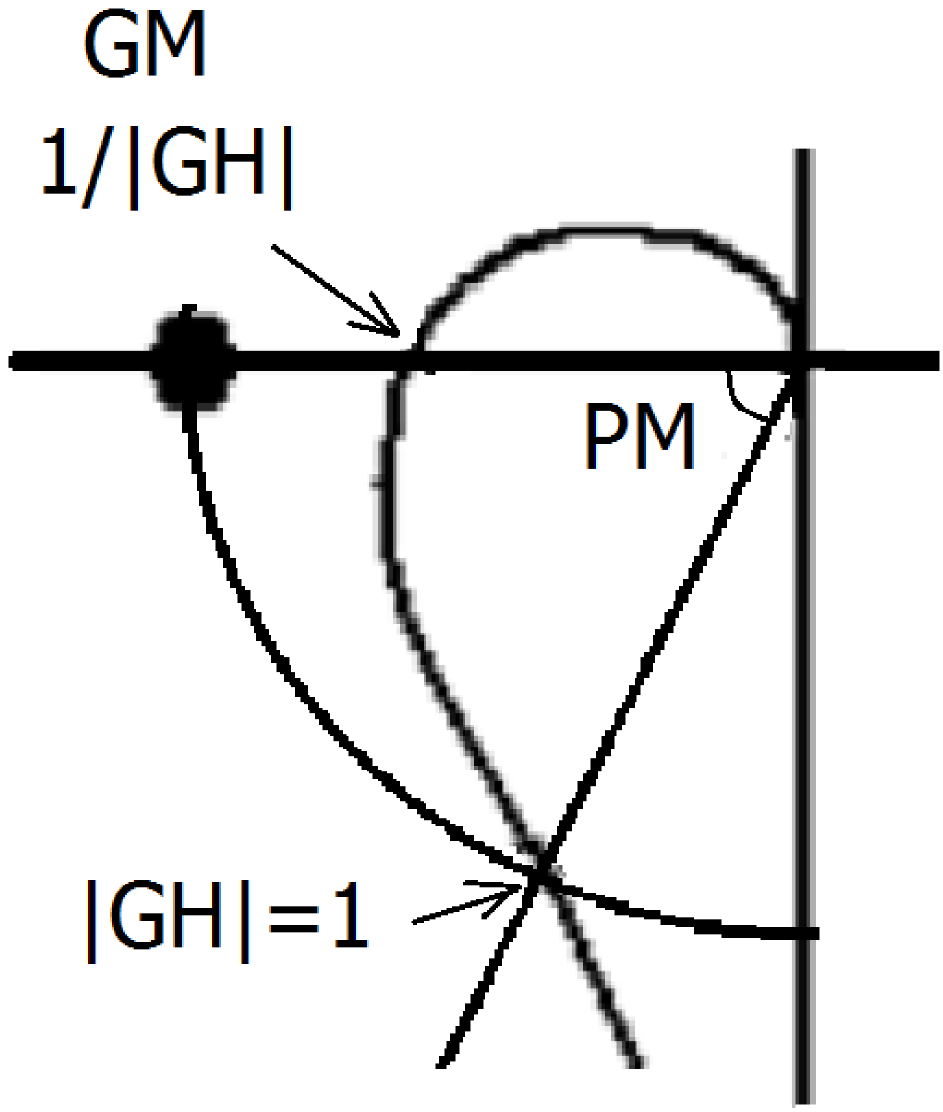
CONTROL SYSTEM DESIGN

STABILITY MARGINS

The degree of stability of a control system depends on how close the GH plot is to the minus one point. Two measures of closeness are the gain margin GM and the phase margin PM. Engineering experience suggests that GM should be at least 2 and PM should be at least 30 degrees.

WEDGE CIRCLE REGION

Most systems have a dominant pair of roots which control how stable it is. Theory shows that the damping factor associated with these roots is constant along radial lines drawn from the origin in the S plane while the undamped natural frequency is constant along semi circles with center at the origin of the S plane. The wedge circle region is where roots should be located to get good damping and speed of response.


GM
1/|GH|
PM
|GH|=1

S PLANE