JOUKOWSKY FOIL ILLUSTRATION

This note gives an illustration of how to map points in a circle plane to a foil plane using the Joukowsky mapping function. It also shows how to calculate the pressure at the points on the foil and the lift on the foil.

The circle to be mapped has a radius R equal to 1.0m: its n offset is 0.2m and its m offset is 0.4m. The foil is moving through water with density ρ equal 1000kg/m^3. It has a speed S equal to 10m/s and an angle of attack Θ equal to 10°.

The mapping from the circle plane to the foil plane makes use of 3 circle plane coordinate systems. These are shown in the sketch on the next page. The foil coordinates in terms of circle coordinates are

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

Geometry gives

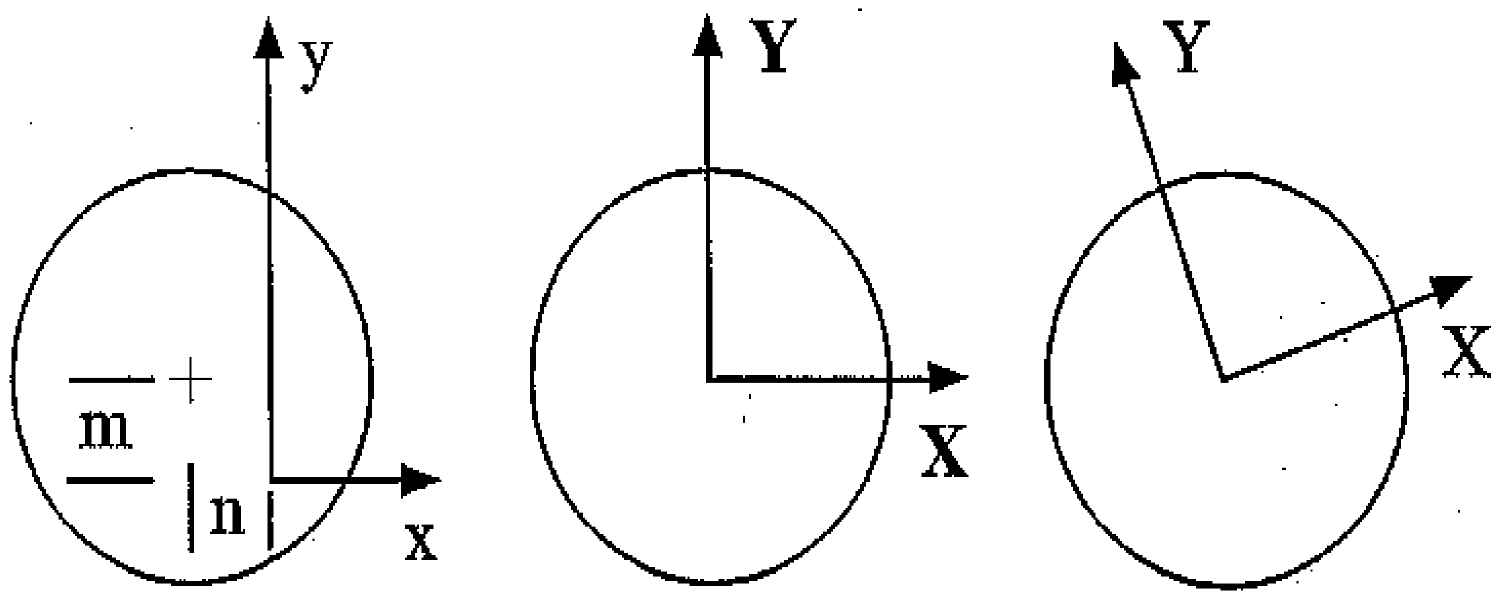
$$X = \mathbf{X}\ \mathrm{Cos}\Theta + \mathbf{Y}\ \mathrm{Sin}\Theta \qquad Y = \mathbf{Y}\ \mathrm{Cos}\Theta - \mathbf{X}\ \mathrm{Sin}\Theta$$

$$\mathbf{X} = x + n \qquad \mathbf{Y} = y - m$$

$$\mathbf{X} = -\ R\ \mathrm{Cos}\Upsilon \qquad \mathbf{Y} = +\ R\ \mathrm{Sin}\Upsilon$$

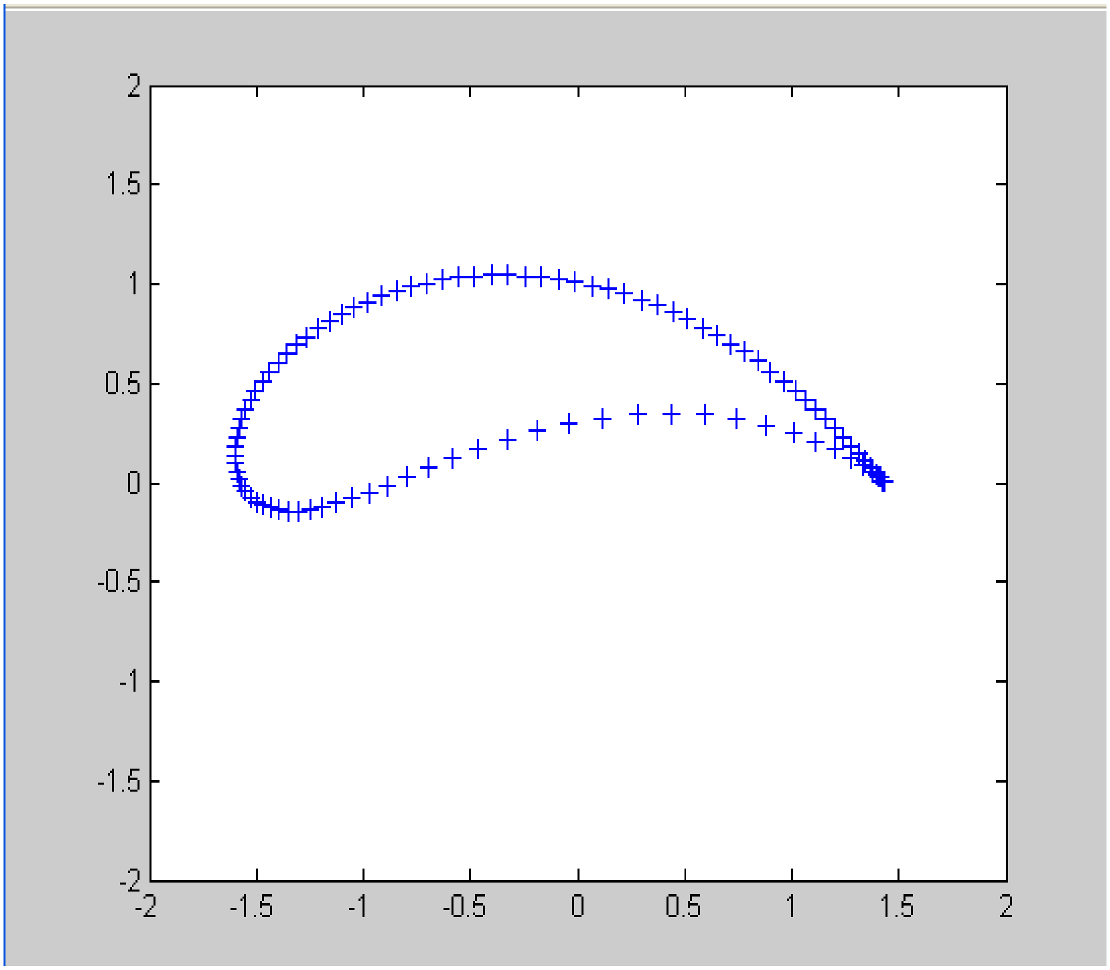
$$a = \sqrt{[R^2-m^2]} - n$$

where Θ is the angle of attack of the foil and Υ is a clockwise angle over the circle from front to back.



$z = x + yj$ $\qquad$ $Z = X + Yj$ $\qquad$ $Z = X + Yj$

The complete foil geometry is shown below. Here only 5 points on the circle will be mapped to foil plane.



Geometry shows that

$$[a + n]^2 + m^2 = R^2$$

$$a = \sqrt{[R^2 - m^2]} - n$$

Substitution into this gives a=0.72m.

For point #1 Υ=0. Substitution into

$$\mathbf{X} = - R \cos\Upsilon \qquad \mathbf{Y} = + R \sin\Upsilon$$

gives $\mathbf{X} = -1.0$ and $\mathbf{Y} = 0.0$. Substitution into

$$\mathbf{X} - n = x \qquad \mathbf{Y} + m = y$$

gives $x = -1.2$ and $y = +0.4$. Substitution into

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

gives $\alpha = -1.59$ and $\beta = +0.27$.

For point #2 Υ=90. Substitution into

$$\mathbf{X} = - R \cos\Upsilon \qquad \mathbf{Y} = + R \sin\Upsilon$$

gives $\mathbf{X} = 0.0$ and $\mathbf{Y} = +1.0$. Substitution into

$$\mathbf{X} - n = x \qquad \mathbf{Y} + m = y$$

gives $x = -0.2$ and $y = +1.4$. Substitution into

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

gives $\alpha = -0.25$ and $\beta = +1.04$.

For point #3 Υ=180. Substitution into

$$\mathbf{X} = -R\,\mathrm{Cos}\Upsilon \qquad \mathbf{Y} = +R\,\mathrm{Sin}\Upsilon$$

gives $\mathbf{X} = +1.0$ and $\mathbf{Y} = 0.0$. Substitution into

$$\mathbf{X} - n = x \qquad \mathbf{Y} + m = y$$

gives $x = +0.8$ and $y = +0.4$. Substitution into

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

gives $\alpha = +1.31$ and $\beta = +0.14$.

For point #4 Υ=270. Substitution into

$$\mathbf{X} = -R\,\mathrm{Cos}\Upsilon \qquad \mathbf{Y} = +R\,\mathrm{Sin}\Upsilon$$

gives $\mathbf{X} = 0.0$ and $\mathbf{Y} = -1.0$. Substitution into

$$\mathbf{X} - n = x \qquad \mathbf{Y} + m = y$$

gives $x = -0.2$ and $y = -0.6$. Substitution into

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

gives $\alpha = -0.46$ and $\beta = +0.17$.

The point where the x axis hits the circle in the circle plane maps to the trailing edge of the foil in the foil plane. For this point inspection of the geometry equations shows that x=+a and y=0.0. Substitution into

$$\alpha = x + xa^2/(x^2+y^2) \qquad \beta = y - ya^2/(x^2+y^2)$$

gives $\alpha = +1.44$ and $\beta = 0.0$. It is left as an exercise to plot the 5 points onto the complete foil sketch.

Superposition of a stream and a doublet and a vortex generates the circle plane flow. On the circle the potential function due to this superposition is

$$\varphi = 2\ S\ X + \Gamma/[2\pi]\ \sigma$$

where S is the stream speed, Γ is the vortex strength and σ is a clockwise angle over the circle. It turns out that when points are mapped to the foil plane they carry with them their circle plane potential function values.

To make the flow look realistic around the foil, the trailing edge must be a stagnation point. It must also be a stagnation point at the corresponding point in the circle plane. Setting the speed $\partial\varphi/\partial c$ to zero there in the circle plane gives $\Gamma = 4\pi SR\ \mathrm{Sin}\kappa$ where $\kappa = \Theta+\varepsilon$ where $\varepsilon = \tan^{-1}[m/(n+a)]$. Substitution into these equations gives:

$$\varepsilon = \tan^{-1}[m/(n+a)] = 23$$

$$\Gamma = 4\pi SR\ \mathrm{Sin}[\Theta+\varepsilon] = 70$$

The theoretical lift on the foil is:

$$\rho\Gamma S = 1000*70*10 = 700000 \text{ Newtons}$$

An estimate of lift can also be obtained numerically using pressures at points on the foil. The Bernoulli equation gives for pressure on the foil:

$$\rho/2\ [\ S^2 - (\partial\varphi/\partial c)^2\ ]$$

Midway between mapped points this gives:

$$\rho/2\ [\ S^2 - (\Delta\varphi/\Delta c)^2\ ]$$

where $\Delta c = \sqrt{[\Delta\alpha^2+\Delta\beta^2]}$. In the circle plane

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma$$

Here $\Delta\Upsilon$=90. In this case $\Delta\sigma=\pi/2$. One gets ΔX from

$$X = \mathbf{X}\ \text{Cos}\Theta + \mathbf{Y}\ \text{Sin}\Theta$$

The incremental lift and drag are:

$$P\Delta c\ \text{Sin}(\boldsymbol{\theta}-\Theta) \qquad P\Delta c\ \text{Cos}(\boldsymbol{\theta}-\Theta)$$

where $\boldsymbol{\theta}$ is the foil normal. The foil normal is

$$\boldsymbol{\theta} = \tan^{-1} [-\Delta\alpha/+\Delta\beta]$$

Summation gives the total lift and drag:

$$L = \Sigma\Delta L \qquad D = \Sigma\Delta D$$

Midway between points 1 and 2

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma = +40.5$$

$$\Delta c = \sqrt{[\Delta\alpha^2+\Delta\beta^2]} = 1.65$$

The pressure is thus

$$\rho/2\ [\ S^2 - (\Delta\varphi/\Delta c)^2\ ] = -251000 \text{ Pascals}$$

The foil normal is

$$\boldsymbol{\theta} = \tan^{-1} [-\Delta\alpha/+\Delta\beta] = 300$$

The incremental lift is

$$P\Delta c\ \mathrm{Sin}(\boldsymbol{\theta}-\Theta) = +401848 \text{ Newtons}$$

Midway between points 2 and 3

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma = +33.7$$

$$\Delta c = \sqrt{[\Delta\alpha^2+\Delta\beta^2]} = 1.81$$

The pressure is thus

$$\rho/2 \ [\ S^2 - (\Delta\varphi/\Delta c)^2\ ] = -123508 \text{ Pascals}$$

The foil normal is

$$\boldsymbol{\theta} = \tan^{-1}[-\Delta\alpha/+\Delta\beta] = 240$$

The incremental lift is

$$P\Delta c \ \mathrm{Sin}(\boldsymbol{\theta}-\Theta) = +223535 \text{ Newtons}$$

Midway between points 3 and 4

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma = -5.5$$

$$\Delta c = \sqrt{[\Delta\alpha^2+\Delta\beta^2]} = 1.77$$

The pressure is thus

$$\rho/2 \ [\ S^2 - (\Delta\varphi/\Delta c)^2\ ] = +47350 \text{ Pascals}$$

The foil normal is

$$\boldsymbol{\theta} = \tan^{-1}[-\Delta\alpha/+\Delta\beta] = 89$$

The incremental lift is

$$P\Delta c \ \mathrm{Sin}(\boldsymbol{\theta}-\Theta) = +78755 \text{ Newtons}$$

Midway between points 4 and 1

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma = +1.3$$

$$\Delta c = \sqrt{[\Delta\alpha^2+\Delta\beta^2]} = 1.24$$

The pressure is thus

$$\rho/2\ [\ S^2 - (\Delta\varphi/\Delta c)^2\ ] = +49450 \text{ Pascals}$$

The foil normal is

$$\boldsymbol{\theta} = \tan^{-1}\ [-\Delta\alpha/+\Delta\beta] = 85$$

The incremental lift is

$$P\Delta c\ \mathrm{Sin}(\boldsymbol{\theta}-\Theta) = +55573 \text{ Newtons}$$

The total lift is

$$401848 + 223535 + 78755 + 55573$$
$$= 759711 \text{ Newtons}$$

This is close to the theoretical lift 700000 Newtons.