POTENTIAL FLOWS

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IDEAL FLOWS

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PREAMBLE

Potential flow is based on two major assumptions. First, the fluid is taken to be inviscid, which means it has zero viscosity. Second, the fluid motion is taken to be irrotational, which means each fluid particle does not spin on its own internal axis. Particles move through space like the carts on a Ferris Wheel. The fluid is said to be ideal. It can be compressible or incompressible. For hydrodynamic flows, the fluid can be taken to be incompressible.

When a body moves at steady speed through an ideal fluid, theory shows that the net load acting on the body is zero. This includes bodies that in reality have lift and drag forces acting on them. This is known as D'Alemberts Paradox. So it appears that ideal fluid theory is of little practical value.

POTENTIAL FLOW EQUATIONS

The derivation of the equations for potential or ideal flow start with the conservation laws of mass and momentum for an incompressible fluid.

$$\nabla.\mathbf{v} = 0$$

$$\rho\partial\mathbf{v}/\partial t + \rho\ \mathbf{v}.\nabla\ \mathbf{v} + \nabla P + \nabla\rho gz - \mu\nabla^2\mathbf{v} = 0$$

where $\mathbf{v} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$ is the velocity vector, **ijk** are unit vectors in xyz, ρ is density, P is pressure, μ is viscosity and g is gravity.

For ideal flows, we assume that the fluid has zero viscosity or is inviscid. With this assumption, the conservation laws become

$$\nabla.\mathbf{v} = 0$$

$$\rho\partial\mathbf{v}/\partial t + \rho\ \mathbf{v}.\nabla\ \mathbf{v} + \nabla P + \nabla\rho gz = 0$$

For ideal fluid flows, we also assume that fluid motion is irrotational. This means that fluid particles do not spin on internal axes: mathematically this means that the spin vector $\mathbf{\Omega}$ is zero.

One can show that the spin vector $\mathbf{\Omega}$ is half the vorticity vector $\boldsymbol{\omega}$. So, for an irrotational flow, the vorticity vector is zero. One can write this as:

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega} = \nabla \times \mathbf{v} = 0$$

For any scalar φ, one can show that

$$\nabla \times \nabla \varphi = 0$$

This suggests that for an irrotational flow

$$\mathbf{v} = \nabla \varphi$$

Substitution $\mathbf{v} = \nabla \varphi$ into the conservation of mass equation gives after some manipulation:

$$\nabla . \nabla \varphi = \nabla^2 \varphi = 0$$

A vector identity shows that

$$\mathbf{v}.\nabla \mathbf{v} = \nabla(\mathbf{v}.\mathbf{v}/2) - \mathbf{v} \times \boldsymbol{\omega}$$

With this conservation of momentum becomes

$$\rho \partial \mathbf{v}/\partial t + \rho \nabla(\mathbf{v}.\mathbf{v}/2) + \nabla P + \nabla \rho g z = 0$$

Substitution $\mathbf{v} = \nabla \varphi$ into the conservation of momentum equation gives after some manipulation:

$$\partial \varphi/\partial t + (\nabla \varphi . \nabla \varphi)/2 + P/\rho + gz = C$$

The primitive variable equations

$$\nabla . \mathbf{v} = 0$$

$$\rho \partial \mathbf{v} / \partial t + \rho \, \mathbf{v} . \nabla \, \mathbf{v} + \nabla P + \nabla \rho g z = 0$$

where

$$\mathbf{v} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$$

have become

$$\nabla^2 \varphi = 0$$

$$\partial \varphi / \partial t + (\nabla \varphi . \nabla \varphi)/2 + P/\rho + gz = C$$

where

$$\mathbf{v} = \nabla \varphi$$

The last equation implies

$$U = \partial \varphi / \partial x \qquad V = \partial \varphi / \partial y \qquad W = \partial \varphi / \partial z$$

SUPERPOSITION OF ELEMENTAL FLOWS

Potential or ideal flows around bodies are usually obtained by superposition of certain basic or elemental flows. Superposition produces in the flow a stream surface that separates inner and outer flows. The stream surface mimics a thin shell body in the flow that deflects inner and outer flows. We are usually interested in the outer flow. The most elemental flow is a stream. This is usually uniform, meaning that all fluid particles are moving in the same direction at the same speed. Another elemental flow is a source. Here all fluid particles are moving outwards from a center. The center is a line in 2D and a point in 3D. At the center the fluid is moving at infinite speed! The inverse of a source is a sink. Here all fluid particles are moving inwards to a center. Superposition of a strong source and a strong sink of equal strength very close together produces the elemental flow known as a doublet. The final elemental flow is known as a potential vortex. Here all fluid particles are moving along circular streamlines. The speed of the particles is inversely proportional to the streamline radius, so particles at the center of the vortex move at infinite speed! Points in a flow where fluid particles are moving at infinite speed are known as singularities. Such points do not exist in reality!

DRAG ON CYLINDERS

Superposition of a 2D stream and a 2D doublet gives approximately the flow pattern around a cylinder. The potential function for the flow is

$$\varphi = S\ X + S\ XR^2/[X^2+Y^2]$$

where R is the cylinder radius and S is the stream speed. To calculate the loads on the cylinder, we need the pressure at points on it. The Bernoulli Equation applied between points on the cylinder and a point in the flow well upstream gives pressure. For this, we need the speed of the fluid over the cylinder. On the cylinder, where $\sqrt{[X^2+Y^2]}$ is equal to R, the potential function reduces to

$$\varphi = 2\ S\ X$$

On the cylinder, geometry gives

$$X = -\ R\ \text{Cos}\sigma$$

where σ is a clockwise angle over the cylinder. The circumferential distance over the cylinder is $c=R\sigma$. This allows us to rewrite the potential function as

$$\varphi = -2\ S\ R\ \text{Cos}[c/R]$$

The speed of the fluid over the cylinder is

$$\partial\varphi/\partial c = 2\ S\ \text{Sin}\sigma$$

Application of Bernoulli gives

$$P/\rho \; + \; (\partial\varphi/\partial c)^2/2 \; = \; S^2/2$$

Manipulation gives

$$P = \rho/2 \; [\; S^2 - (\partial\varphi/\partial c)^2 \;]$$

$$P = \rho/2 \; S^2 \; [\; 1 - 4 \; \mathrm{Sin}^2\sigma \;]$$

This is only good up to separation. In the wake downstream, the pressure is approximately constant and is approximately equal to the pressure where the boundary layer separates. At a point on the cylinder, pressure acts over the incremental area : $dA = R d\sigma$.

This gives the incremental force :

$$dF = P \; dA = P \; R \; d\sigma$$

and the following incremental drag and lift forces

$$dD = + \; dF \; \mathrm{Cos}\sigma$$

$$dL = - \; dF \; \mathrm{Sin}\sigma$$

Integration gives the total drag and lift.

DRAG ON SPHERES

Superposition of a 3D stream with a 3D doublet produces the ideal flow around a sphere. The potential function is

$$\varphi = - S\ r\ \mathrm{Cos}\sigma \ - \ S/2\ R^3/r^2\ \mathrm{Cos}\sigma$$

where R is the radius of the sphere, r is a distance out from the center of the sphere, S is the stream speed and σ is a clockwise angle over the sphere. To calculate the loads on the sphere, we need the pressure at points on it. For this, we need flow speed. On the sphere, where r=R and c=Rσ, the potential function reduces to

$$\varphi = - 3/2\ S\ R\ \mathrm{Cos}[c/R]$$

The speed of the fluid over the sphere is

$$\partial\varphi/\partial c = 3/2\ S\ \mathrm{Sin}\sigma$$

Application of Bernoulli gives

$$P/\rho \ + \ (\partial\varphi/\partial c)^2/2 \ = \ S^2/2$$

Manipulation gives

$$P = \rho/2 \; [\; S^2 - (\partial\varphi/\partial c)^2 \;]$$

$$P = \rho/2 \; S^2 \; [\; 1 - 9/4 \; \mathrm{Sin}^2\boldsymbol{\sigma} \;]$$

As it was for the cylinder, this is only good up to separation. At a point on the sphere, pressure acts over the incremental area

$$dA = R \; d\sigma \; \mathbf{R} \; d\Theta \qquad \mathbf{R} = R \; \mathrm{Sin}\sigma$$

This gives the incremental force

$$dF = P \; dA = P \; R^2 \; \mathrm{Sin}\sigma \; d\sigma \; d\Theta$$

and the following incremental drag and lift forces

$$dD = dF \; \mathrm{Cos}\sigma$$

$$dL = - \; dF \; \mathrm{Sin}\sigma \; \mathrm{Cos}\Theta$$

Integration gives the total drag and lift.

LIFTING BODIES

Ideal fluid theory predicts that for a body shaped like a foil the fluid is able to turn the sharp corner at the trailing edge and move back over the top of the foil to join with fluid that moved around the leading edge and over the top. The two bits of fluid would pass through two stagnation or zero velocity points: one on the bottom and one on the top. In reality, the fluid cannot turn the sharp corner at the rear. The fluid has to undergo infinite deceleration and acceleration to turn such a corner. Associated with this is an infinite suction pressure. As a real fluid tries to moves away from this into a higher pressure region on top of the foil, it moves inside a boundary layer. Within it, energy is taken from the fluid by viscous drag forces. The low to high pressure is known as an adverse pressure gradient. It turns out that fluid in a boundary layer would not be able to move into such a strong gradient and would be stopped at the trailing edge. The fluid is said to separate. The trailing edge becomes a stagnation point and a separation point. The fluid can be seen to leave the trailing edge smoothly. It turns out that the loads on the foil in this case are not zero. Note that this happens because of the behavior of a boundary layer, which is a mainly viscous phenomenon. This suggests that without viscosity wings would not work and present day airplanes would not be able to fly!

One can use a potential vortex to force the ideal flow over a foil to mimic a real flow. The vortex drags the stagnation point normally on top of the foil back to the trailing edge. When this is done, loads are no longer zero.

Superposition of a 2D stream and a 2D doublet with a potential line vortex gives approximately the flow pattern around a spinning cylinder. The potential function for the flow is

$$\varphi = S\ X + S\ XR^2/[X^2+Y^2] \ + \ \Gamma/[2\pi]\ \sigma$$

where R is the cylinder radius, S is the stream speed, Γ is the vortex strength and σ is a clockwise angle over the cylinder. On the circle this becomes

$$\varphi = 2\ S\ X + \ \Gamma/[2\pi]\ \sigma$$

It turns out that the flow around the cylinder can be mapped into flow around a foil shape. The foil coordinates in terms of circle coordinates are

$$\alpha = x + xa^2/(x^2+y^2)$$

$$\beta = y - ya^2/(x^2+y^2)$$

Geometry gives

$$X = \mathbf{X} \cos\Theta + \mathbf{Y} \sin\Theta$$

$$Y = \mathbf{Y} \cos\Theta - \mathbf{X} \sin\Theta$$

$$\mathbf{X} = x + n \qquad \mathbf{Y} = y - m$$

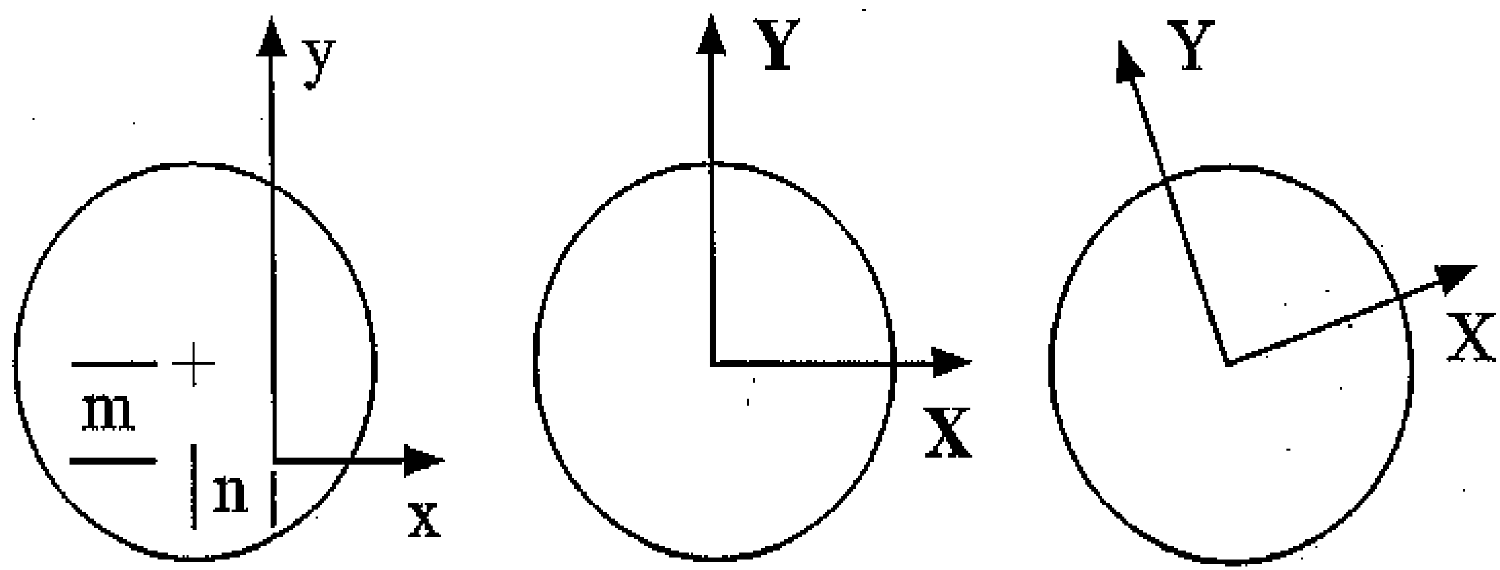
$$a = \sqrt{[R^2 - m^2]} - n$$

where Θ is the angle of attack of the foil and n and m are offsets. To make the flow look realistic around the foil, the trailing edge must be a stagnation point. It turns out that the point where the x axis hits the circle in the circle plane maps to the trailing edge of the foil in the foil plane, and this point is a stagnation point in both planes. Setting the speed to zero there in the circle plane shows that the circulation must be:

$$\Gamma = 4\pi SR \sin\kappa$$

$$\kappa = \Theta + \varepsilon \qquad \varepsilon = \tan^{-1} [m/(n+a)]$$

One can show that the theoretical lift on the foil is $\rho S\Gamma$ while the theoretical drag is zero. Note that lift is zero when the vortex strength is zero. The vortex mimics viscosity. One can also estimate the lift and drag numerically. The Bernoulli equation gives for pressure at any point on the foil:


y
x
m
n
Y
X
Y
X

$$\rho/2 \; [\; S^2 - (\partial\varphi/\partial c)^2 \;]$$

A finite difference approximation is:

$$\rho/2 \; [\; S^2 - (\Delta\varphi/\Delta c)^2 \;]$$

where

$$\Delta\varphi = 2\ S\ \Delta X + \Gamma/[2\pi]\ \Delta\sigma$$

$$\Delta X = \Delta\mathbf{X}\ \mathrm{Cos}\Theta + \Delta\mathbf{Y}\ \mathrm{Sin}\Theta$$

$$\Delta c = \sqrt{[\Delta\alpha^2 + \Delta\beta^2]}$$

The incremental lift and drag are:

$$P\Delta c\ \mathrm{Sin}(\boldsymbol{\theta}-\Theta)$$

$$P\Delta c\ \mathrm{Cos}(\boldsymbol{\theta}-\Theta)$$

where $\boldsymbol{\theta}$ is the foil normal. Summation gives

$$L = \Sigma\Delta L$$

$$D = \Sigma\Delta D$$