CONSERVATION

LAWS

HINCHEY

CONSERVATION LAWS FOR A POINT IN A FLOW

LAGRANGIAN VS EULERIAN FORMULATIONS

The Lagrangian Formulation focuses on a specific group of fluid particles in a flow. It is the most natural way to develop the governing equations but it not very practical from a mathematical point of view because there are just too many groups in a flow to follow. The Eulerian Formulation focuses on a specific region in space. Mathematically this control volume approach is much more practical. Here we start with the Lagrangian Formulation but use the Transport Theorem to switch to the Eulerian Formulation. For the derivations we assume that fluid is continuous. This means that no matter how much we zoom in on a bit of fluid we do not see any molecules. We also assume that the fluid is Newtonian.

CONSERVATION OF MASS

Consider an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain mass ρdV where ρ is the fluid density. Integration over the volume gives the total mass of the group. According to Conservation of Mass, the time rate of change of the mass of the group is zero. Mathematically we can write

$$D/Dt \int_V \rho \, dV = 0$$

The Transport Theorem allows us to rewrite the integral as

$$\int_V [\ \partial\rho/\partial t + \nabla.(\rho\mathbf{v})\ ]\ dV = 0$$

where **v** is the fluid velocity and ∇ is the del operator. For an arbitrary bit of mass, the integrand inside the square brackets must be zero. Setting it to zero gives:

$$\partial\rho/\partial t + \nabla.(\rho\mathbf{v}) = 0$$

With **v** = Ui + Vj + Wk, this becomes

$$\partial\rho/\partial t + \partial(\rho U)/\partial x + \partial(\rho V)/\partial y + \partial(\rho W)/\partial z = 0$$

$$\partial\rho/\partial t + \rho\partial U/\partial x + \rho\partial V/\partial y + \rho\partial W/\partial z$$
$$+ U\partial\rho/\partial x + V\partial\rho/\partial y + W\partial\rho/\partial z = 0$$

For incompressible flow, it reduces to

$$\partial U/\partial x + \partial V/\partial y + \partial W/\partial z = 0$$

Mathematically velocity is divergence free. The Conservation of Mass equation is often called the Continuity equation.

CONSERVATION OF MOMENTUM

Consider again an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain momentum ρdV**v**. Integration over V gives the total momentum of the group. According to Conservation of

Momentum, the time rate of change of the momentum of the group is equal to the net force acting on it. The forces acting can be of two types: surface forces and body forces. Surface forces in turn can be of two types: pressure and viscous traction. Body forces are generally due only to gravity. Mathematically we can write

$$D/Dt \int_V \rho \mathbf{v}\, dV = \int_S \boldsymbol{\sigma}\, dS + \int_V \rho \mathbf{b}\, dV$$

where $\boldsymbol{\sigma}$ is a vector representing the stress or force per unit area at any point on the surface S and $\mathbf{b}$ is a vector representing the body force per unit mass at any point within the volume V. The Transport Theorem allows us to rewrite the integral as

$$\int_V [\partial(\rho \mathbf{v})/\partial t + \nabla.(\rho \mathbf{v}\mathbf{v})]\, dV = \int_S \boldsymbol{\sigma}\, dS + \int_V \rho \mathbf{b}\, dV$$

The surface integral can be converted to a volume integral. Once this is done, all the volume integrals can be combined into one. For an arbitrary bit of mass, the integrand of this single integral must be zero. Setting the integrand to zero gives:

$$X\, i + Y\, j + Z\, k = 0$$

$$X = 0 \quad Y = 0 \quad Z = 0$$

The details of this manipulation are beyond the scope of this note. The X = 0 equation gives x momentum equation:

$$\rho\ (\partial U/\partial t + U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z) = -\ \partial P/\partial x$$

$$+\ \partial/\partial x\ (\lambda\ [\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x\ (\mu\ [\partial U/\partial x + \partial U/\partial x])$$

$$+\ \partial/\partial y\ (\mu\ [\partial V/\partial x + \partial U/\partial y]) + \partial/\partial z\ (\mu\ [\partial W/\partial x + \partial U/\partial z])$$

The Y = 0 equation gives y momentum equation:

$$\rho\ (\partial V/\partial t + U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z) = - \partial P/\partial y$$

$$+ \partial/\partial y\ (\lambda\ [\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x\ (\mu\ [\partial V/\partial x + \partial U/\partial y])$$

$$+ \partial/\partial y\ (\mu\ [\partial V/\partial y + \partial V/\partial y]) + \partial/\partial z\ (\mu\ [\partial W/\partial y + \partial V/\partial z])$$

The Z = 0 equation gives z momentum equation:

$$\rho\ (\partial W/\partial t + U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z) = - \partial P/\partial z - \rho g$$

$$+ \partial/\partial z\ (\lambda\ [\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x\ (\mu\ [\partial W/\partial x + \partial U/\partial z])$$

$$+ \partial/\partial y\ (\mu\ [\partial V/\partial z + \partial W/\partial y]) + \partial/\partial z\ (\mu\ [\partial W/\partial z + \partial W/\partial z])$$

Stokes' Hypothesis states that $\lambda=-2/3\mu$. For an incompressible fluid with constant viscosity, these equations reduce to:

X Momentum

$$\rho\ (\partial U/\partial t + U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z) = - \partial P/\partial x$$

$$+ \mu\ (\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 + \partial^2 U/\partial z^2)$$

Y Momentum

$$\rho\ (\partial V/\partial t + U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z) = - \partial P/\partial y$$

$$+ \mu\ (\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2)$$

Z Momentum

$$\rho\ (\partial W/\partial t + U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z) = - \partial P/\partial z - \rho g$$

$$+ \mu\ (\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 + \partial^2 W/\partial z^2)$$

These equations are often called the Navier Stokes equations.

CONSERVATION OF ENERGY

Consider once more an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain energy edV where e is the fluid energy density. The energy density consists of internal energy and observable kinetic and potential energies:

$$e = u + \mathbf{v}.\mathbf{v}/2 + gz$$

Integration over the volume gives the total energy of the group. According to Conservation of Energy, the time rate of change of the energy of the group is equal to rate at which heat flows to the group from the surroundings plus the rate at which the surroundings does work on the group. Assuming there is no internal heat generation, mathematically we can write

$$D/Dt \int_V \rho e \; dV = - \int_S \mathbf{q}.\mathbf{n} \; dS + \int_S \mathbf{v}.\boldsymbol{\sigma} \; dS$$

where **q** is the heat flux vector and **n** is the unit outward normal at points on S. Here we assume that **q** is due to conduction only: radiation is ignored. A gravity body force work term is not present above because it has already been accounted for as potential energy in energy density. We could remove potential energy from energy density and add a body force work term after the **v.σ** equation:

$$\int_V \mathbf{v}.\rho\mathbf{b} \; dV$$

One can show that both approaches give the same contribution to the conservation of energy equation. The Transport Theorem allows us to rewrite the energy integral as:

$$\int_V [\partial(\rho e)/\partial t + \nabla.(\rho e\mathbf{v})]\ dV = -\int_S \mathbf{q}.\mathbf{n}\ dS + \int_S \mathbf{v}.\boldsymbol{\sigma}\ dS$$

The surface integrals can be converted to volume integrals. Once this is done, all the volume integrals can be combined into one. For an arbitrary bit of mass, the integrand of this single integral must be zero. Setting the integrand to zero gives:

$$\rho\ C_v\ (\partial T/\partial t + U\partial T/\partial x + V\partial T/\partial y + W\partial T/\partial z) =$$

$$\mu\ \Phi - P\ (\partial U/\partial x + \partial V/\partial y + \partial W/\partial z)$$

$$+ \partial/\partial x\ (k\partial T/\partial x) + \partial/\partial y\ (k\partial T/\partial y) + \partial/\partial z\ (k\partial T/\partial z)$$

where T is temperature, k is the fluid thermal conductivity and Φ is a complex function known as the viscous dissipation function. Note that all of the mechanical energy terms have disappeared. The Conservation of Energy equation can also be written as

$$\rho\ C_p\ (\partial T/\partial t + U\partial T/\partial x + V\partial T/\partial y + W\partial T/\partial z) =$$

$$\mu\ \Phi + \partial P/\partial t + (U\partial P/\partial x + V\partial P/\partial y + W\partial P/\partial z)$$

$$+ \partial/\partial x\ (k\partial T/\partial x) + \partial/\partial y\ (k\partial T/\partial y) + \partial/\partial z\ (k\partial T/\partial z)$$

CONSERVATION LAWS FOR A STREAMTUBE

LAGRANGIAN VS EULERIAN FORMULATIONS

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CONSERVATION OF MASS

Consider an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain mass ρdV where ρ is the fluid density. Integration over the volume gives the total mass of the group. According to Conservation of Mass, the time rate of change of the mass of the group is zero. Mathematically we can write

$$D/Dt \int_V \rho \, dV = 0$$

Using the Transport Theorem this can be rewritten as

$$\int_V \partial\rho/\partial t \; dV \quad + \quad \int_S \rho \; \mathbf{v}.\mathbf{n} \; dS \quad = \quad 0$$

where **v** is the fluid velocity and **n** is the unit outward normal at points on S. For steady flow in a streamtube with multiple inlets and outlets conservation of mass reduces to

$$\Sigma \; (\rho CA)_{OUT} \;-\; \Sigma \; (\rho CA)_{IN} \;=\; \Sigma \; \dot{M}_{OUT} \;-\; \Sigma \; \dot{M}_{IN} \;=\; 0$$

where C is the flow speed and A is the tube area.

CONSERVATION OF MOMENTUM

Consider again an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain momentum ρdV**v**. Integration over V gives the total momentum of the group. According to Conservation of Momentum, the time rate of change of the momentum of the group is equal to the net force acting on it. The forces acting can be of two types: surface forces and body forces. Surface forces in turn can be of two types: pressure and viscous traction. Body forces are generally due only to gravity. Mathematically we can write

$$D/Dt \int_V \rho\mathbf{v} \; dV \quad = \quad \int_S \boldsymbol{\sigma} \; dS \quad + \quad \int_V \rho\mathbf{b} \; dV$$

where **σ** is a vector representing the stress or force per unit area at any point on the surface S and **b** is a vector representing the body force per unit mass at any point within the volume V. Using the Transport Theorem the integral can be rewritten as

$$\int_V \partial\rho\mathbf{v}/\partial t \; dV \;+\; \int_S \rho\mathbf{v} \; \mathbf{v}.\mathbf{n} \; dS \;=$$

$$+\; \int_S \boldsymbol{\sigma} \; dS \;+\; \int_V \rho\mathbf{b} \; dV$$

For short streamtubes friction and gravity are often insignificant. In this case for steady flow in a streamtube with multiple inlets and outlets conservation of momentum reduces to

$$\Sigma \; (\rho\mathbf{v}CA)_{OUT} \;-\; \Sigma \; (\rho\mathbf{v}CA)_{IN} \;=\; \Sigma \; (\dot{M}\,\mathbf{v})_{OUT} \;-\; \Sigma \; (\dot{M}\,\mathbf{v})_{IN}$$

$$=$$

$$-\; \Sigma \; (PA\mathbf{n})_{OUT} \;-\; \Sigma \; (PA\mathbf{n})_{IN} \;+\; \mathbf{R}$$

where **R** is the wall force on the fluid in the streamtube.

CONSERVATION OF ENERGY

Consider once more an arbitrary specific group of fluid particles with volume V and surface S anywhere within a flow. A differential volume dV within V would contain energy edV where e is the fluid energy density. The energy density consists of internal energy and observable kinetic and potential energies:

$$e \;=\; u \;+\; \mathbf{v}.\mathbf{v}/2 \;+\; gz$$

Integration over the volume gives the total energy of the group.

According to Conservation of Energy, the time rate of change of the energy of the group is equal to rate at which heat flows to the group from the surroundings plus the rate at which the surroundings does work on the group. Mathematically we can write

$$D/Dt \int_V \rho e \; dV \;=\; - \int_S \mathbf{q}.\mathbf{n} \; dS \;+\; \int_S \mathbf{v}.\boldsymbol{\sigma} \; dS$$

A body force due to gravity work term is not present in this integral because it has already been accounted for as potential energy in energy density. Using the Transport Theorem the integral can be rewritten as

$$\int_V \partial \rho e/\partial t \; dV \;+\; \int_S \rho e \; \mathbf{v}.\mathbf{n} \; dS \;=$$

$$- \int_S \mathbf{q}.\mathbf{n} \; dS \;+\; \int_S \mathbf{v}.\boldsymbol{\sigma} \; dS$$

For steady adiabatic isothermal flow in a streamtube with multiple inlets and outlets conservation of energy becomes

$$\Sigma \; [(\rho CA)(C^2/2+gz+P/\rho)]_{OUT} \;-\; \Sigma \; [(\rho CA)(C^2/2+gz+P/\rho)]_{IN}$$

$$= \; \Sigma \; (\dot{M} gh)_{OUT} \;-\; \Sigma \; (\dot{M} gh)_{IN} \;=\; \dot{T} \;-\; \dot{L}$$

where h is the flow head at inlets and outlets

$$h \;=\; C^2/2g \;+\; P/\rho g \;+\; z$$

and $\dot{L}$ accounts for losses and $\dot{T}$ accounts for shaft work.

REYNOLDS TRANSPORT THEOREM

Consider a small bit of fluid mass and follow it for a short period of time Δt. Let α be any property of the fluid such as its density. Since we have focused on a specific bit of mass, the property α can be only a function of time. The rate of change of the integral of α over the volume V of the mass is

$$D/Dt \int_{V(t)} \alpha(t)\ dV = \lim_{\Delta t \to 0} \left[\int_{V(t^*)} \alpha(t^*)\ dV - \int_{V(t)} \alpha(t)\ dV \right] / \Delta t$$

where $t^* = t + \Delta t$. Now adding and subtracting the integral of $\alpha(t^*)$ over V(t) inside the [] brackets allows us to rewrite the limit as

$$\lim_{\Delta t \to 0} \left[\int_{V(t)} \alpha(t^*)\ dV - \int_{V(t)} \alpha(t)\ dV \right] / \Delta t$$

$$+$$

$$\lim_{\Delta t \to 0} \left[\int_{V(t^*)} \alpha(t^*)\ dV - \int_{V(t)} \alpha(t^*)\ dV \right] / \Delta t$$

The first limit gives the Eulerian local derivative

$$\int_{V(t)} \partial\alpha/\partial t \; dV$$

Geometric considerations give $\Delta V = [\mathbf{v}\Delta t].[\mathbf{n}dS]$ where S(t) is the surface which encloses V(t), **v** is the velocity at any point on this surface and **n** is the unit outward normal at this point. This allows us to replace the second limit with

$$\int_{S(t)} \alpha(t) \; \mathbf{v.n} \; dS$$

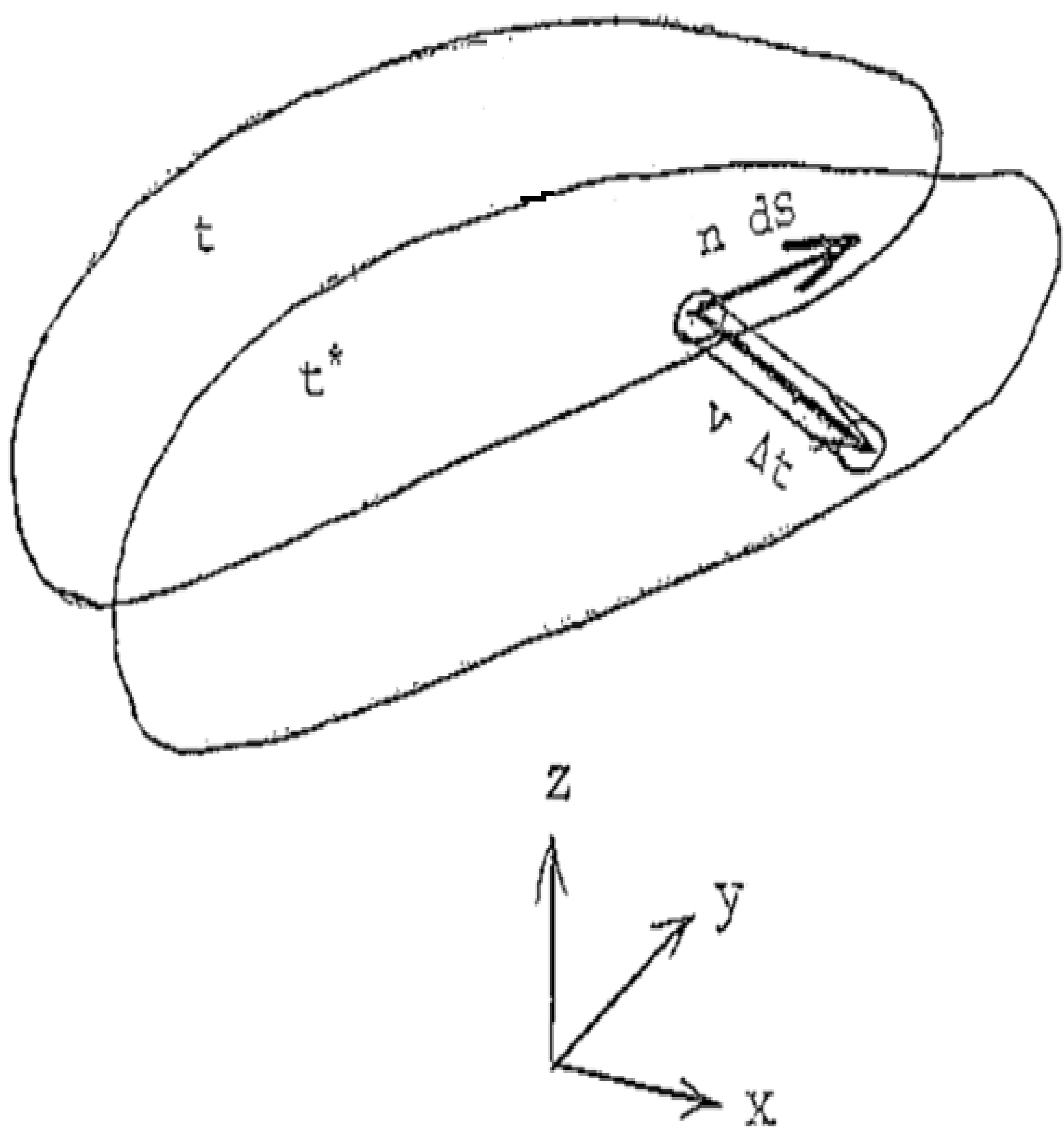
Gauss' Theorem can be used to convert the surface integral to a volume integral. When this is done one gets

$$\int_{S(t)} \alpha(t) \; \mathbf{v.n} \; dS \;=\; \int_{V(t)} \nabla.(\alpha\mathbf{v}) \; dV$$

where $\nabla = \partial/\partial x \; i + \partial/\partial y \; j + \partial/\partial z \; k$. So, one finally gets:

$$D/Dt \int_{V(t)} \alpha(t) \; dV \;=\; \int_{V(t)} [\; \partial\alpha/\partial t + \nabla.(\alpha\mathbf{v}) \;] \; dV$$

This is **Reynolds Transport Theorem**.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