LOW REYNOLDS NUMBER FLOWS

HINCHEY

LUBRICATION FLOWS

Lubrication flows are governed by Reynolds Equation for Pressure. For a Cartesian geometry, its derivation starts with the following simplified form of the Conservation Laws:

$$\partial U/\partial x + \partial V/\partial y + \partial W/\partial z = 0$$

$$\partial P/\partial x = \mu \partial^2 U/\partial z^2$$

$$\partial P/\partial y = \mu \partial^2 V/\partial z^2$$

$$0 = \mu \partial^2 W/\partial z^2$$

Integration of Mass across the gap gives

$$\int [\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]\, dz = 0$$

Manipulation gives

$$\partial I/\partial x + \partial J/\partial y + \partial K/\partial z = 0$$

where

$$I = \int U dz \qquad J = \int V dz$$

$$K = \int W dz$$

Integration of Momentum across the gap gives

$$U = \partial P/\partial x \ (z^2 - zh)/2\mu + (U_T - U_B) z/h + U_B$$

$$V = \partial P/\partial y \ (z^2 - zh)/2\mu + (V_T - V_B) z/h + V_B$$

$$W = (W_T - W_B) z/h + W_B$$

where h is the local gap and the subscripts T and B indicate the velocities of the bearing surfaces.

Substitution into integrated mass gives

$$\partial/\partial x \ (h^3/12\mu \ \partial P/\partial x) + \partial/\partial y \ (h^3/12\mu \ \partial P/\partial y)$$

$$= \partial[h(U_T+U_B)/2]/\partial x + \partial[h(V_T+V_B)/2]/\partial y + (W_T - W_B)$$

This is Reynolds Equation for Pressure. Analytical solutions to this 2D equation are generally impossible and one must use CFD. Manipulation of Reynolds equation gives

$$\partial/\partial x \ (h^3 \ \partial P/\partial x) + \partial/\partial y \ (h^3 \ \partial P/\partial y) = 6 \mu S \ \partial h/\partial x$$

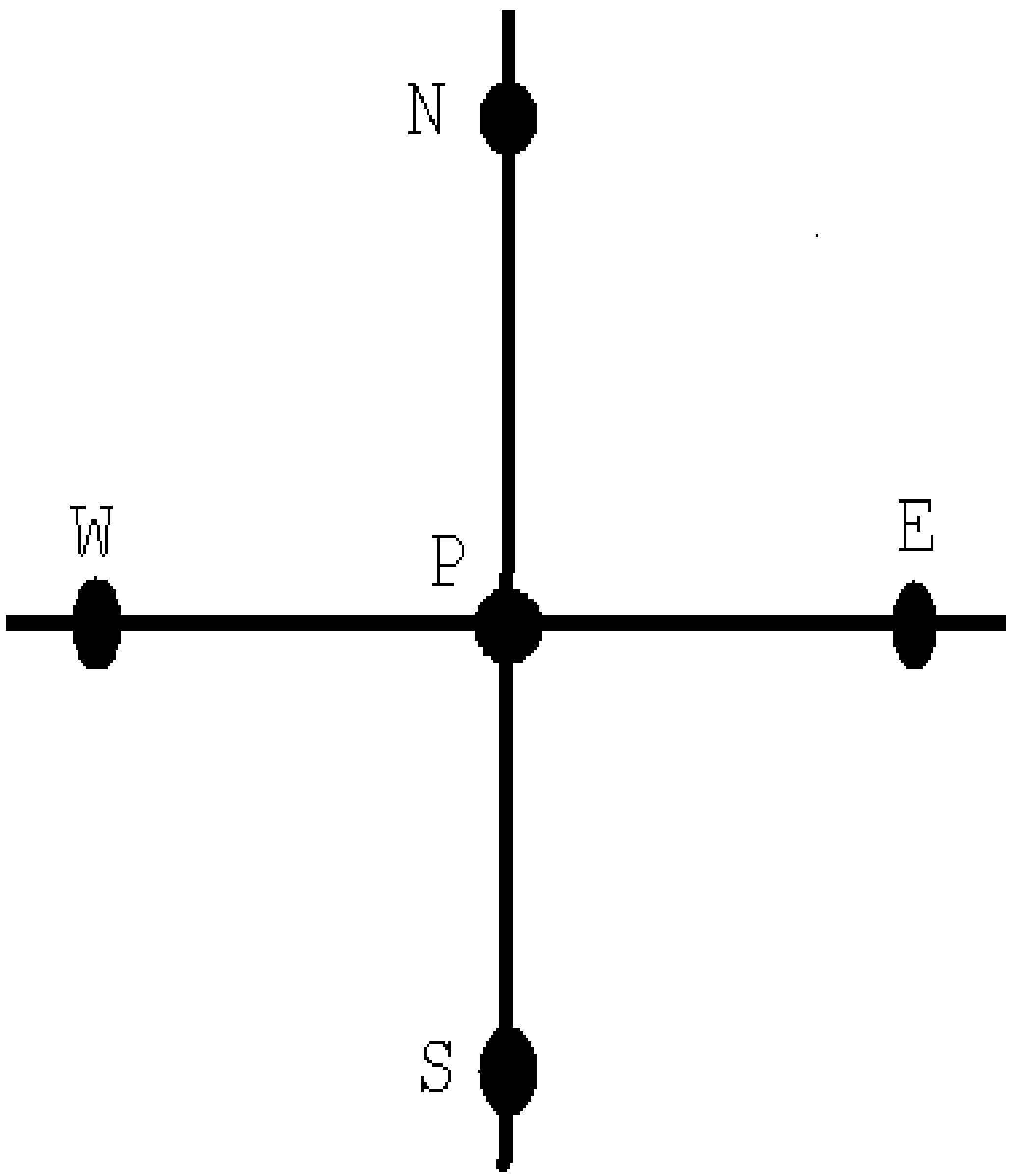
Application of a North South East West CFD scheme to the Cartesian geometry gives

$$[\ [(h_E+h_P)/2]^3 \ (P_E-P_P)/\Delta x - [(h_W+h_P)/2]^3 \ (P_P-P_W)/\Delta x\ ]/\Delta x$$

$$+$$

$$[\ [(h_N+h_P)/2]^3 \ (P_N-P_P)/\Delta y - [(h_S+h_P)/2]^3 \ (P_P-P_S)/\Delta y\ ]/\Delta y$$

$$= 6\mu S \ (h_E - h_W)/[2\Delta x]$$


N
W
P
E
S

Manipulation gives the template

$$P_P = (A\ P_E + B\ P_W + C\ P_N + D\ P_S + H)\ /\ (A + B + C + D)$$

where

$$A = [(h_E+h_P)/2]^3\ /\ [\Delta x]^2$$

$$B = [(h_W+h_P)/2]^3\ /\ [\Delta x]^2$$

$$C = [(h_N+h_P)/2]^3\ /\ [\Delta y]^2$$

$$D = [(h_S+h_P)/2]^3\ /\ [\Delta y]^2$$

$$H = -\ 6\mu\ S\ (h_E-h_W)/[2\Delta x]$$

For a cylindrical geometry, Reynolds Equation is

$$1/r\ \partial/\partial r\ (rh^3/12\mu\partial P/\partial r) + 1/r\ \partial/\partial\Theta\ (h^3/12\mu\ 1/r\ \partial P/\partial\Theta)$$

$$= 1/r\ \partial/\partial r\ [rh(U_T+U_B)/2] + 1/r\ \partial/\partial\Theta\ [h(V_T+V_B)/2] + (W_T-W_B)$$

This can be written as

$$\partial/\partial r\ (rh^3/12\mu\ \partial P/\partial r) + r\ \partial/\partial c\ (h^3/12\mu\ \partial P/\partial c)$$

$$= \partial[rh(U_T+U_B)/2]/\partial r + \partial[h(V_T+V_B)/2]/\partial\Theta + r(W_T-W_B)$$

Manipulation gives

$$r \, \partial/\partial c \; (h^3 \, \partial P/\partial c) + \partial/\partial r \; (r h^3 \, \partial P/\partial r) = 6\mu S \, \partial h/\partial \Theta$$

Application of a North South East West CFD scheme to the cylindrical geometry gives

$$r_P \; [\; [(h_E+h_P)/2]^3 \; (P_E-P_P)/\Delta c \; - \; [(h_W+h_P)/2]^3 \; (P_P-P_W)/\Delta c \;] \; / \; \Delta c$$

$$+$$

$$[[\; [h_P]^3 \; [r_N+r_P]/2] \; (P_N-P_P)/\Delta r \; - \; [\; [h_P]^3 \; [r_S+r_P]/2] \; (P_P-P_S)/\Delta r \;] \; / \; \Delta r$$

$$= \; 6\mu S \; (h_E - h_W)/[2\Delta\Theta]$$

Manipulation gives the template

$$P_P = (A \; P_E + B \; P_W + C \; P_N + D \; P_S + H) \; / \; (A + B + C + D)$$

where

$$A = [(h_E+h_P)/2]^3 \; r_P \; / \; [\Delta c]^2$$

$$B = [(h_W+h_P)/2]^3 \; r_P \; / \; [\Delta c]^2$$

$$C = [h_P]^3 \; [(r_N+r_P)/2] \; / \; [\Delta r^2]$$

$$D = [h_P]^3 \; [(r_S+r_P)/2] \; / \; [\Delta r^2]$$

$$H = - \; 6\mu \; r_P\omega \; (h_E-h_W)/[2\Delta\Theta]$$

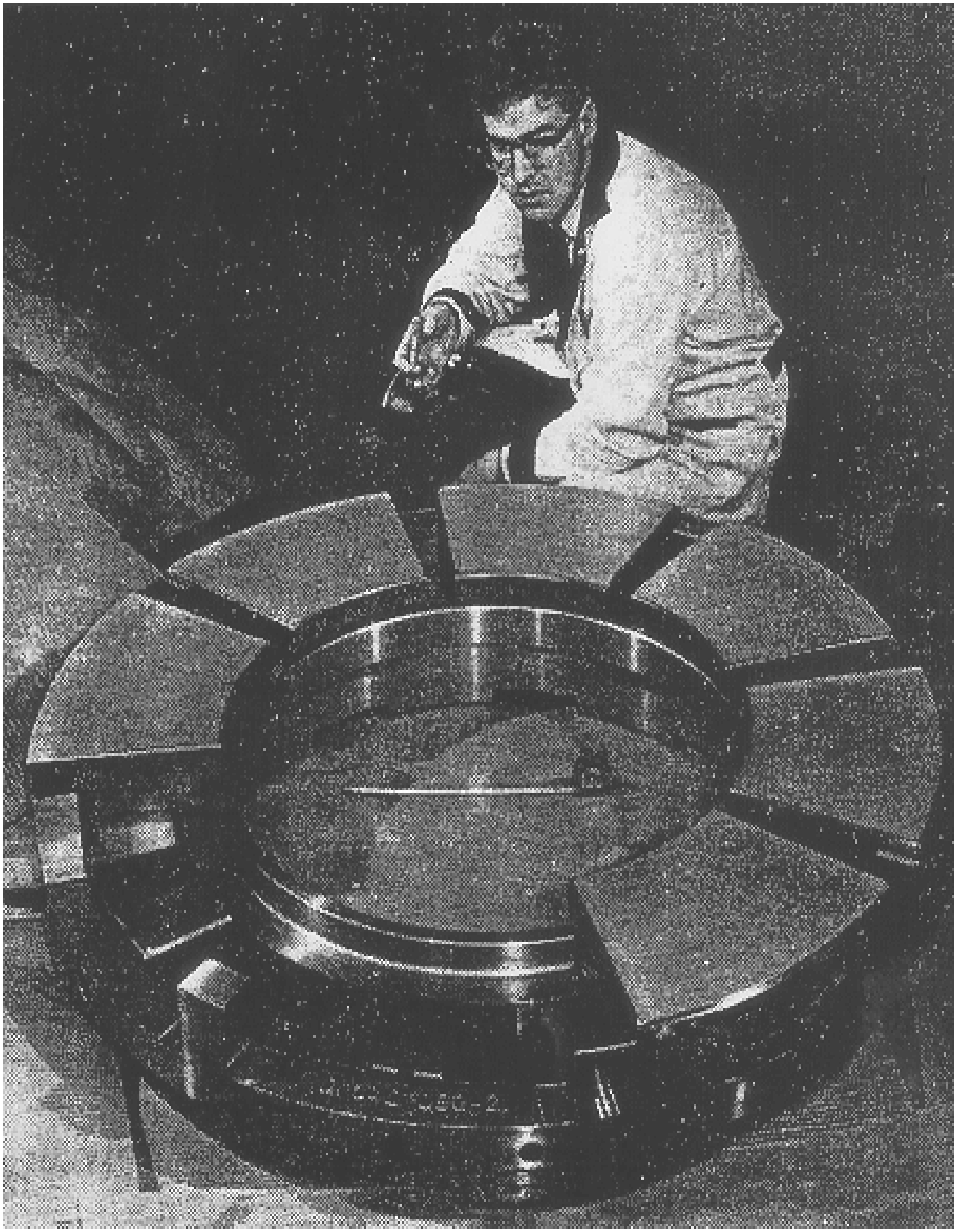
Probably the most important application of the Cartesian formulation is hydrodynamic lubrication journal bearings. A journal bearing consists of a shaft or rotor which rides inside a cylindrical sleeve. It turns out that the curvature of the geometry is not important and one can analyze the bearing by rolling it out flat. During operation, there is a minimum gap between the rotor and the sleeve. It turns out that downstream of this gap, where the fluid moves into a diverging wedge shaped space, the fluid film breaks down and the pressure is approximately atmospheric or zero gage. Upstream of the minimum gap, where the fluid moves into a converging wedge shaped space, high pressures are generated. Once pressures are calculated using the rolled out geometry, they can be rolled back onto the shaft and the load supported by the bearing can be calculated. Probably the most important application of the cylindrical formulation is hydrodynamic lubrication thrust bearings. These are often used on ships and submarines to isolate the engine from the propeller shaft load. A photograph of such a bearing is given at the back. A matlab m code for it and a pressure plot produced by the code are also given on the next few pages.

Although analytical solutions are generally not possible for 2D geometries, they are possible for certain 1D geometries. For wide bearings, the Cartesian Reynolds Equation becomes

$$d/dx\ (h^3\ dP/dx) = 6\mu\ S\ dh/dx = H\ dh/dx$$

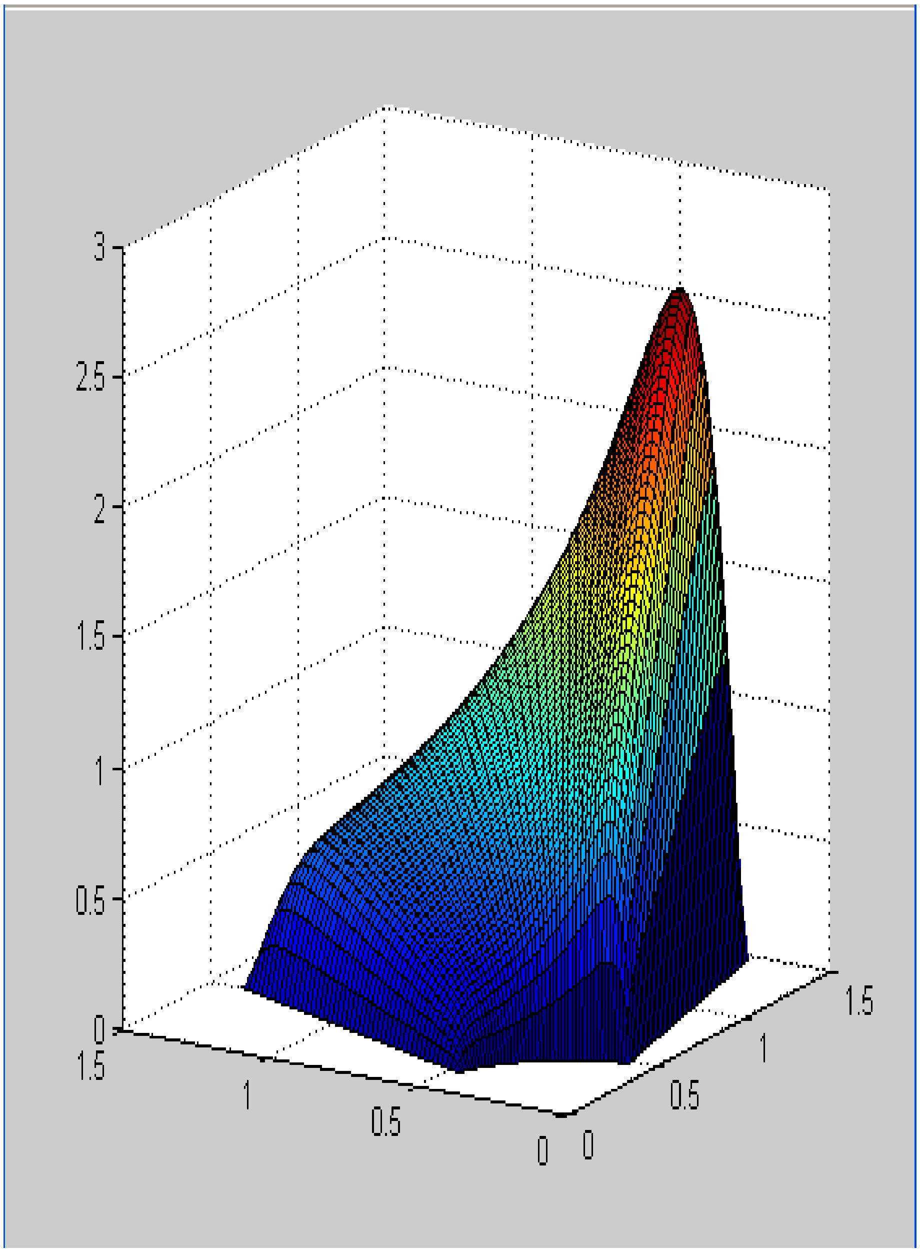
Integration of this equation gives

$$h^3\ dP/dx = H\ h + A$$



```
%
%        HYDRODYNAMIC THRUST BEARING
%
clear all
NR=51;NA=51;NIT=22;
MR=NR-1;MA=NA-1;
PI=3.14159; DENSITY=880.0;
GRAVITY=9.81; VISCOSITY=0.1;
RPM=100.0; RPS=RPM/60.0;
RIN=0.5;ROUT=1.5;
AIN=+10.0;AOUT=+70.0;
AIN=AIN/180.0*PI;
AOUT=AOUT/180.0*PI;
ONE=0.001;TWO=0.002;
DELR=(ROUT-RIN)/MR;
DELA=(AOUT-AIN)/MA;
GAP=TWO-ONE;
SPAN=AOUT-AIN;
SLOPE=GAP/SPAN;
PRESSURE=0.0;
RNODE=RIN;
for JJ=1:NR
ANODE=AIN;
for II=1:NA
R(II,JJ)=RNODE;
HEAD(II,JJ)=0.0;
CHANGE=ANODE-AIN;
P(II,JJ)=PRESSURE;
X(II,JJ)=+RNODE*cos(ANODE);
Y(II,JJ)=+RNODE*sin(ANODE);
H(II,JJ)=ONE+SLOPE*CHANGE;
ANODE=ANODE+DELA;
end
RNODE=RNODE+DELR;
end
```

```
THRUST=0.0;
for IT=1:NIT
for JJ=2:MR
for II=2:MA
DELC=DELA*R(II,JJ);
AREA=DELR*DELC;
SPEED=RPS*2.0*PI*R(II,JJ);
A=((H(II+1,JJ)+H(II,JJ))/2.0)^3/DELC^2;
B=((H(II,JJ)+H(II-1,JJ))/2.0)^3/DELC^2;
C=((H(II,JJ+1)+H(II,JJ))/2.0)^3/DELR^2;
D=((H(II,JJ)+H(II,JJ-1))/2.0)^3/DELR^2;
C=C*(R(II,JJ+1)+R(II,JJ))/2.0;
D=D*(R(II,JJ)+R(II,JJ-1))/2.0;
A=A*R(II,JJ); B=B*R(II,JJ);
S=6.0*VISCOSITY*SPEED*SLOPE;
AA=A*P(II+1,JJ); BB=B*P(II-1,JJ);
CC=C*P(II,JJ+1); DD=D*P(II,JJ-1);
P(II,JJ)=(S+AA+BB+CC+DD)/(A+B+C+D);
DELP=P(II,JJ)-PRESSURE;
HEAD(II,JJ)=DELP/DENSITY/GRAVITY;
FORCE=(P(II,JJ)-PRESSURE)*AREA;
if(IT==NIT) THRUST=THRUST+FORCE;end;
end
end
end
THRUST=THRUST*2.0
surf(X,Y,HEAD)
```



3
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0
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1
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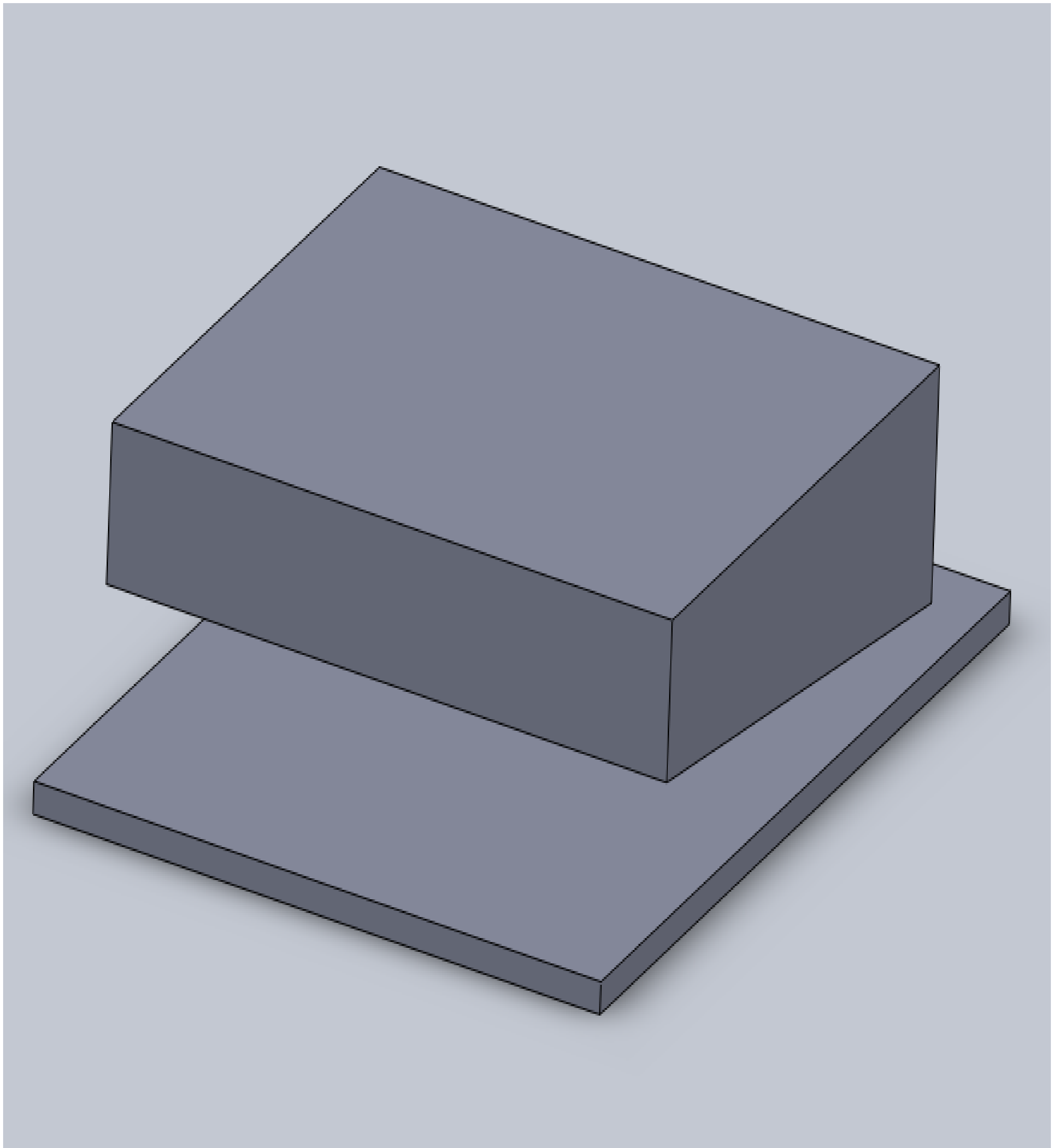
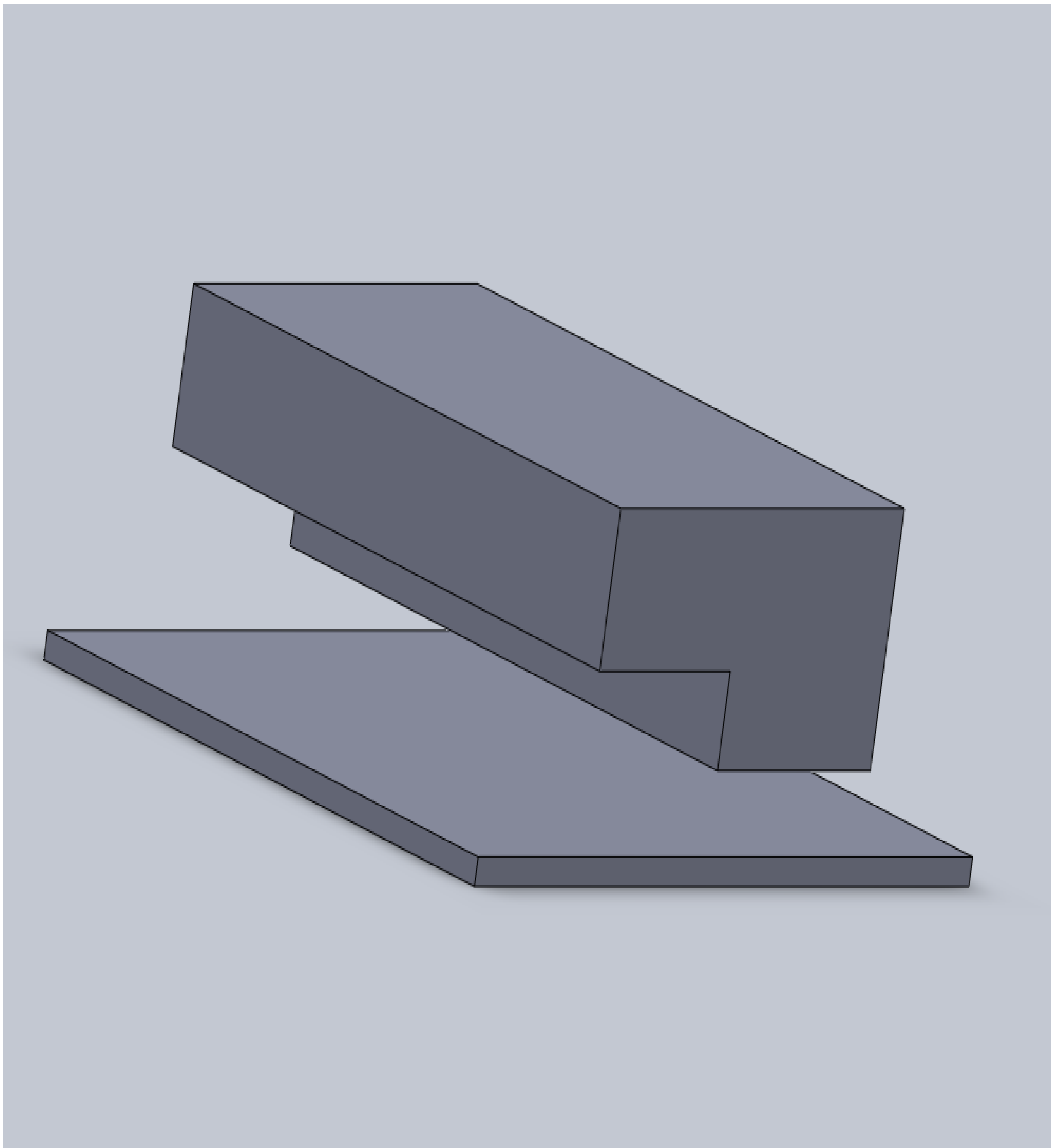




where A is a constant of integration. Manipulation gives

$$dP/dx = H/h^2 + A/h^3$$

For a linear gap variation

$$h = s\ x + b$$

where s is the bearing slope and b is the front gap. If a is the back gap, then $s=(a-b)/d$ where d is the bearing length. Substitution into the gradient equation gives

$$dP/dx = H/(sx+b)^2 + A/(sx+b)^3$$

Another integration gives

$$P = -H/[s(sx+b)] - A/[2s(sx+b)^2] + B$$

At the edges of the bearing, pressure is atmospheric pressure **P.** Application of these boundary conditions gives

$$A = 2H\ [a^2b-b^2a]\ /\ [b^2-a^2]$$

$$B = \mathbf{P} + H\ /\ [s(b+a)]$$

For a step bearing, where there is a jump in gap from one constant level to another constant level, the pressure gradient equation shows that the gradient is constant in the constant gap regions. This means that the pressure variation is linear in the constant gap regions. Let the peak pressure be **P**. Let the gap front of the step be b and let the gap at the back be a. Let the length of the front region be w and the length of the back region be v. Across the step, one can write

$$\Delta [h^3 \, dP/dx] = H \, \Delta h$$

Substitution into this gives

$$a^3 \, [0-\mathbf{P}]/v - b^3 \, [\mathbf{P}-0]/w = H \, [a-b]$$

Manipulation gives

$$\mathbf{P} = H \, [b-a] \, / \, [a^3/v + b^3/w]$$

For narrow bearings, Reynolds Equation reduces to

$$d/dy \, (h^3 \, dP/dy) = 6\mu \, S \, dh/dx = H \, dh/dx$$

This equation ignores leakage due to pressure gradients in the x direction. For a wedge bearing, manipulation gives

$$d/dy(dP/dy) = H/h^3 \, dh/dx = G$$

Integration gives

$$P = G/2 \, y^2 + Ay + B$$

where A and B are constants of integration.

DRUM VISCOMETER

A drum viscometer consists of a drum which rotates inside a sleeve. A liquid fills the gap between the drum and the sleeve. Let the gap be h. The torque required to rotate the drum is:

$$T = R\ \mu\ R\omega/h\ 2\pi\ RL$$

With known geometry and measured torque, one gets

$$\mu = [\ T\ h\ ] / [\ 2\pi\ R^3\ L\ \omega\ ]$$

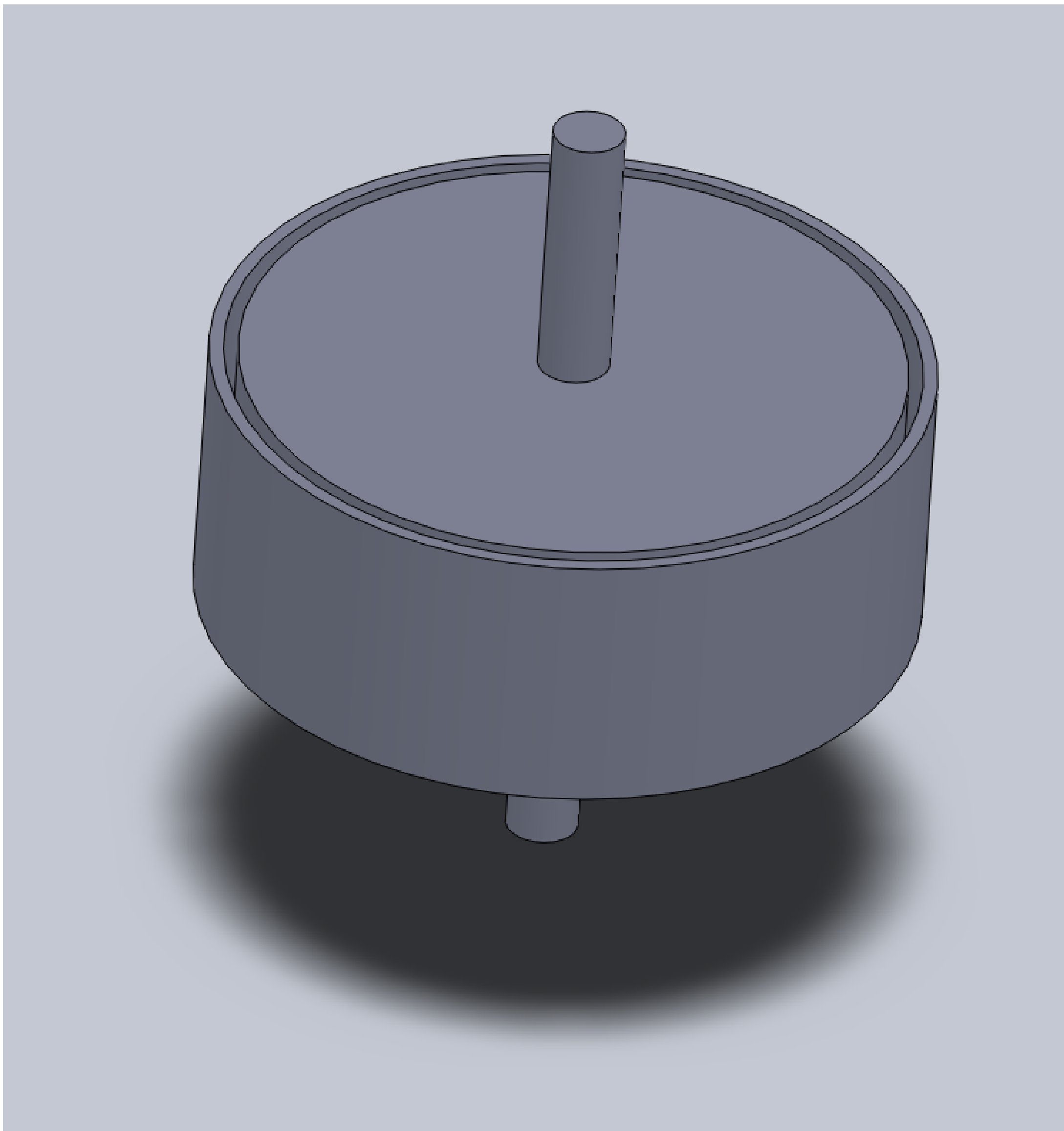
DISK VISCOMETER

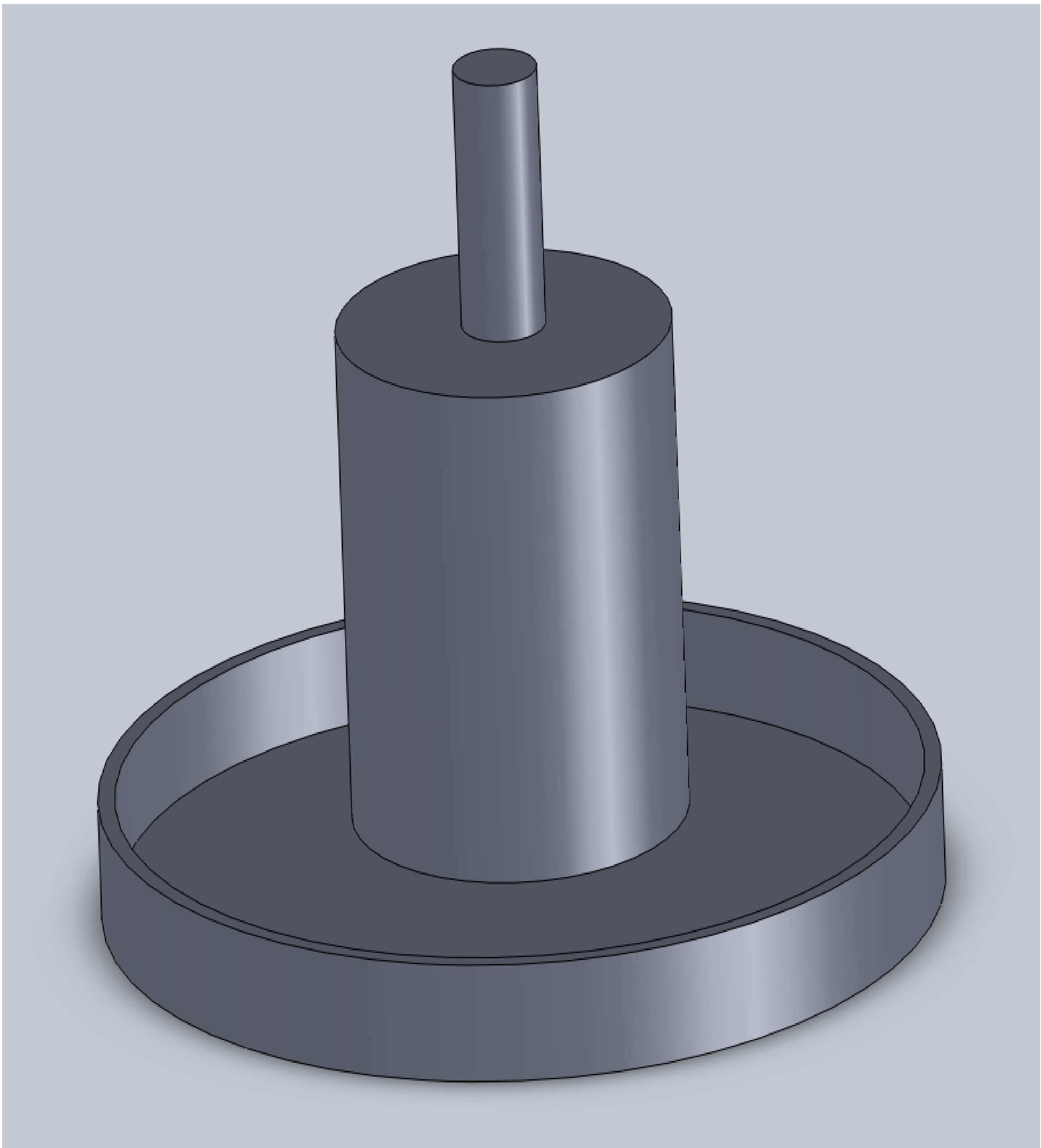
A disk viscometer consists of a disk which rotates inside a can. A liquid fills the gap between the disk and the can. Let the gap be h. The torque required to rotate the disk is:

$$T = \int r\ \mu\ r\omega/h\ 2\pi\ r\ dr$$

With known geometry and measured torque, one gets

$$\mu = [\ 2\ T\ h\ ] / [\pi\ R^4\ \omega]$$





CAPILLARY FLOWS

The pressure driven flow through a small diameter tube is known as capillary flow. The small size makes the flow inside the tube laminar. Conservation of Mass considerations give

$$\partial U/\partial s = 0$$

while Conservation of Momentum considerations give

$$\partial P/\partial s = 1/r \ \partial/\partial r \ (r \ \mu \ \partial U/\partial r)$$

Integration of Momentum gives

$$-\partial P/\partial s \ r^2/2 \ + \ r \ \mu \ \partial U/\partial r \ + \ K \ = \ 0$$

where K must be zero because r can be zero. Manipulation gives

$$\partial U/\partial r = r/[2\mu] \ \partial P/\partial s$$

Integration of this equation gives

$$U = r^2/[4\mu] \ \partial P/\partial s \ + \ C$$

At r equal to R, U is zero. So U becomes

$$U = - \ [R^2-r^2]/[4\mu] \ \partial P/\partial s$$

Integration gives the volumetric flow rate

$$Q = \int U \; 2\pi r \; dr$$

$$= - \int [R^2-r^2]/[4\mu] \; \partial P/\partial s \; 2\pi r \; dr \quad = - [\pi R^4]/[8\mu] \; \partial P/\partial s$$

For a tube L meters long open at both ends with its outlet H meters below its inlet, this equation becomes

$$Q = - [\pi R^4]/[8\mu] \; [-\rho g H]/L \; = [\pi R^4]/[8\mu] \; [\rho g H]/L$$

Manipulation of this equation gives

$$\mu = [\rho g H][\pi R^4] \; / \; [8QL]$$

This is the equation for a tube viscometer.

For a steady flow the head loss is H. Solving for H gives

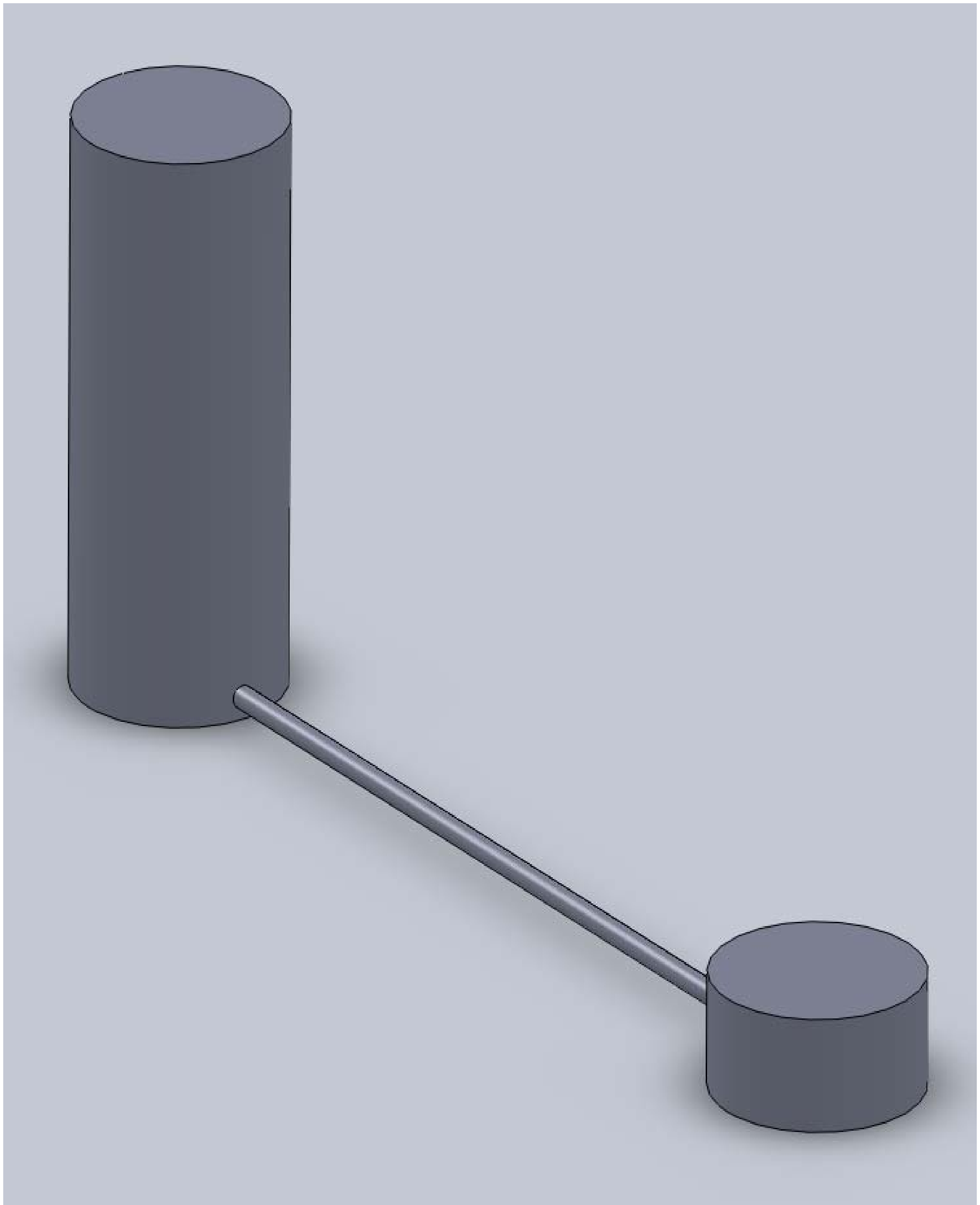
$$H = \mu \; [8QL] \; / \; [\rho g \; \pi R^4]$$

In terms of average flow speed **U** the flow is

$$Q = \mathbf{U} \; \pi R^2 = \mathbf{U} \; \pi D^2/4$$

With this head becomes

$$H = 64/Re \; L/D \; \mathbf{U}^2/[2g] \; = \; f \; L/D \; \mathbf{U}^2/[2g]$$



DRUM PUMP

A drum pump consists of a drum which rotates inside a sleeve. The rotation drags liquid from an inlet to an outlet. Let the drum rotational speed be Ω. Let the drum radius be R. Let the gap between the drum and the sleeve be h. Let the direction across the gap be s and the circumferential direction be c.

Conservation of Momentum considerations give

$$\partial P/\partial c = \mu\ \partial^2 U/\partial s^2$$

Integration across the gap gives

$$U = \partial P/\partial c\ [\ s^2/[2\mu] - sh/[2\mu]\ ] - R\Omega s/h + R\Omega$$

Integration gives the volumetric flow rate

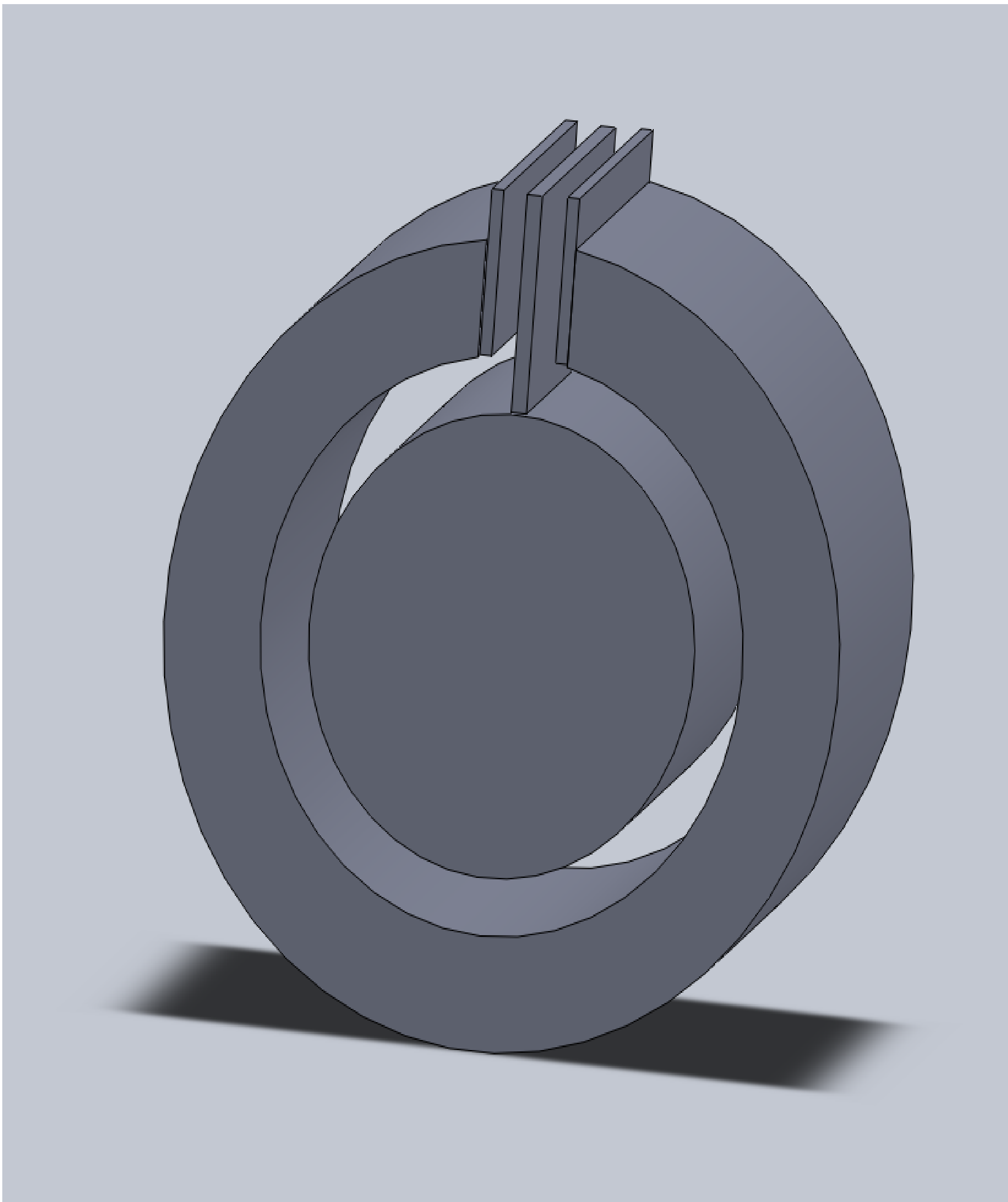
$$Q = \int U ds = - h^3/[12\mu]\ \partial P/\partial c + R\Omega\ h/2$$

This can be written as

$$\Delta Q = - h^3/[12\mu]\ \Delta P/[2\pi R] + R\Omega\ h/2$$

Manipulation gives a characteristic of the form

$$\Delta P = A + B\ \Delta Q$$



BELT PUMP

A belt pump consists of a belt which moves vertically through a bath of liquid. The motion drags liquid vertically upward.

Conservation of Momentum considerations give

$$d/ds \; (\mu \; dW/ds) - \rho g = 0$$

Integration gives

$$\mu \; dW/ds - \rho g \; s + A = 0$$

$$\mu \; W - \rho g \; s^2/2 + As + B = 0$$

The boundary conditions are

$$dW/ds = 0 \quad \text{at } s=h \qquad\qquad W = \mathbf{W} \quad \text{at } s=0$$

These give the constants of integration

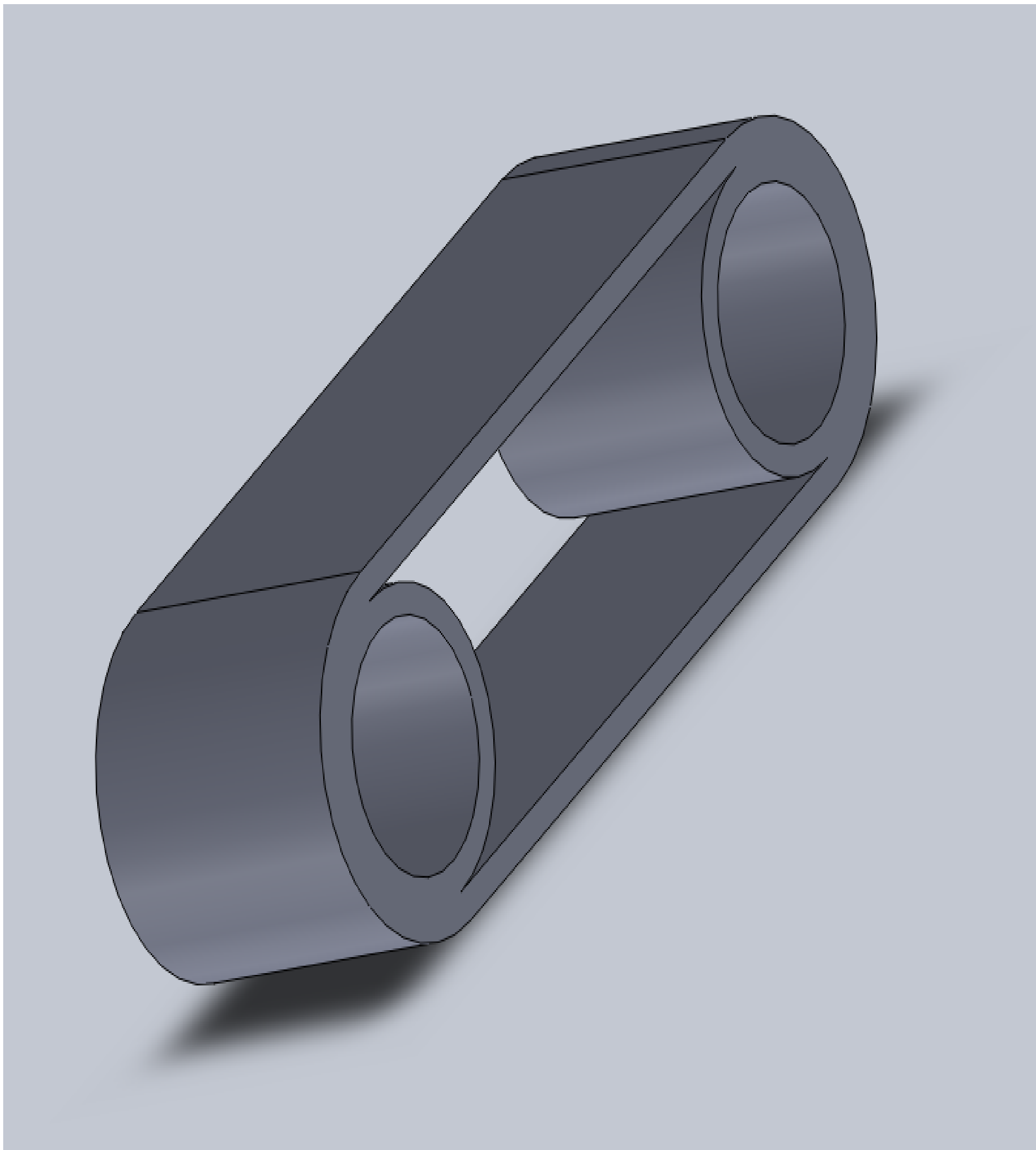
$$A = \rho g \; h \qquad B = - \mu \; \mathbf{W}$$

Substitution gives

$$W = \mathbf{W} - \rho g/[\mu] \; s \; [h-s/2]$$

Integration gives

$$Q = \int W ds = \mathbf{W} \; h - \rho g/[3\mu] \; h^3$$



WIRE COATING

Assume that the die is long and there is no axial variation inside it. In this case Conservation of Momentum is:

$$0 = 1/r \, d/dr \, [\, r \, \mu \, dU/dr \,]$$

Integration gives

$$U = W \ln[r/R_D] \, / \, \ln[R_W/R_D]$$

The volumetric flow rate far downstream is

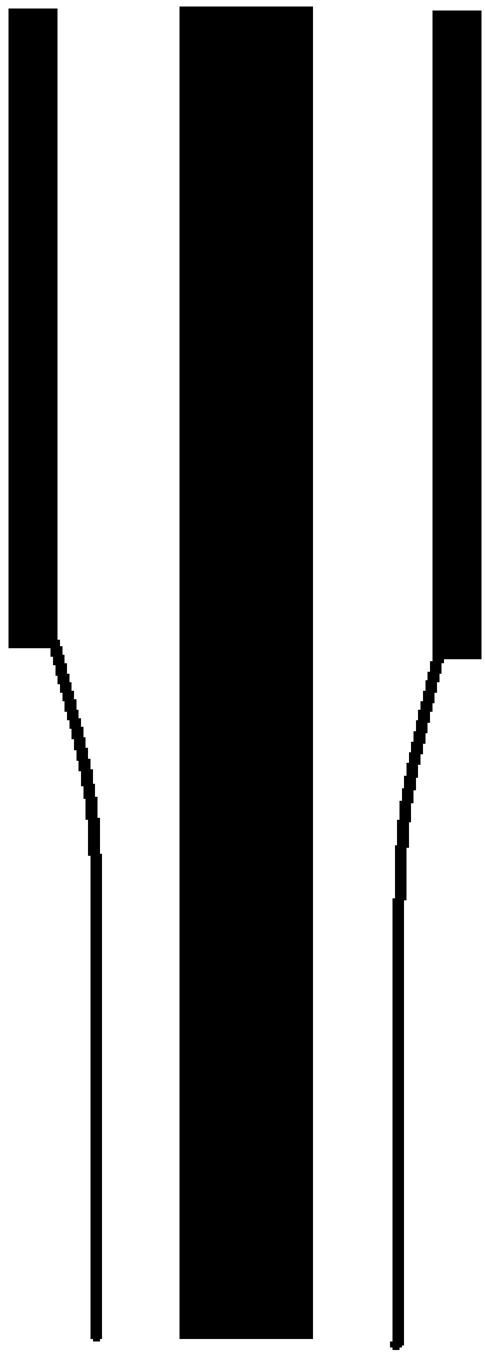
$$Q = W \, (\, \pi \, R_C \, R_C \; - \; \pi \, R_W \, R_W \,)$$

The volumetric flow rate within the die is

$$Q = \int 2\pi r \, U \, dr$$

Equating these flow rates gives

$$R_C \, R_C \; = [\, R_D \, R_D \; - \; R_W \, R_W \,] \, / \, [\, 2 \ln[R_D/R_W] \,]$$



POROUS MEDIA FLOWS

The Darcy Law gives

$$\mathbf{v} = - K \nabla P$$

where $\mathbf{v}$ is the velocity vector and P is pressure. The parameter K is equal to the permeability k divided by the viscosity μ

$$K = k/\mu$$

Conservation of Mass gives

$$\nabla . \mathbf{v} = 0$$

Substitution into Mass gives

$$\nabla . [K \nabla P] = 0$$

Manipulation gives

$$\partial/\partial x\ [K\ \partial P/\partial x] + \partial/\partial y\ [K\ \partial P/\partial y] + \partial/\partial z\ [K\ \partial P/\partial z] = 0$$

For a rectangular slab, central differencing gives

$$[(K_E+K_P)/2\ (P_E-P_P)/\Delta x - (K_W+K_P)/2\ (P_P-P_W)/\Delta x\ ]\ /\Delta x$$

$$+$$

$$[(K_N+K_P)/2\ (P_N-P_P)/\Delta y - (K_S+K_P)/2\ (P_P-P_S)/\Delta y\ ]\ /\Delta y$$

$$+$$

$$[(K_J+K_P)/2\ (P_J-P_P)/\Delta z - (K_I+K_P)/2\ (P_P-P_I)/\Delta z\ ]\ /\Delta z$$

$$= 0$$

Manipulation gives the template

$$P_P = \frac{(A\ P_E + B\ P_W + C\ P_N + D\ P_S + G\ P_J + H\ P_I)}{(A + B + C + D + G + H)}$$

$$A = [(K_E+K_P)/2]/[\Delta x^2] \qquad B = [(K_W+K_P)/2]/[\Delta x^2]$$

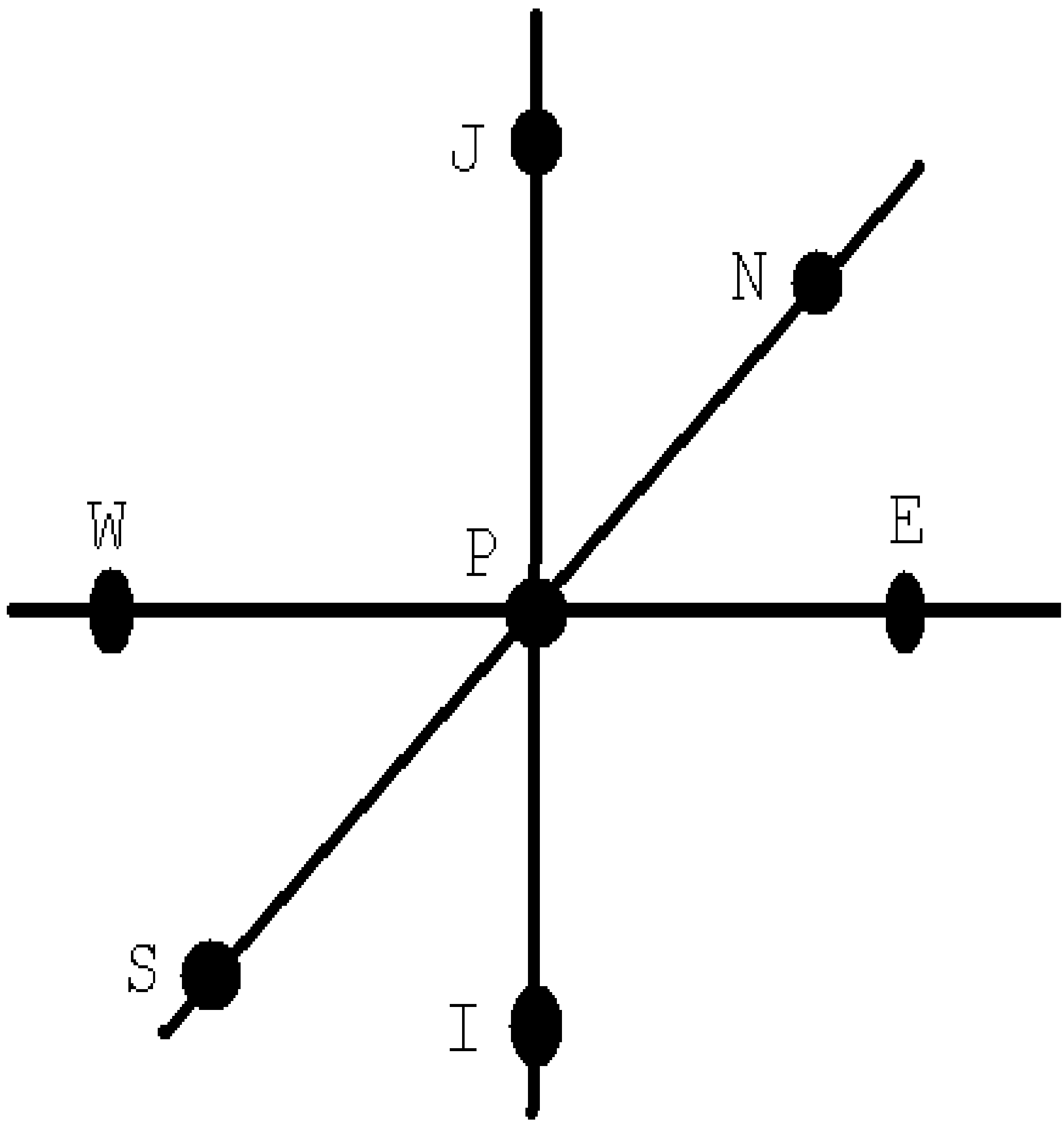
$$C = [(K_N+K_P)/2]/[\Delta y^2] \qquad D = [(K_S+K_P)/2]/[\Delta y^2]$$

$$G = [(K_J+K_P)/2]/[\Delta z^2] \qquad H = [(K_I+K_P)/2]/[\Delta z^2]$$

The blocked sides CFD template for one point is

$$P_P = (A\ P_E + B\ P_W)\ /\ (A + B)$$

For a blocked sides case, the pressure equation is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$$\frac{d}{dx}\left[K \frac{dP}{dx}\right] = 0$$

Integration gives

$$K \frac{dP}{dx} = G \qquad \frac{dP}{dx} = G/K$$

For a linear variation in the parameter K

$$K = ax + b$$

The pressure gradient equation becomes

$$\frac{dP}{dx} = G / [ax + b]$$

Integration gives

$$P = G/a \ln[ax+b] + H$$

The boundary conditions are

$$P=P_{IN} \text{ at } x=0 \qquad P=P_{OUT} \text{ at } x=d$$

This allows one to find the constants of integration.

SLIGHTLY TAPERED SLAB (TRANSFORMATION)

Consider an xyz slab which is slightly tapered in the x direction. This can be transformed into a cubical $\alpha\beta\varepsilon$ slab. The transformation equations are:

$$x = a\alpha \qquad y = b\beta \qquad z = n\varepsilon(1+m\alpha) = c\varepsilon$$

With this the pressure equation becomes

$$1/a\ \partial/\partial\alpha[K/a\ \partial P/\partial\alpha] + 1/b\ \partial/\partial\beta[K/b\ \partial P/\partial\beta] + 1/c\ \partial/\partial\varepsilon[K/c\ \partial P/\partial\varepsilon] = 0$$

This can be solved numerically using central differences.

ARBITRARY SLAB (ISOPARAMETRIC FINITE ELEMENTS)

The governing equation is:

$$\partial/\partial x\ [K\ \partial P/\partial x] + \partial/\partial y\ [K\ \partial P/\partial y] + \partial/\partial z\ [K\ \partial P/\partial z] = 0$$

For a Galerkin finite element analysis, we assume that pressure can be given as a sum of scaled shape functions:

$$P = \Sigma\ M\ m$$

where m is pressure at a node and M is a shape function. In terms of nodal values, the parameter K is

$$K = \Sigma\ M\ k$$

Substitution of the assumed form for P into the governing equation gives a residual. In a Galerkin analysis, weighted averages of this residual throughout the slab are set to zero. After some manipulation, one gets

$$\int [\nabla W \ K \ \nabla P] \ dV \ - \ \int W \ K \ \partial P/\partial n \ dS \ = \ 0$$

where W is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. Notice the integration by parts of the space derivative terms in the weighted residual integral. This introduces slope boundary conditions into the formulation. It also allows us to use linear shape functions. Without it we would have to use quadratic or higher order shape functions. Linear shape functions for a brick finite element are:

$$M_1 = 1/8 \ (1-\varepsilon) \ (1-\alpha) \ (1-\beta) \qquad M_2 = 1/8 \ (1-\varepsilon) \ (1+\alpha) \ (1-\beta)$$
$$M_3 = 1/8 \ (1-\varepsilon) \ (1-\alpha) \ (1+\beta) \qquad M_4 = 1/8 \ (1-\varepsilon) \ (1+\alpha) \ (1+\beta)$$
$$M_5 = 1/8 \ (1+\varepsilon) \ (1-\alpha) \ (1-\beta) \qquad M_6 = 1/8 \ (1+\varepsilon) \ (1+\alpha) \ (1-\beta)$$
$$M_7 = 1/8 \ (1+\varepsilon) \ (1-\alpha) \ (1+\beta) \qquad M_8 = 1/8 \ (1+\varepsilon) \ (1+\alpha) \ (1+\beta)$$

where α β ε are local coordinates. In the global coordinate system, the elements are isoparametric. Details of the local global connection are beyond the scope of this note.

After performing the integrations numerically using Gaussian Quadrature, one gets a set of algebraic equations for the nodal pressures. Details of it are beyond the scope of this note.