HIGH SPEED

GAS DYNAMICS

HINCHEY

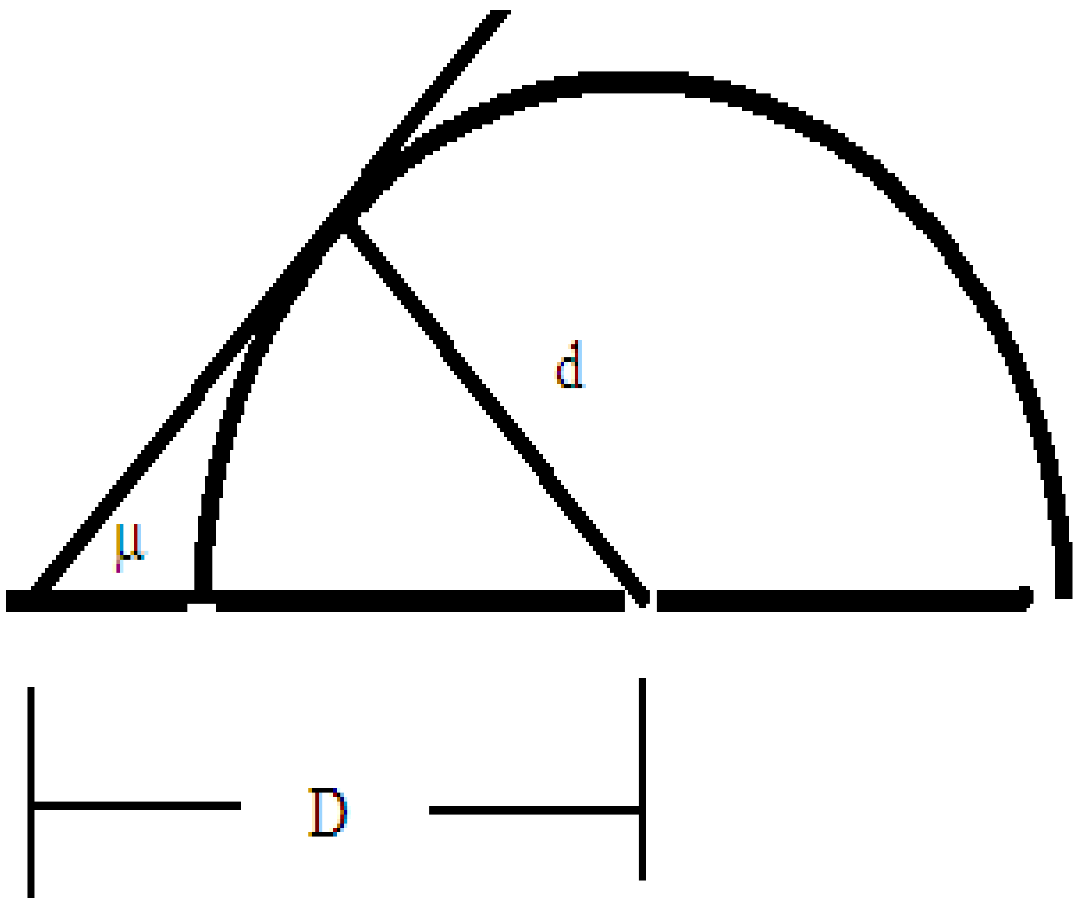
MACH WAVES

Mach Number is the speed of something divided by the local speed of sound. When an infinitesimal disturbance moves at a steady speed, at each instant in time it generates a sound wave which moves radially outward from it in all directions. When the speed of the disturbance is supersonic, these waves can be found inside a cone which extends back from the disturbance. Half of this is shown in the sketch on the next page. Consider a disturbance which was at the center of the half circle but is now at the tip of the cone. Let the distance between these points be D. If the speed of the disturbance is U, then it took T equal to D/U seconds for the disturbance to move from the center to the tip. During this time, the wave travelled a distance d equal to a T where a is the speed of sound. Geometry shows that the cone half angle is:

$$\mathrm{Sin}[\mu] = d/D = [aT]/[UT] = 1/M$$

$$\mu = \mathrm{Sin}^{-1}[1/M]$$

Inside the cone is known as the zone of action: outside the cone is known as the zone of silence. An Expansion Wave is made up of an infinite number of Mach Waves in a fan like structure. A Shock Wave is where an infinite number of Mach Waves pile up into a single wave.


d
μ
D

THERMODYNAMIC CONNECTIONS

According to thermodynamics

$$h = u + Pv = u + P/\rho$$

$$h = C_P\ T \qquad u = C_V\ T \qquad k = C_P\ /\ C_V$$

$$Pv = P/\rho = R\ T$$

Substitution into enthalpy gives

$$C_P\ T = C_V\ T + R\ T$$

$$C_P = C_V + R$$

Manipulation gives

$$C_P/C_P = C_V/C_P + R/C_P$$

$$1 - 1/k = (k-1)/k = R/C_P$$

$$C_P = k/(k-1)\ R$$

CONSERVATION OF ENERGY

For a streamtube, Conservation of Energy gives

$$h + U^2/2 = K$$

For two points in the tube

$$h_1 + U_1U_1 / 2 = h_2 + U_2U_2 / 2$$

Thermodynamics and manipulation gives

$$C_P T_1 + [U_1U_1] / 2 =$$
$$C_P T_2 + [U_2U_2] / 2$$

$$T_1 (1 + [U_1U_1] / [2C_PT_1]) =$$
$$T_2 (1 + [U_2U_2] / [2C_PT_2])$$

$$T_1 (1 + (k-1)/2 [U_1U_1]/[kRT_1]) =$$
$$T_2 (1 + (k-1)/2 [U_2U_2]/[kRT_2])$$

$$T_1 (1 + (k-1)/2 [U_1U_1]/[a_1a_1]) =$$
$$T_2 (1 + (k-1)/2 [U_2U_2]/[a_2a_2]])$$

$$T_1 (1 + (k-1)/2 M_1M_1) =$$
$$T_2 (1 + (k-1)/2 M_2M_2)$$

$$T_2 / T_1 = (1 + (k-1)/2 M_1M_1) / (1 + (k-1)/2 M_2M_2)$$

ISENTROPIC PROCESSES

The Second Law of Thermodynamics gives

$$T\ ds = du + Pdv \qquad T\ ds = dh - vdP$$

Manipulation gives:

$$ds = C_v\ dT/T + R\ dv/v \qquad ds = C_p\ dT/T - R\ dP/P$$

Integration from one state to another state gives:

$$s_2 - s_1 = \Delta s = C_v \ln(T_2/T_1) + R \ln(v_2/v_1)$$

$$s_2 - s_1 = \Delta s = C_p \ln(T_2/T_1) - R \ln(P_2/P_1)$$

For an isentropic process, the last equation gives

$$\ln(P_2/P_1) = C_p/R \ln(T_2/T_1) = [k/(k-1)]R/R \ln(T_2/T_1)$$

$$P_2/P_1 = [T_2/T_1]^{k/(k-1)}$$

For an isentropic process, the Δs equations give

$$\ln(T_2/T_1) = - R/C_v \ln(v_2/v_1) = + R/C_p \ln(P_2/P_1)$$

Manipulation of these equations gives

$$P_2/P_1 = (v_1/v_2)^k = (\rho_2/\rho_1)^k$$

$$P_2 / [\rho_2]^k = P_1 / [\rho_1]^k$$

$$P / \rho^k = K \qquad P = K \rho^k$$

NOZZLE FLOWS

A nozzle is a short length of pipe or tube with a variable cross sectional area. For a flow of gas in a nozzle, conservation of mass considerations require that

$$\rho\ A\ U$$

must be constant along the nozzle, where ρ is the gas density, A is the nozzle area and U is the flow velocity. For two points very close together in a nozzle:

$$\rho\ A\ U = (\rho+\Delta\rho)\ (A+\Delta A)\ (U+\Delta U)$$

Manipulation gives

$$A\ U\ \Delta\rho + \rho\ U\ \Delta A + \rho\ A\ \Delta U = 0$$

$$\Delta\rho/\rho + \Delta A/A + \Delta U/U = 0$$

Conservation of energy considerations require that

$$h + U^2/2$$

must be constant along the nozzle, where h is enthalpy. Thermodynamics shows that

$$h = C_P\ T = k/(k-1)\ RT = k/(k-1)\ P/\rho$$

With this, energy becomes

$$k/(k-1)\ P/\rho \ + \ U^2/2$$

For two points very close together in a nozzle

$$k/(k-1)\ P/\rho \ + \ U^2/2 \ =$$

$$k/(k-1)\ [P+\Delta P]/[\rho+\Delta\rho] \ + \ [U+\Delta U]^2/2$$

Expansion gives

$$k/(k-1)\ P/\rho \ + \ U^2/2 \ =$$

$$k/(k-1)\ [P+\Delta P]/[\rho+\Delta\rho]\ [\rho-\Delta\rho]/[\rho-\Delta\rho] \ + \ [U+\Delta U]^2/2$$

$$k/(k-1)\ [\rho\Delta P - P\Delta\rho]/\rho^2 \ + U\ \Delta U \ = 0$$

$$k/(k-1)\ [\rho\ kRT\ \Delta\rho - \ \rho RT\ \Delta\rho]/\rho^2 \ + U\ \Delta U \ = 0$$

$$k/(k-1)\ [(k-1)RT]\ \Delta\rho/\rho \ + U\ \Delta U \ = 0$$

$$kRT\ \Delta\rho/\rho \ + U\ \Delta U \ = 0$$

$$a^2\ \Delta\rho/\rho \ + U\ \Delta U \ = 0$$

Manipulation gives

$$\Delta\rho/\rho \;=\; - U/a^2\ \Delta U \;=\; - U^2/a^2\ \Delta U/U$$

Energy into mass gives

$$- U^2/a^2\ \Delta U/U + \Delta A/A + \Delta U/U = 0$$

Manipulation gives

$$- U^2/a^2\ \Delta U + U/A\ \Delta A + \Delta U = 0$$

$$(1- M^2)\ \Delta U + U/A\ \Delta A = 0$$

$$\Delta U = U\ \Delta A\ /\ [A\ (M^2-1)]$$

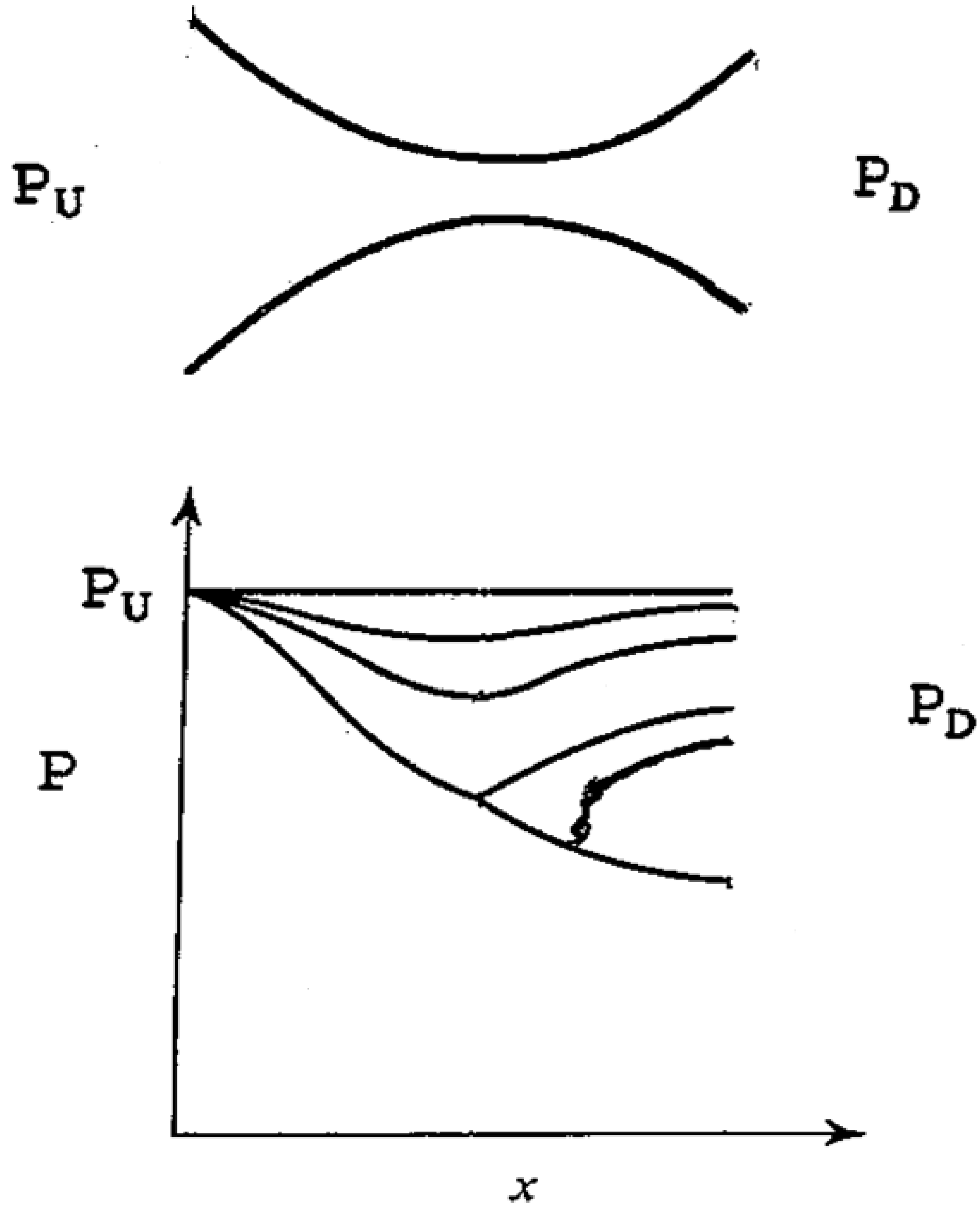
This equation shows that, if M is less than unity or flow is subsonic, then flow speed decreases when area increases and increases when area decreases. However, if M is greater than unity or flow is supersonic, then flow speed increases when area increases and decreases when area decreases. One can write the last equation as follows:

$$\Delta U\ (M^2-1) = U\ \Delta A/A$$

When there is a transition from subsonic upstream of a throat to supersonic downstream, this equation suggests that the flow is sonic at the throat.

CHOKED FLOW

Consider gas flow down the converging/diverging tube shown on the back of this page. Consider the case where the upstream pressure P_U is fixed and the downstream pressure P_D is gradually lowered below the upstream level. Initially, with P_D slightly less than P_U, gas flow would be subsonic throughout the tube. Gas would speed up as it moved through the converging section and it would slow down as it moved through the diverging section. However, at some point as P_D is reduced, the speed at the throat would become sonic. Further reduction in P_D would create supersonic flow downstream of the throat: flow would remain sonic at the throat. As information waves cannot propagate faster than the speed of sound, they would be swept downstream by the supersonic flow and the mass flow rate would become independent of P_D. The flow is said to be choked. Usually, when flow is choked, shock waves form downstream of the throat. However, for a given P_U, there is an optimum P_D where they do not form.


P_U
P_D
P_U
P_D
P
x

NORMAL SHOCK WAVES

When there is supersonic flow around a blunt object, a normal shock wave can be found directly in front of the object. The flow could be generated by the speed of the object or by a blast. Consider flow through a small bit of area ΔA of the shock face. Conservation of mass gives

$$\dot{M} = \rho_1 U_1 \Delta A = \rho_2 U_2 \Delta A$$

while conservation of momentum gives

$$\dot{M} (U_2 - U_1) = (P_1 - P_2) \Delta A$$

Mass into momentum gives

$$\rho_2 U_2 \Delta A U_2 - \rho_1 U_1 \Delta A U_1 = (P_1 - P_2) \Delta A$$

$$P_1 + \rho_1 U_1 U_1 = P_2 + \rho_2 U_2 U_2$$

Manipulation gives

$$P_1 \; (1 + \rho_1/P_1 \; [U_1U_1]) \; = \; P_2 \; (1 + \rho_2/P_2 \; [U_2U_2])$$

$$P_1 \; (1 + [U_1U_1]/[RT_1]) \; = \; P_2 \; (1 + [U_2U_2]/[RT_2])$$

$$P_1 \; (1 + k \; [U_1U_1]/[kRT_1]) \; = \; P_2 \; (1 + k \; [U_2U_2]/[kRT_2])$$

$$P_1 \; (1 + k \; [U_1U_1]/[a_1a_1]) \; = \; P_2 \; (1 + k \; [U_2U_2]/[a_2a_2])$$

$$P_1 \; (1 + k \; M_1M_1) \; = \; P_2 \; (1 + k \; M_2M_2)$$

Conservation of energy gives

$$h_1 \; + \; [U_1U_1]/2 \; = \; h_2 \; + \; [U_2U_2]/2$$

Manipulation gives

$$T_1 \; (1 + (k-1)/2 \; M_1M_1) \; = \; T_2 \; (1 + (k-1)/2 \; M_2M_2)$$

Recall conservation of mass

$$\rho_1 \ U_1 \ \Delta A \ = \ \rho_2 \ U_2 \ \Delta A$$

Manipulation gives

$$\rho_2 \ / \ \rho_1 = \ U_1 \ / \ U_2$$

$$= [\ a_1 \ U_1/a_1 \ ] \ / \ [\ a_2 \ U_2/a_2 \ ] \ = \ a_1/a_2 \ \ M_1/M_2$$

$$= \ \sqrt{[kRT_1]} \ / \ \sqrt{[kRT_2]} \ \ M_1/M_2 \ = \ \sqrt{T_1/T_2} \ \ M_1/M_2$$

$$= \sqrt{\{[1 + (k-1)/2 \ M_2M_2] \ / \ [1 + (k-1)/2 \ M_1M_1]\}} \ \ M_1/M_2$$

The Ideal Gas Law gives

$$P_1 \ / \ [\rho_1 \ R \ T_1] \ = \ P_2 \ / \ [\rho_2 \ R \ T_2]$$

Manipulation gives

$$\rho_2/\rho_1 \ T_2/T_1 \ = \ P_2/P_1$$

Substitution of ratios into this gives

$$M_2M_2 = [(k-1)\ M_1M_1 + 2] \ / \ [2k\ M_1M_1 - (k-1)]$$

With this, the important ratio becomes

$$P_2/P_1 = 1 + 2k/(k+1)\ (M_1M_1 - 1)$$

$$T_2/T_1 = ([1+(k-1)/2\ M_1M_1][2k\ M_1M_1-(k-1)])/[(k+1)^2/2\ M_1M_1]$$

$$\rho_2/\rho_1 = [(k+1)\ M_1M_1] \ / \ [2 + (k-1)\ M_1M_1]$$

For air where k=7/5 the ratios become

$$M_2M_2 = [M_1M_1 + 5] \ / \ [7\ M_1M_1 - 1]$$

$$P_2/P_1 = [7\ M_1M_1 - 1] \ / \ 6$$

$$T_2/T_1 = ([M_1M_1 + 5]\ [7\ M_1M_1 - 1]) \ / \ [36\ M_1M_1]$$

$$\rho_2/\rho_1 = [12\ M_1M_1] \ / \ [10 + 2\ M_1M_1]$$

The Mach Number connection predicts compression shocks with M_1 greater than unity and M_2 less than unity and expansion shocks with M_1 less than unity and M_2 greater than unity. The Second Law of Thermodynamics shows that compression shocks are possible but expansion shocks are impossible. The Second Law gives for entropy

$$S_2 - S_1 = C_P \ln[T_2/T_1] - R \ln[P_2/P_1]$$

where

$$T_2 / T_1 = [2 + (k-1) M_1M_1] / [2 + (k-1) M_2M_2]$$

$$P_2 / P_1 = (1 + k M_1M_1) / (1 + k M_2M_2)$$

For a compression shock this shows that ΔS is greater than zero, which is possible, whereas for an expansion shock it shows that ΔS is less than zero, which is impossible. In other words, a supersonic flow can suddenly go subsonic, but a subsonic flow cannot suddenly go supersonic.

OBLIQUE SHOCK WAVES

Oblique shock waves form on objects such as supersonic foils at a low angle of attack. The shock turns the flow and makes it parallel to the foil. The sketch on the next page shows the oblique shock geometry. Only the component of the flow normal to the shock is changed by passage through the shock. Because pressure is basically constant either side of the shock, the tangential component of the flow is not changed. The normal component is changed by the amount needed to make the flow align with the foil. Properties depend on the Mach Numbers based on the normal component of the flow. These are:

$$N_1 = U_{1N}/a_1 = U_1 Sin[\beta]/a_1 = M_1 \ Sin[\beta]$$

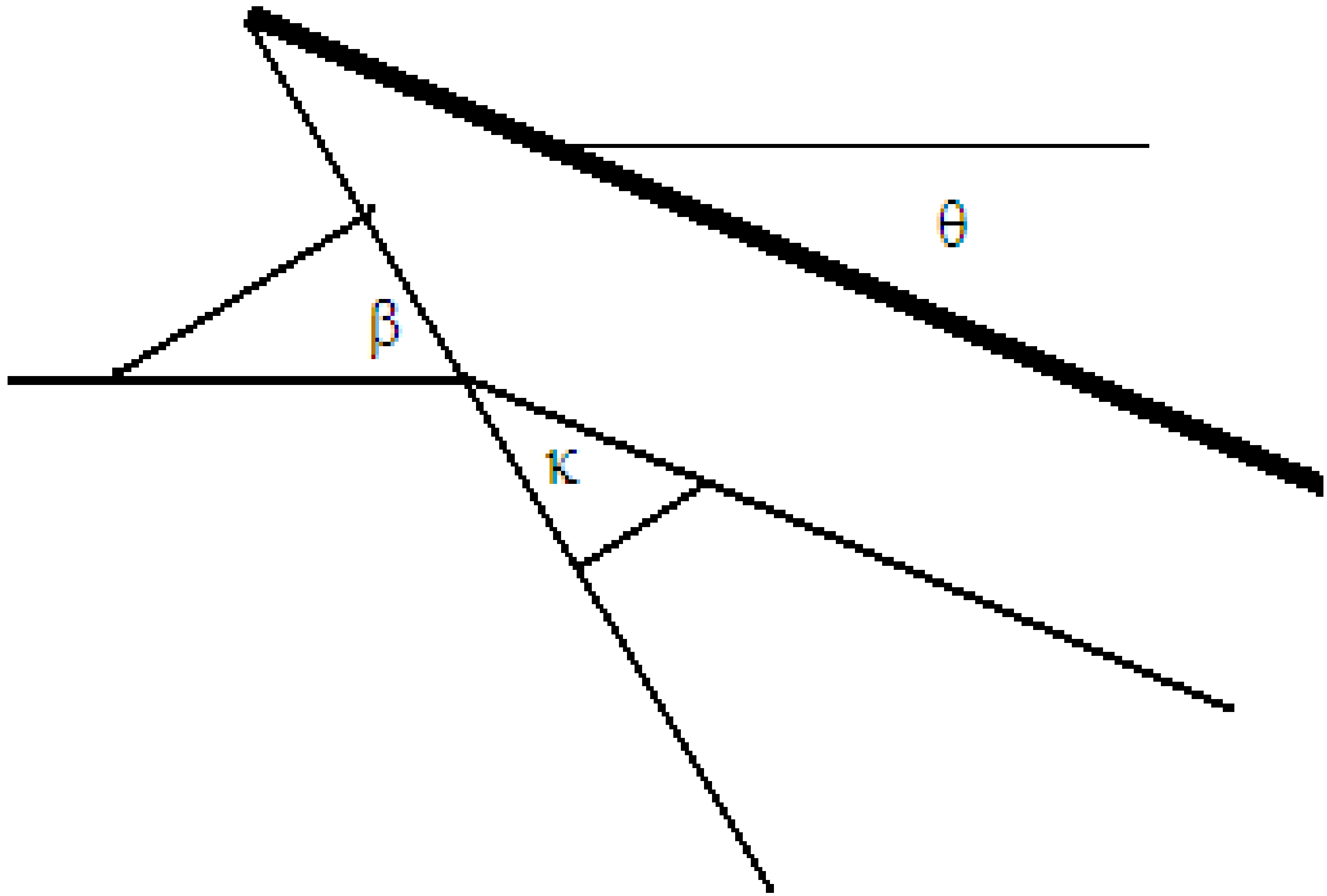
$$N_2 = U_{2N}/a_2 = U_2 Sin[\beta-\theta]/a_2 = M_2 \ Sin[\kappa]$$

where β is the shock angle and θ is the angle of attack of the foil. The property ratio equations become:

$$P_2/P_1 = 1 + 2k/(k+1) \ (N_1N_1 - 1)$$

$$T_2/T_1 = ([1+(k-1)/2 \ N_1N_1][2k \ N_1N_1-(k-1)])/[(k+1)^2/2 \ N_1N_1]$$

$$\rho_2/\rho_1 = [(k+1) \ N_1N_1] \ / \ [2 + (k-1) \ N_1N_1]$$


θ
β
κ

The Mach Number connection becomes:

$$N_2N_2 = [(k-1)\ N_1N_1 + 2] / [2k\ N_1N_1 - (k-1)]$$

Conservation of mass gives

$$\rho_1\ U_{1N}\ \Delta A = \rho_2\ U_{2N}\ \Delta A$$

$$\rho_2 / \rho_1 = U_{1N} / U_{2N}$$

Geometry gives

$$U_{1N}/U_{1T} = \mathrm{Tan}[\beta] \qquad U_{2N}/U_{2T} = \mathrm{Tan}[\kappa]$$

$$U_{1N} / U_{2N} = \mathrm{Tan}[\beta] / \mathrm{Tan}[\kappa]$$

Thermodynamics gives

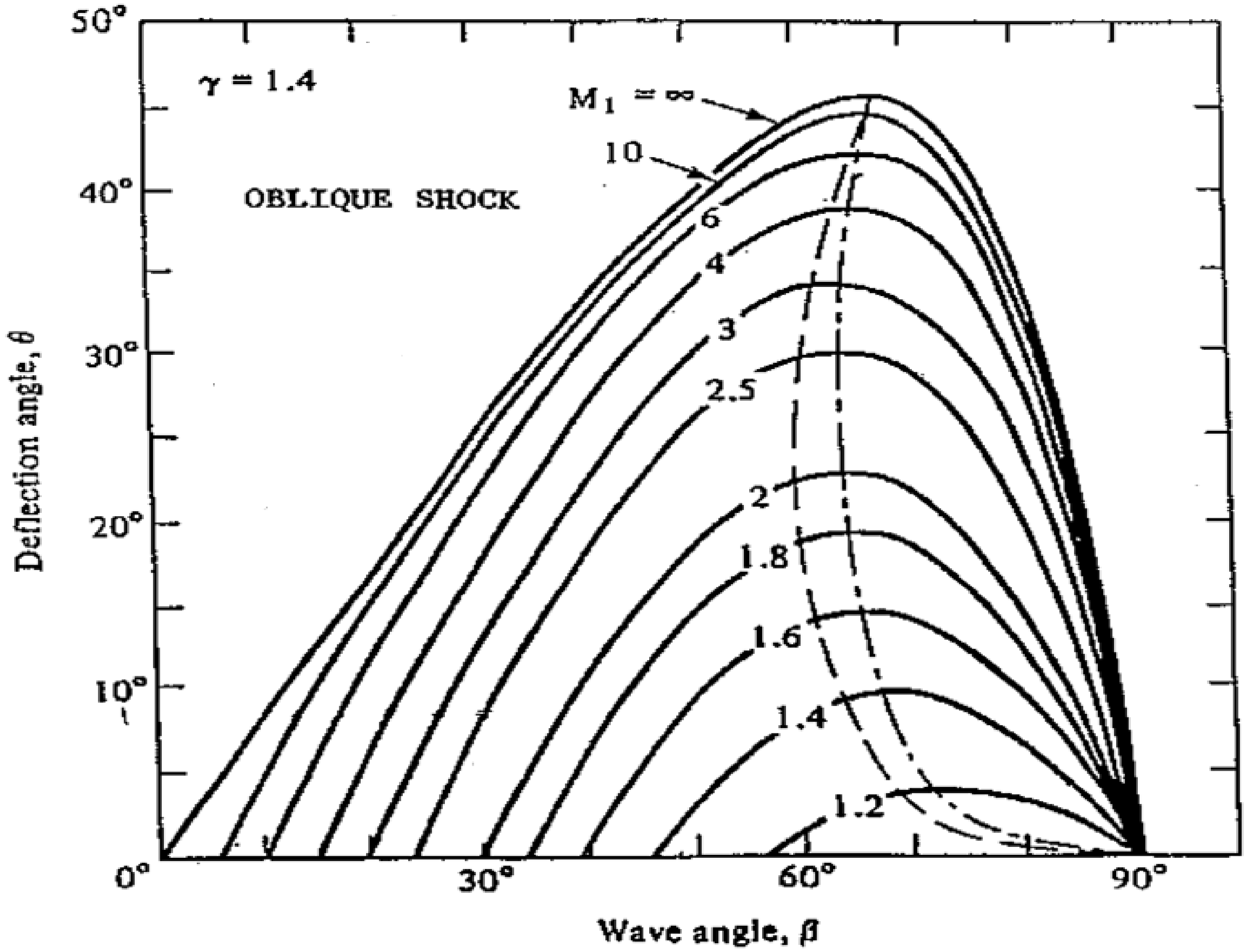
$$\rho_2/\rho_1 = [(k+1)\ N_1N_1] / [2 + (k-1)\ N_1N_1]$$

Manipulation gives

$$\mathrm{Tan}[\beta]/ \mathrm{Tan}[\kappa] = (k+1)\ N_1N_1 / [(k-1)\ N_1N_1 + 2]$$

$$= (k+1)\ M_1M_1\ \mathrm{Sin}^2[\beta] / [(k-1)\ M_1M_1\ \mathrm{Sin}^2[\beta] + 2]$$

A plot of this equation is given on the next page.



50°
40°
30°
20°
10°
0°
30°
60°
90°
γ = 1.4
OBLIQUE SHOCK
M₁ = ∞
10
6
4
3
2.5
2
1.8
1.6
1.4
1.2
Deflection angle, θ
Wave angle, β

EXPANSION WAVES

An expansion wave is basically a fan of Mach waves. Processes within such waves are isentropic. Expansion theory considers an expansion to be made up of an infinite number of infinitesimal expansions. The sketch on the next page shows the geometry of a Mach wave. The pressure either side of such a wave is basically constant. So the tangential component of fluid motion is unchanged when fluid crosses it. Only the normal component is changed. Geometry gives for the tangential component:

$$U \, \mathrm{Cos}[\mu] = (U+\Delta U) \, \mathrm{Cos}[\mu+\Delta\upsilon]$$

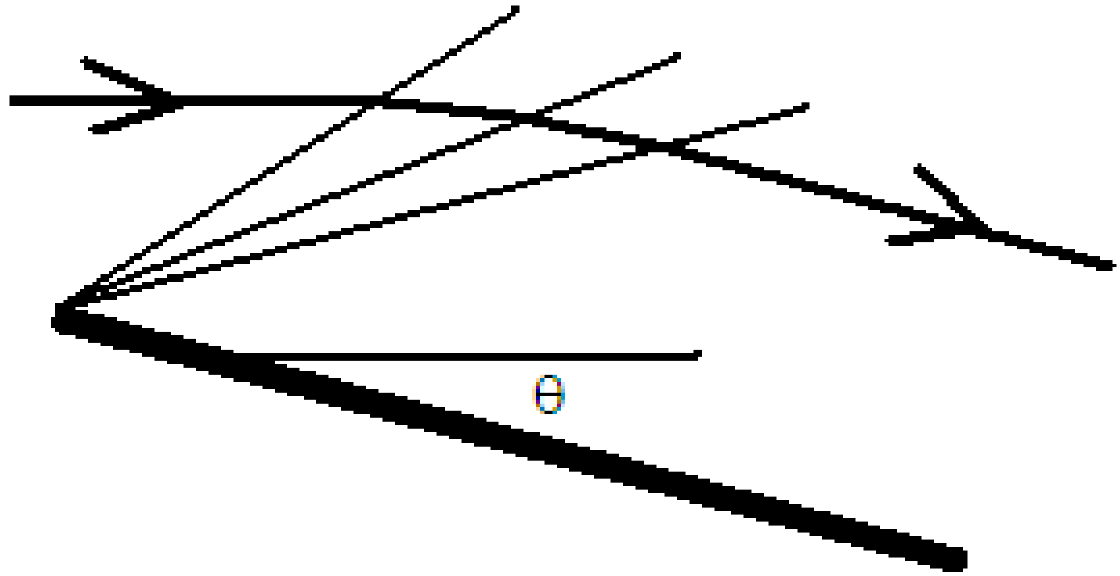
Trigonometry gives

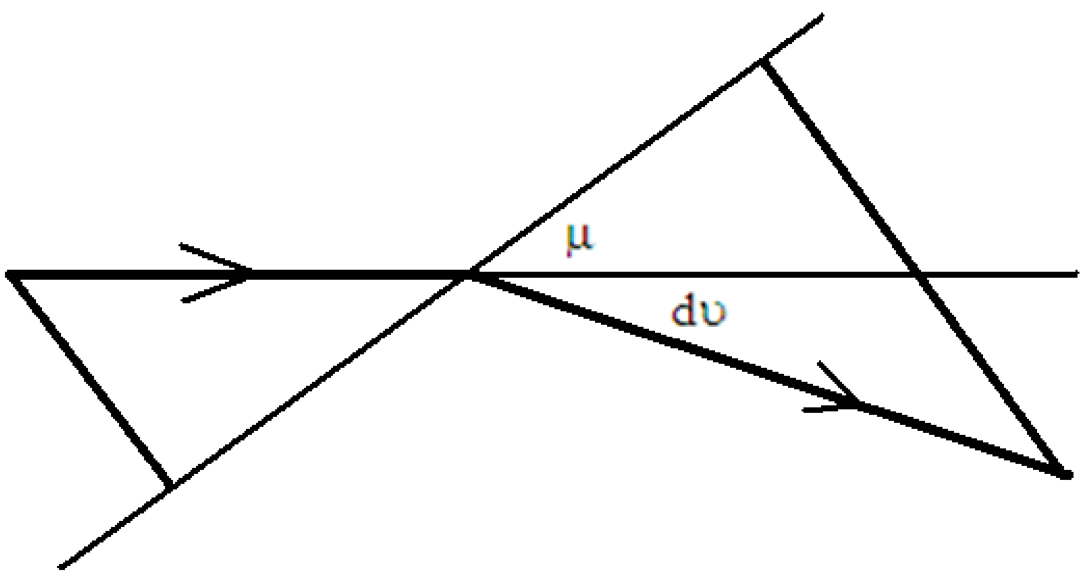
$$\mathrm{Cos}[\mu+\Delta\upsilon] = \mathrm{Cos}[\mu] \, \mathrm{Cos}[\Delta\upsilon] - \mathrm{Sin}[\mu] \, \mathrm{Sin}[\Delta\upsilon]$$

Small angle approximation gives

$$\mathrm{Cos}[\mu+\Delta\upsilon] = \mathrm{Cos}[\mu] \; 1 - \mathrm{Sin}[\mu] \; \Delta\upsilon$$

$$= \mathrm{Cos}[\mu] - \Delta\upsilon \,/\, M$$


θ


μ
dυ

The tangential equation becomes

$$U \cos[\mu] = (U+\Delta U)\ (\cos[\mu] - \Delta\upsilon/M)$$

Manipulation gives

$$\Delta\upsilon = M \cos[\mu]\ \Delta U/U$$

$$= M \sqrt{(1-\sin^2[\mu])}\ \Delta U/U = \sqrt{(M^2-M^2/M^2)}\ \Delta U/U$$

$$= \sqrt{(M^2-1)}\ \Delta U/U$$

The Mach Number is

$$M = U/a = U / \sqrt{kRT}$$

Manipulation gives

$$U = M \sqrt{kRT}$$

Differentiation gives

$$\Delta U = \Delta M \sqrt{kRT} + M / [2\sqrt{kRT}] \; kR \; \Delta T$$

Manipulation gives

$$\Delta U/U = \Delta M/M + 1/2 \; \Delta T/T$$

Conservation of energy gives

$$U^2/2 + h = K$$

$$= U^2/2 + C_P \; T$$

$$= U^2/2 + kR/(k-1) \; T$$

Differentiation gives

$$U \; \Delta U + kR/(k-1) \; \Delta T = 0$$

Manipulation gives

$$\Delta U/U \quad + \quad 1/(k-1)\ [kRT]/U^2\ \ \Delta T/T \quad = \quad 0$$

$$\Delta U/U \quad + \quad 1/(k-1)\ 1/M^2\ \Delta T/T \quad = \quad 0$$

$$\Delta T/T \quad = \ -\ (k-1)\ M^2\ \Delta U/U$$

Substitution into the $\Delta U/U$ equation gives

$$\Delta U/U \ = \ \Delta M/M \quad - \quad (k-1)/2\ M^2\ \Delta U/U$$

$$\Delta U/U \ = \ 1\ /\ [\ 1 + (k-1)/2\ M^2\ ]\ \ \Delta M/M$$

Substitution into the geometry equation gives

$$\Delta \upsilon \ = \ \sqrt{(M^2-1)} \quad \Delta U/U$$

$$= \ \sqrt{(M^2-1)}\ \ /\ [\ 1 + (k-1)/2\ M^2\ ]\ \ \Delta M/M$$

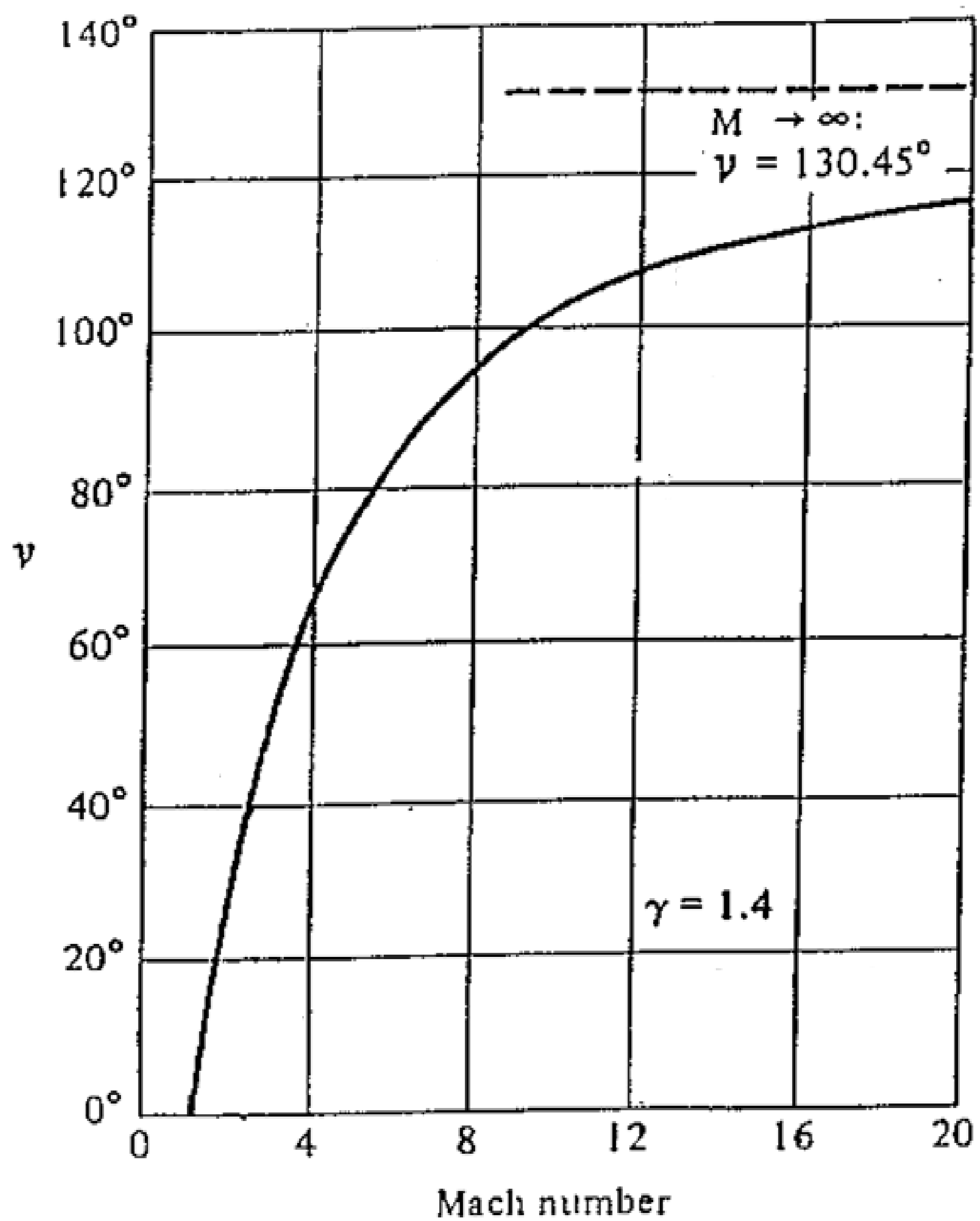
Integration of this equation gives

$$\upsilon = \sqrt{K}\ \tan^{-1}\sqrt{[(M^2-1)/K]} - \tan^{-1}\sqrt{[M^2-1]}$$

$$K = (k+1)/(k-1)$$

A plot of this equation is given on the next page.

EXPANSION WAVE



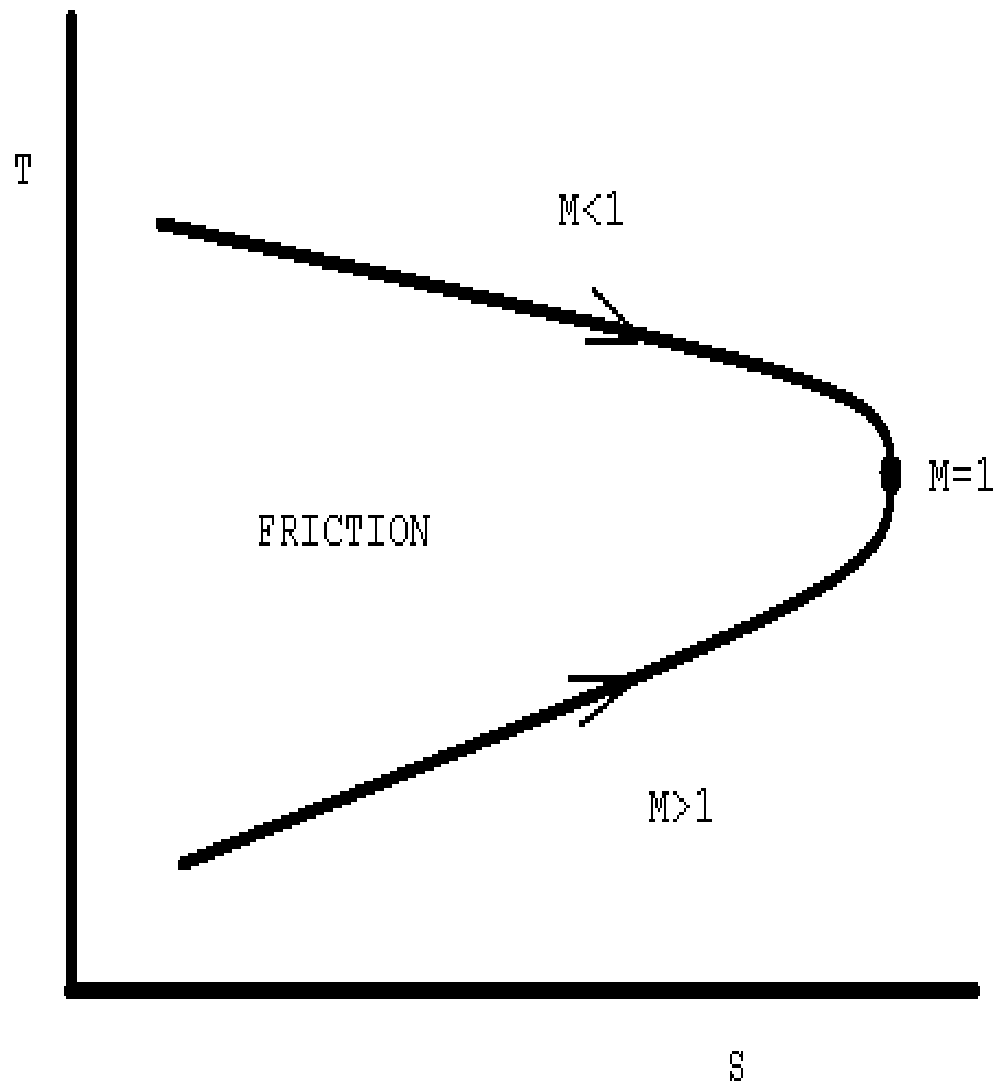
STEADY COMPRESSIBLE FLOW IN LONG PIPES

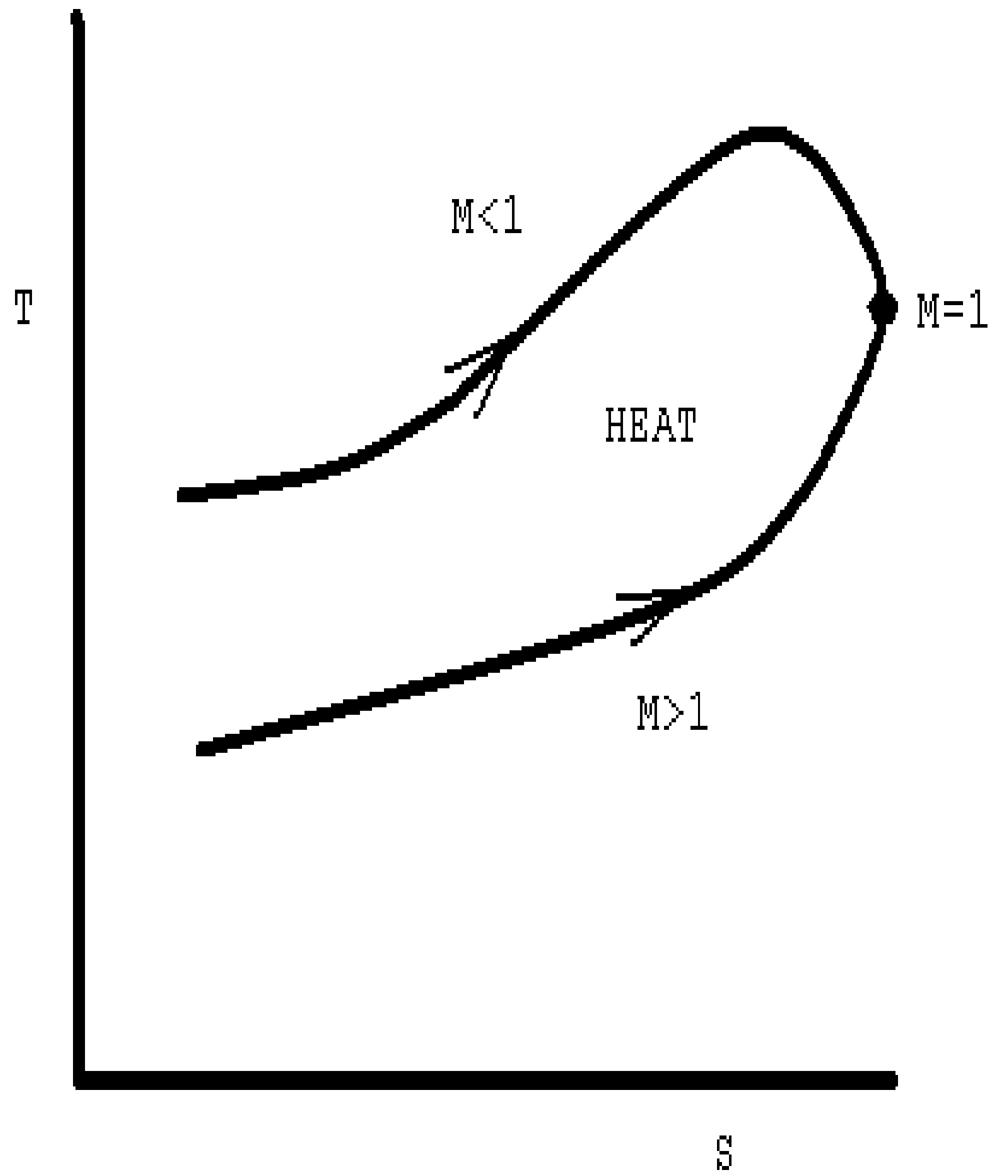
FANNO LINE FLOW

When a gas is flowing in a pipe, wall friction can cause the flow to choke. For subsonic flow, M increases to 1 while for supersonic flow, it decreases to 1. For subsonic flow, extra length beyond that needed to choke flow causes flow reduction. For supersonic flow, extra length causes a shock wave to form in pipe.

RAYLEIGH LINE FLOW

When heat is added to the flow in a pipe, one finds that it can cause the flow to choke. Once the flow is choked, more heat addition causes a flow reduction. Heat removal reverses the process. It is interesting that heat addition for a subsonic flow can cause its temperature to drop.


T
M<1
M=1
FRICTION
M>1
S


M<1
T
M=1
HEAT
M>1
S

PIPE FLOWS WITH FRICTION

For compressible flow in a pipe, the differential equations are

$$dP/P = -kM^2[1+(k-1)\ M^2]/[2(1-M^2)]\ fdx/D$$

$$dT/T = -k(k-1)M^4/[2(1-M^2)]\ fdx/D$$

$$d\rho/\rho = -kM^2/[2(1-M^2)]\ fdx/D$$

$$dM^2/M^2 = kM^2[1+(k-1)/2M^2]\ /[1-M^2]\ fdx/D$$

These equations can be integrated numerically to get how the various unknowns change down the pipe. The changes are usually tabulated in fluids texts. One finds that if $M<1$ it gradually increases to 1 whereas if $M>1$ it gradually decreases to 1. One finds that there is critical length L^* that the flow must travel to become sonic. For adiabatic flow the critical length follows from

$$fL^*/D = (1-M^2)/(kM^2) + (k+1)/(2k)\ \ln[(k+1)M^2/(2+(k-1)M^2)]$$

while for isothermal flow it follows from

$$fL^*/D = (1-kM^2)/(kM^2) + \ln[kM^2]$$