FLUIDS NOTES

These notes give an overview of basic fluid mechanics. They cover Fluids at Rest or Fluid Statics and Fluids in Motion or Fluid Dynamics.

FLUIDS AT REST

CONCEPTS

PRESSURE DEPTH LAW

Consider an imaginary vertical cylinder of water extending down from an interface between air and water. A schematic is shown on the next page. Let the cross sectional area of the cylinder is A and let its height be h. Let the pressure at the top be P_o and the pressure at the bottom be P.

The weight of the cylinder is

$$W = \rho\ g\ V = \rho\ g\ A\ h$$

The pressure load on the cylinder is

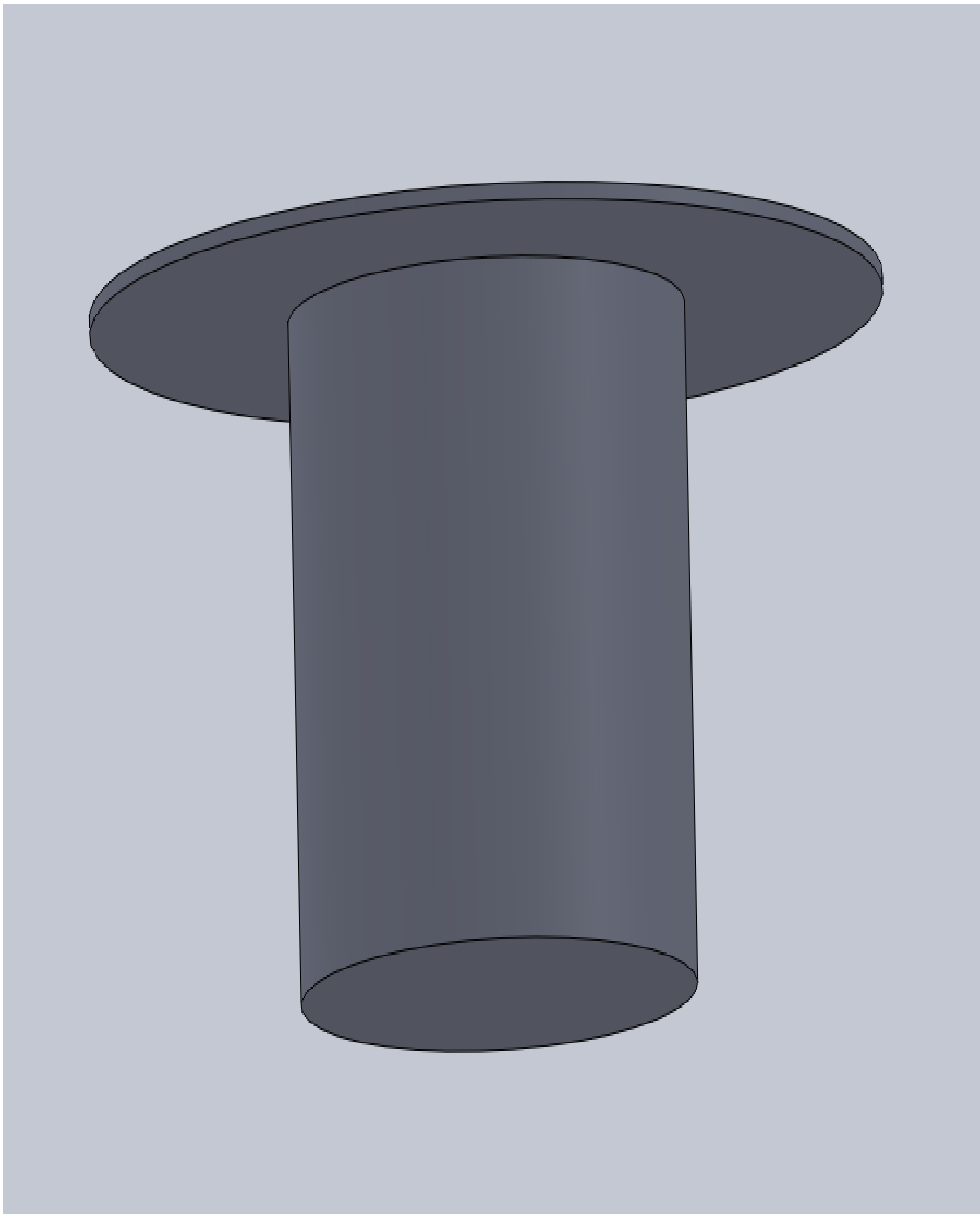
$$P\ A - P_o\ A$$

A load balance gives

$$P\ A - P_o\ A - \rho\ g\ A\ h\ =\ 0$$

$$\Delta P = P - P_o = \rho\ g\ h$$

This is the pressure depth law for water.



BUOYANCY

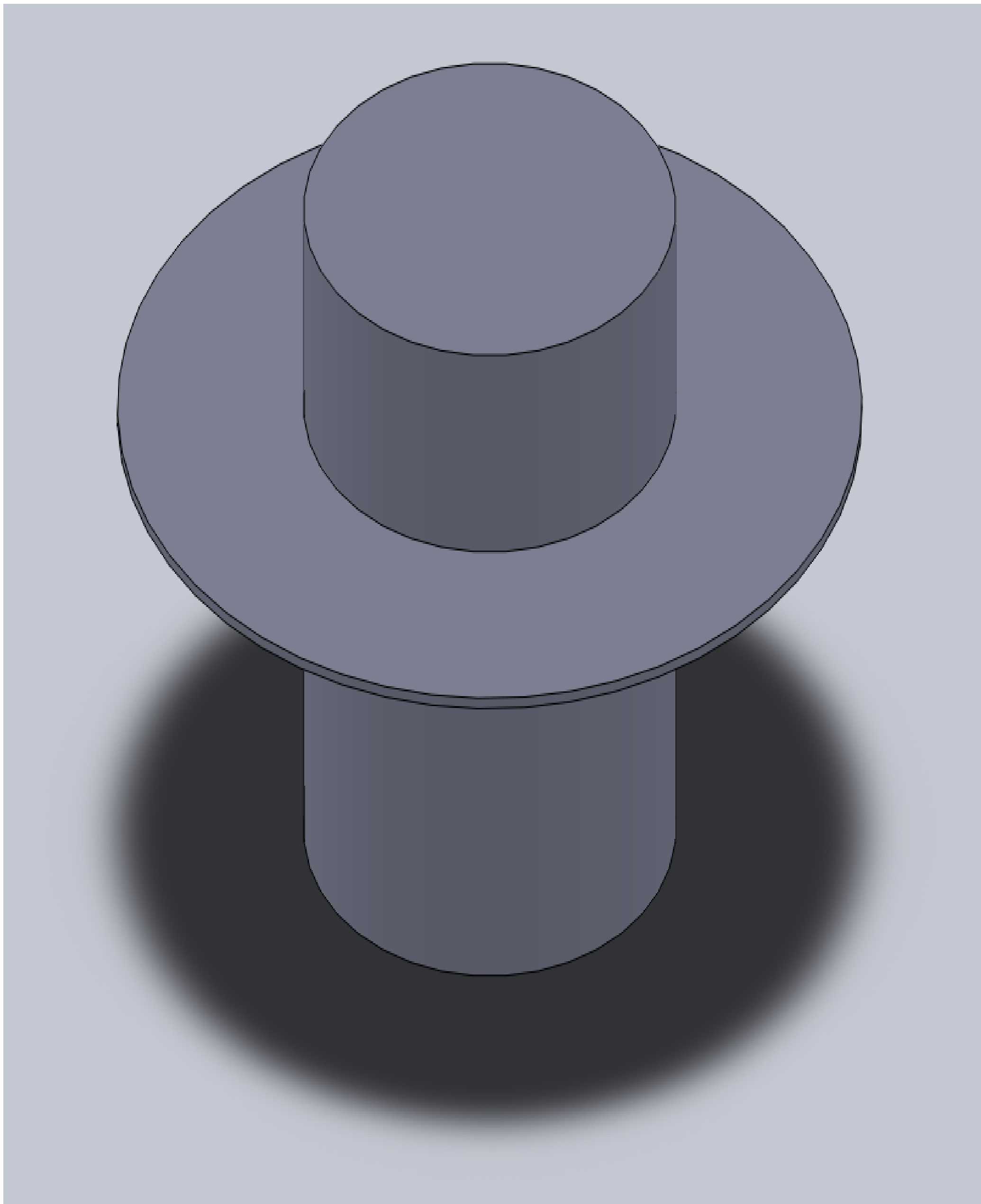
Consider an imaginary vertical cylinder of water extending down from an interface between air and water. A schematic is shown on the previous page. Let the cross sectional area of the cylinder be A and let its height be h. Let the pressure at the top be P_o and the pressure at the bottom be P.

The pressure load on the cylinder is

$$B = P\ A - P_o\ A = \Delta P\ A$$

$$= \rho g h\ A = \rho g\ V = W$$

The pressure load is known as the buoyancy. It is equal to the weight of the displaced volume of water. A floating cylinder would have a part below water and a part above water. A schematic is shown on the next page. The part below water would displace a volume of water with a buoyancy force equal to the body weight.



BUOYANCY SPRING

The buoyancy force can be written as

$$B = \rho g A\ h$$

This resembles a spring

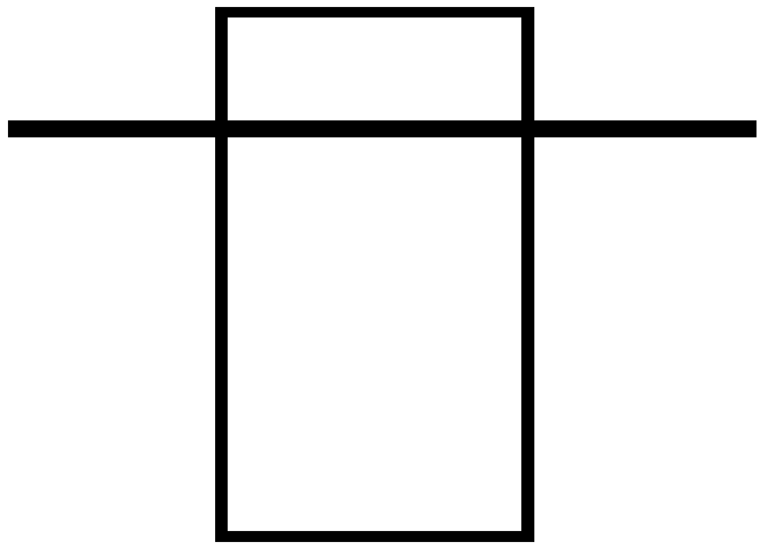
$$F = K\ x$$

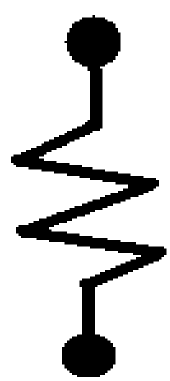
The buoyancy spring constant is

$$K = \rho g A$$

A schematic of the analogy is on the next page.

Pushing the cylinder downwards increases the buoyancy force. It compresses the buoyancy spring. It pushes the bottom down to a level where the pressure is higher and that creates an increase in the force upwards.





STABILITY

To study the concept of stability, consider a rig with 4 legs that are spaced far apart. A schematic is shown on the next page. Let the spacing of the legs be 2H. Let the area of each leg be A and let the depth of submergence be h. Because H is large, when the rig rolls an angle θ, the displaced volume at the bottom of each leg is approximately a cylinder with area A and height Hθ. The torque due to buoyancy is

$$\Delta T = 4\ H\ \rho g\ A\ H\theta$$

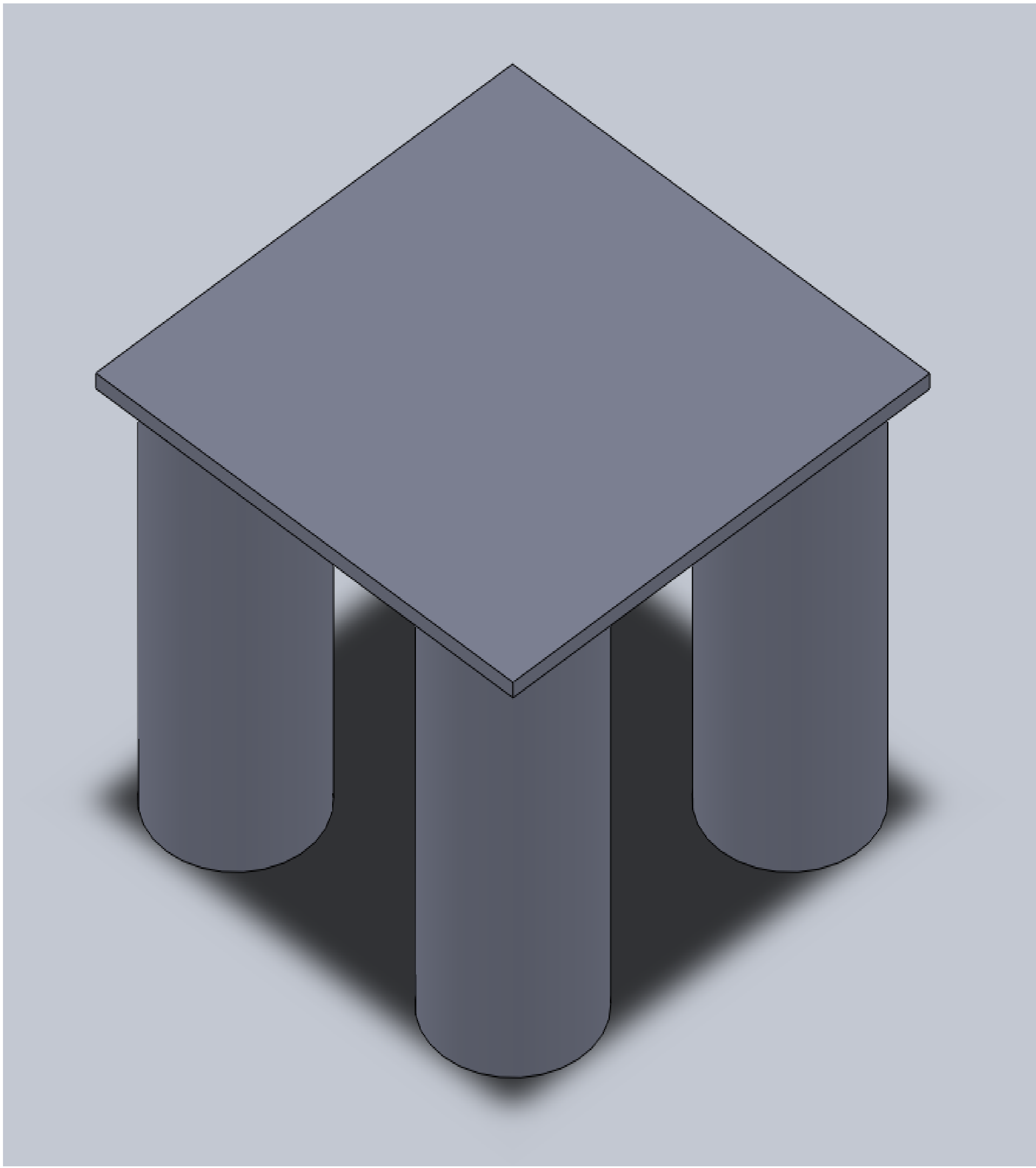
The rotation causes a shift S in the center of the displaced volume and the torque due to this is

$$\Delta T = S\ 4\ \rho g\ Ah$$

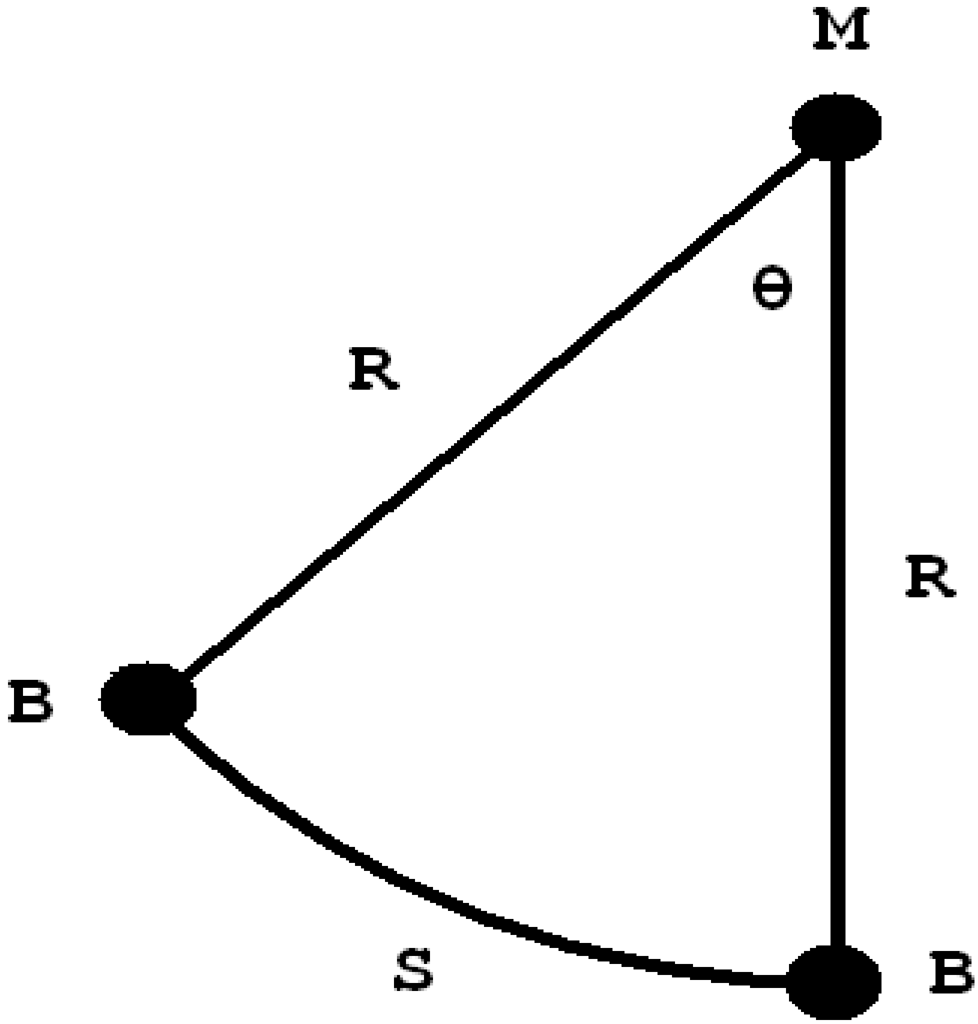
A torque balance gives

$$S\ 4\ \rho g\ Ah = 4\ H\ \rho g\ A\ H\theta$$

$$S = H^2/h\ \theta = R\ \theta$$



This suggests that the center of buoyancy has moved along a circular arc with radius R. A schematic of this shift is shown in the sketch on the next page. The radius is known as the metacentric radius. The center of rotation is known as the metacenter. The line of action of the buoyancy force always passes through this point. Knowing the location of the metacenter allows us to locate the line of action of the buoyancy force. If the center of gravity is below the metacenter, the weight and buoyancy forces create a restoring moment and the rig is stable. If the center of gravity is above the metacenter, the weight and buoyancy forces create an overturning moment and the rig is unstable.


M
θ
R
R
B
S
B

FLUIDS AT REST

SUBMERGED

SURFACES

HYDRAULIC GATES

To get loads on hydraulic gates one can break its surface up into an infinite number of infinitesimal bits of surface. The force on an infinitesimal bit of surface is:

$$d\mathbf{F} = P\ ds\ \mathbf{n}$$

where **n** is the inward normal on the surface and P is the pressure acting on it. The normal **n** is:

$$\mathbf{n} = n_x\ \mathbf{i} + n_y\ \mathbf{j} + n_z\ \mathbf{k}$$

where **ijk** indicates unit normal vectors. The force can be broken down into xyz components

$$d\mathbf{F} = dF_x\ \mathbf{i} + dF_y\ \mathbf{j} + dF_z\ \mathbf{k}$$

$$= P\ ds\ n_x\ \mathbf{i} + P\ ds\ n_y\ \mathbf{j} + P\ ds\ n_z\ \mathbf{k}$$

The pressure depth law gives

$$P = \rho g\ h$$

The total force can be obtained by integration of the component forces over the total surface:

$$F_x = \int P\ n_x\ ds \qquad F_y = \int P\ n_y\ ds \qquad F_z = \int P\ n_z\ ds$$

The total force is

$$\mathbf{F} = F_x\ \mathbf{i} + F_y\ \mathbf{j} + F_z\ \mathbf{k}$$

$$|\mathbf{F}| = \sqrt{\ [\ [F_x]^2 + [F_x]^2 + [F_x]^2\ ]}$$

Moment balances give the location of the forces.

The panel method for hydraulic gates starts by subdividing the surface of the gate into a finite number of finite size flat panels. The pressure depth law gives the pressure at the centroid of each panel. Pressure times panel area gives the force at the centroid. The unit normal pointing at the panel allows one to break the force into components. Summation gives the total force on the gate in each direction.

$$F_x = \sum P \, n_x \, \Delta s \qquad F_y = \sum P \, n_y \, \Delta s \qquad F_z = \sum P \, n_z \, \Delta s$$

The total force is

$$\mathbf{F} = F_x \, \mathbf{i} + F_y \, \mathbf{j} + F_z \, \mathbf{k}$$

$$|\mathbf{F}| = \sqrt{\ } \, [\, [F_x]^2 + [F_x]^2 + [F_x]^2 \,]$$

Moment balances give the location of the forces.

The pressure/weight method for hydraulic gates starts by boxing the gate with vertical and horizontal surfaces. The fluid within these surfaces is considered frozen to the gate. Then the horizontal and vertical pressure forces on the box surfaces are calculated. Force balances, which subtract the weight frozen to the gate, then give the horizontal and vertical forces on the gate. The total force is

$$\mathbf{F} = F_x \, \mathbf{i} + F_y \, \mathbf{j} + F_z \, \mathbf{k}$$

$$|\mathbf{F}| = \sqrt{\ } \, [\, [F_x]^2 + [F_x]^2 + [F_x]^2 \,]$$

Moment balances give the location of the forces.

HORIZONTAL FLAT GATE

The pressure acting on the gate is

$$\rho g\ H$$

The total force on the gate is:

$$\rho g\ H\ A$$

VERTICAL RECTANGULAR FLAT GATE

For a horizontal slice of the gate, the pressure is

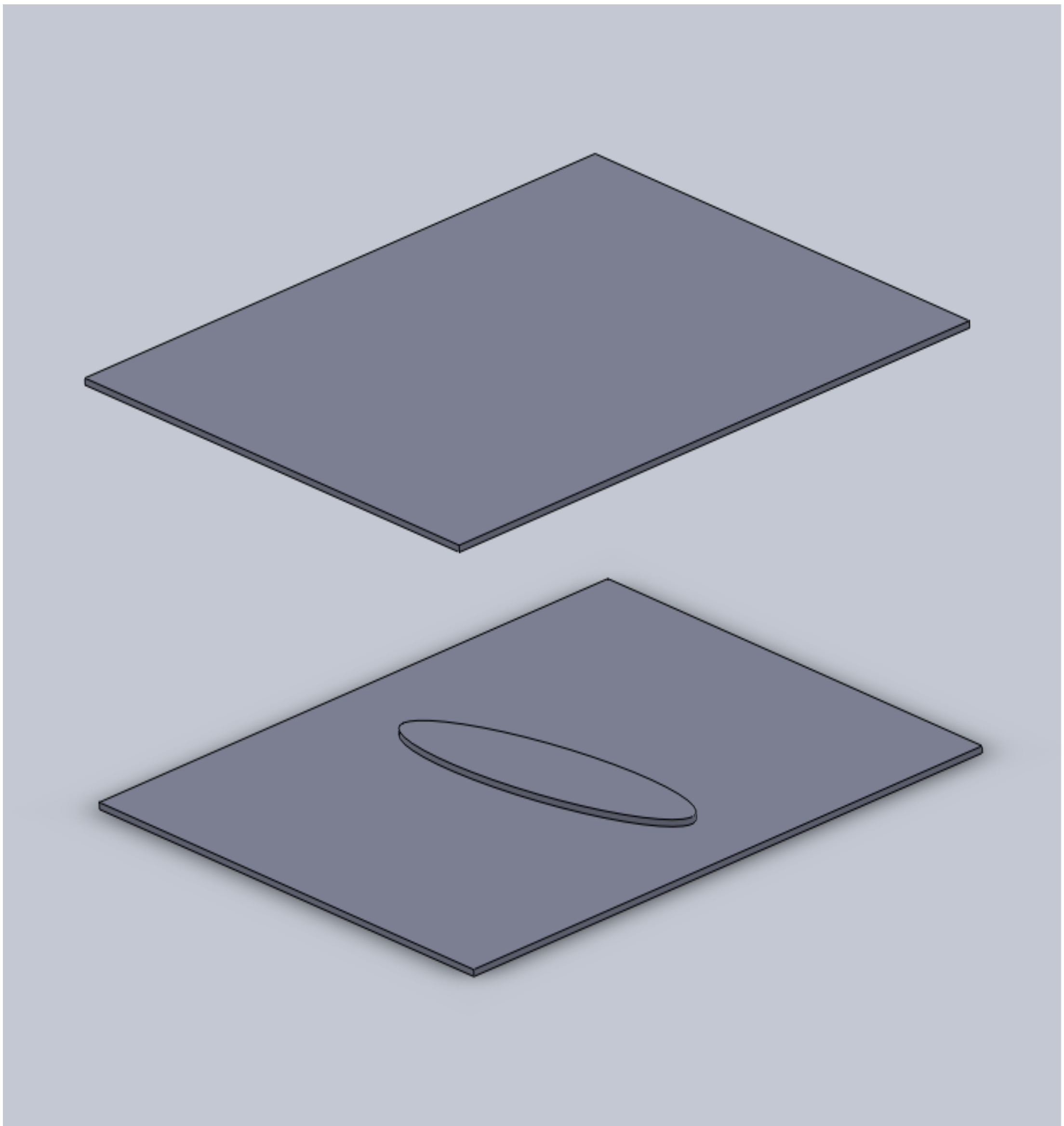
$$\rho g\ (H + r)$$

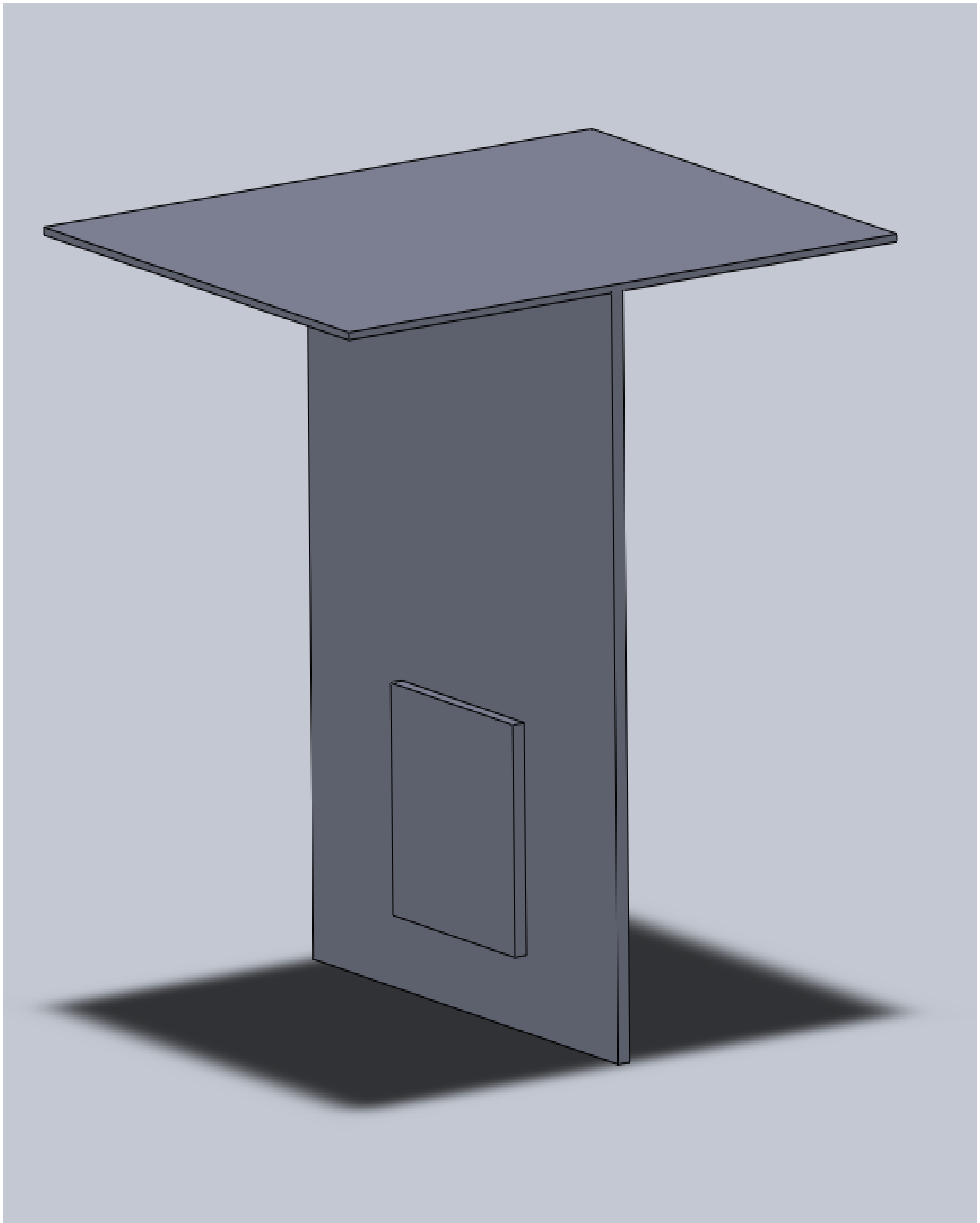
The area the pressure acts over is:

$$W\ dr$$

The total force on the gate is:

$$\int_{-G}^{+G} W\ \rho g\ (H + r)\ dr$$





Evaluation of the integral gives

$$\rho g\ H\ 2G\ W$$

VERTICAL CIRCULAR FLAT GATE

For a horizontal slice of the gate the pressure is

$$\rho g\ (H + r)$$

The area the pressure acts over is:

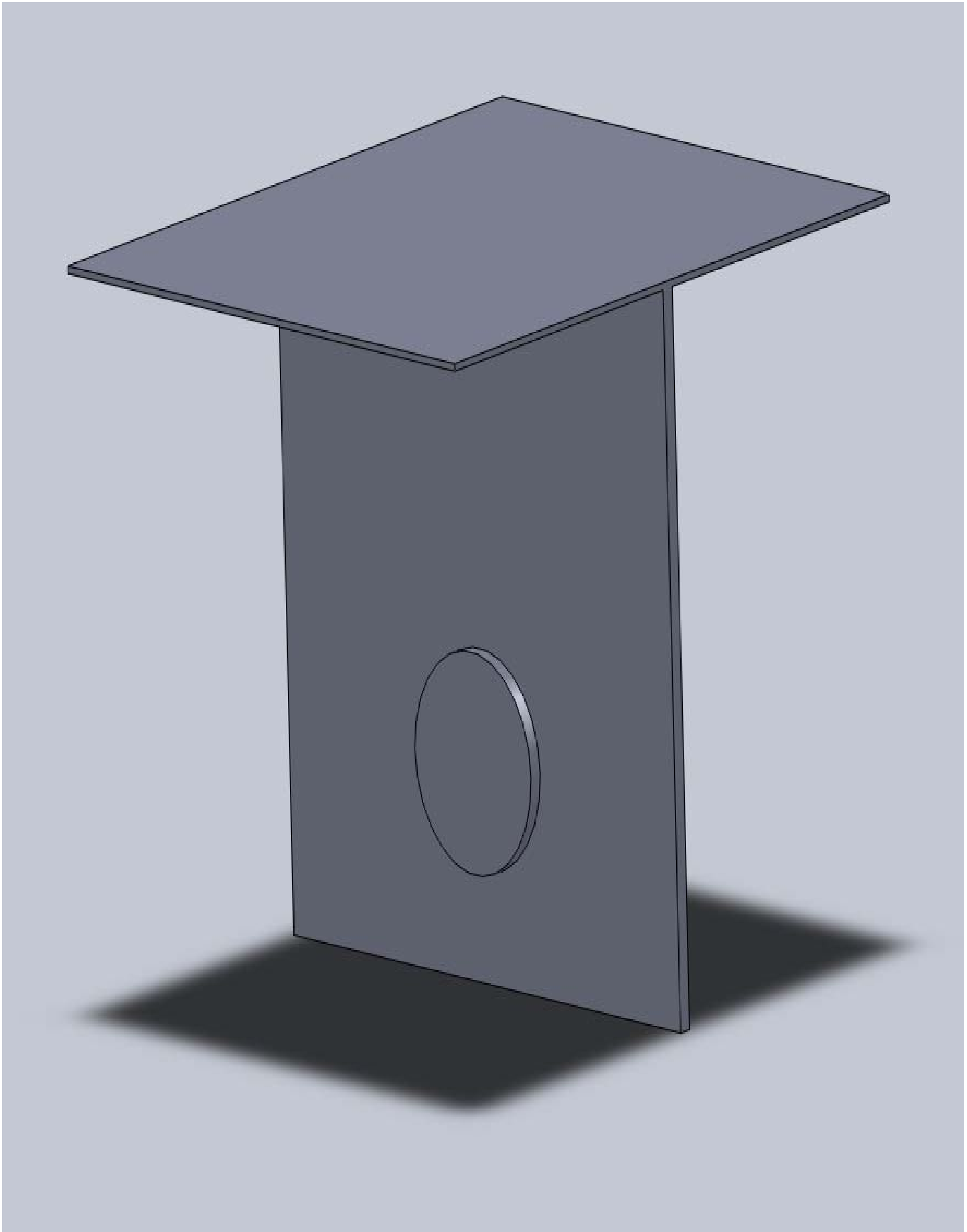
$$2\ \sqrt{[G^2-r^2]}\ \ dr$$

The total force on the gate is:

$$\int_{-G}^{+G} \rho g\ (H + r)\ 2\ \sqrt{[G^2-r^2]}\ \ dr$$

Evaluation of the r integral gives

$$\rho g\ H\ \pi G^2$$



HEMISPHERICAL SIDE GATE

For a horizontal slice of the gate the pressure is

$$\rho g \; (H - G \cos\theta)$$

Angle θ is measured from top to bottom (like latitude on earth). The area the pressure acts over is:

$$G \, d\theta \;\; G \sin\theta \;\; \pi$$

The total vertical force on the gate is:

$$\int_0^{\pi} [\; \rho g \; (H - G \cos\theta) \;\; G \;\; G \sin\theta \;\; \pi \; [-\cos\theta] \;] \; d\theta$$

Evaluation of the integral gives

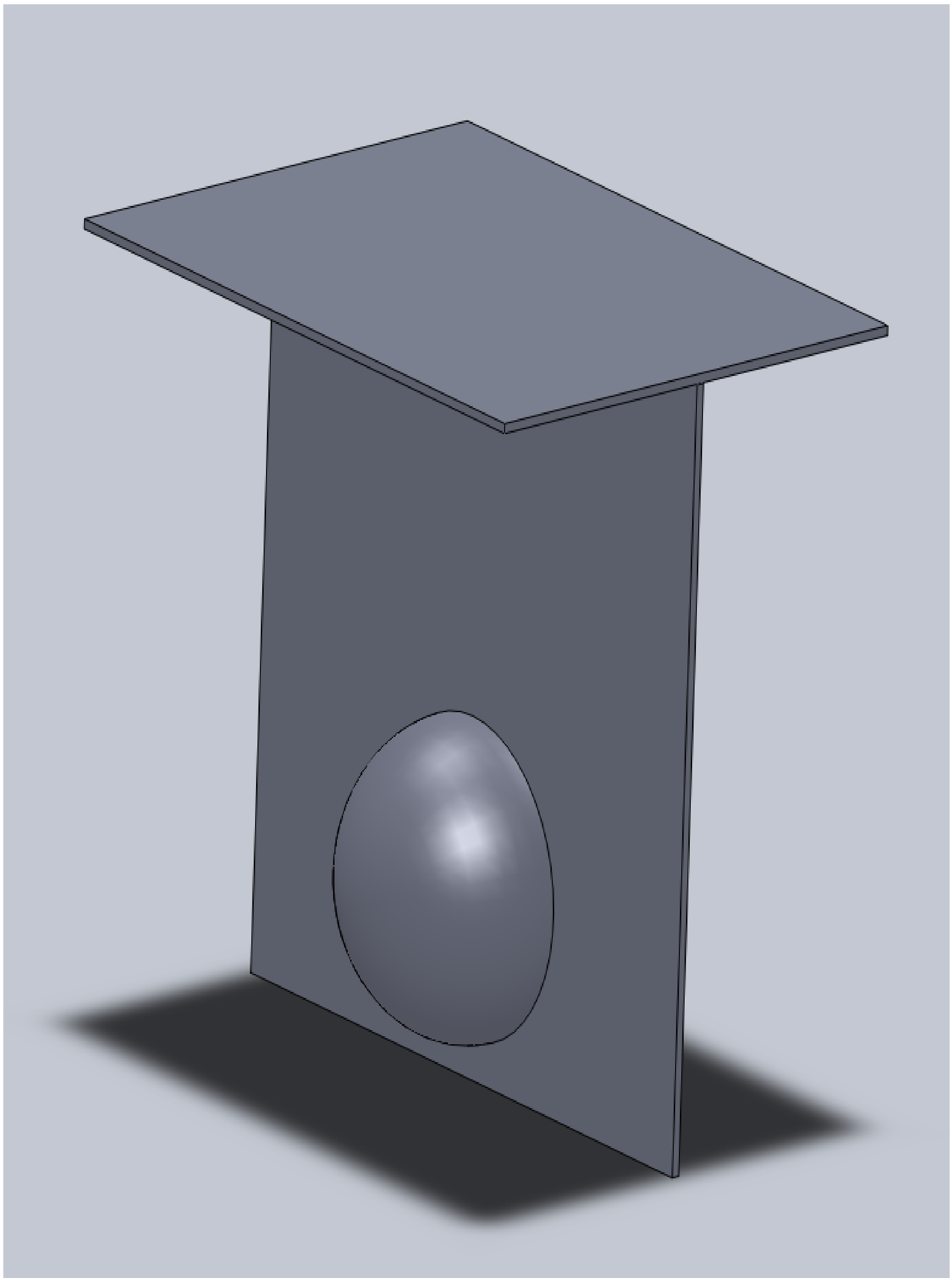
$$\rho g \; [4/3 \; \pi \; G^3] \; / \; 2$$

The horizontal force on the gate is:

$$\int_{-\pi/2}^{+\pi/2} \int_0^{\pi} [\; \rho g \; (H - G \cos\theta) \; G \;\; G \sin\theta \;\; [+\sin\theta] \; d\theta \;] \; \cos\sigma \; d\sigma$$

Angle σ is measured around the slice (like longitude on earth). Evaluation of the integral gives

$$\rho g \; H \; \pi \; G^2$$



HEMISPHERICAL WATER TANK

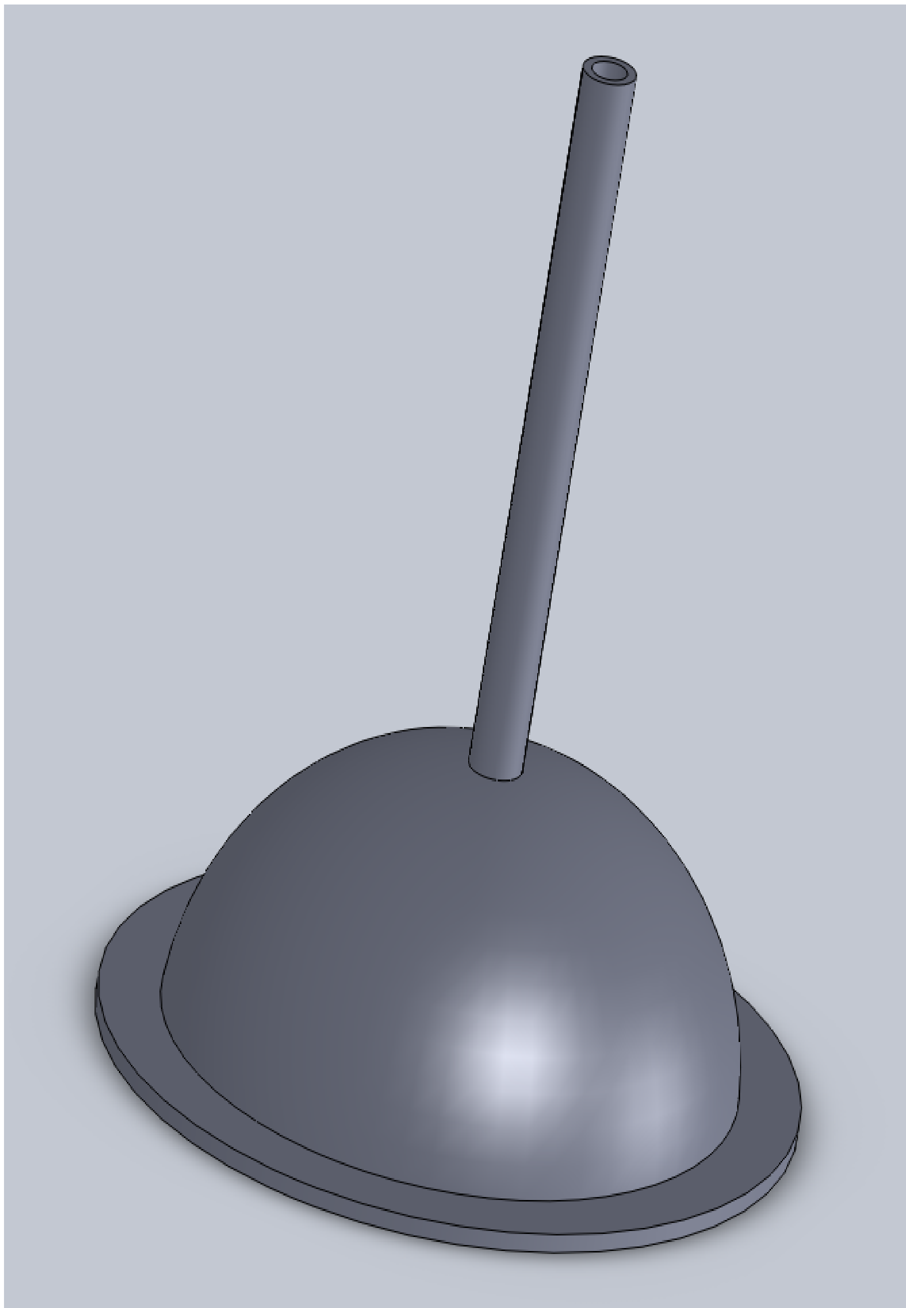
A certain hemispherical water tank sits on a concrete foundation. The tank diameter is 5m. At the top of the tank, there is a small diameter vertical fill tube that is open at the bottom to the tank and open at the top to the atmosphere. The water level in the tube is 5m above the top of the tank. Using the Pressure Weight Method, calculate the vertical force in wall at the base of the tank needed to counteract hydrostatic pressure load. Check your answer using the Panel Method and also using Analytical Integration. The tank wall is 1cm thick and is made out of steel. Calculate the force on the concrete foundation.

Pressure Weight Method: Imagine the tank wall is cut just where it joins the bottom plate. A free body diagram shows that the force balance on tank and water above the cut gives: wall force plus pressure force minus water weight minus wall weight must total to zero. The forces are:

$$\text{pressure force} : \rho g\ H\ \pi\ G^2$$

$$\text{water weight} : \rho g\ [4/3\ \pi\ G^3]\ /\ 2$$

$$\text{wall weight} : \sigma g\ [4\pi\ G^2]\ t\ /\ 2$$



Analytical Integration Method: For a horizontal slice of the tank the pressure is:

$$\rho g \ (H - G \ Cos\theta)$$

Angle θ is measured from top to bottom (like latitude on earth). The area the pressure acts over is:

$$G \ d\theta \quad G \ Sin\theta \quad 2\pi$$

The vertical direction is $+Cos\theta$. The vertical force is:

$$\int_{0}^{+\pi/2} \rho g \ (H - G \ Cos\theta) \quad G \quad G \ Sin\theta \quad 2\pi \ Cos\theta \quad d\theta$$

Evaluation of the integral gives:

$$\rho g \ H \ \pi \ G^2 \ - \ \rho g \ [4/3 \ \pi \ G^3] \ / \ 2$$

Panel Method: The panel method replaces the integral with a sum. The tank is broken into flat panels. The pressure is evaluated at the centroid of each panel. The area of each panel is the length of the panel times 2π times the radius out to the centroid. The panel normal is $+Cos\theta$.

Force on Concrete Foundation: This is just the weight of the steel in the tank walls plus the weight of the water.

```
%
%   HEMISPHERICAL WATER TANK
%
    PANELS=20;PI=3.14159;
    RADIUS=2.5;DEPTH=5.0;
    LENGTH=DEPTH+RADIUS;
    CHANGE=[PI/2]/PANELS;
    FLAT=2.0*RADIUS*sin(CHANGE/2.0);
    OUT=sqrt(RADIUS^2-(FLAT/2)^2);
    GRAVITY=9.81;DENSITY=1000.0;
    WEIGHT=GRAVITY*DENSITY;
%
%   INTEGRATION METHOD
    ONE=+WEIGHT*LENGTH*PI*RADIUS^2;
    TWO=-WEIGHT*PI*RADIUS^3*2/3;
    INTEGRAL=ONE+TWO
%
%   PANEL METHOD
    LIFT=0.0;
    ANGLE=CHANGE/2.0;
    for STEPS=1:PANELS
    HEIGHT=LENGTH-OUT*cos(ANGLE);
    AREA=FLAT*OUT*sin(ANGLE)*2*PI;
    PRESSURE=WEIGHT*HEIGHT;
    NORMAL=+cos(ANGLE);
    BIT=PRESSURE*AREA*NORMAL;
    LIFT=LIFT+BIT;
    ANGLE=ANGLE+CHANGE;
    end
    PANEL=LIFT
%
%   PRESSURE WEIGHT METHOD
    LOAD=+WEIGHT*LENGTH*PI*RADIUS^2;
    WATER=+WEIGHT*PI*RADIUS^3*2/3;
    BOX=LOAD-WATER
```

FLUIDS AT REST

HYDROSTATIC

STABILITY

METACENTER

When a neutrally buoyant body is rotated, it will return to its original orientation if buoyancy and weight create a restoring moment. For a submerged body, this occurs when the center of gravity is below the center of buoyancy. The center of buoyancy will act like a pendulum pivot. For a floating body, a restoring moment is generated when the center of gravity is below a point known as the metacenter. In this case, the metacenter acts like a pendulum pivot. The moments of the wedge shaped volumes generated by rotation is equal to the moment due to the shift in the center of volume. These moments are:

$$M_W = K\ \theta \qquad M_V = V\ S$$

Equating the two moments gives

$$S = K\ \theta\ /\ V$$

The shift in the center of volume can also be related to rotation about the meta center:

$$S = BM\,\theta$$

Equating the two shifts gives

$$BM\,\theta = K\,\theta / V$$

$$BM = K / V$$

For a general case, the moment of the wedges is

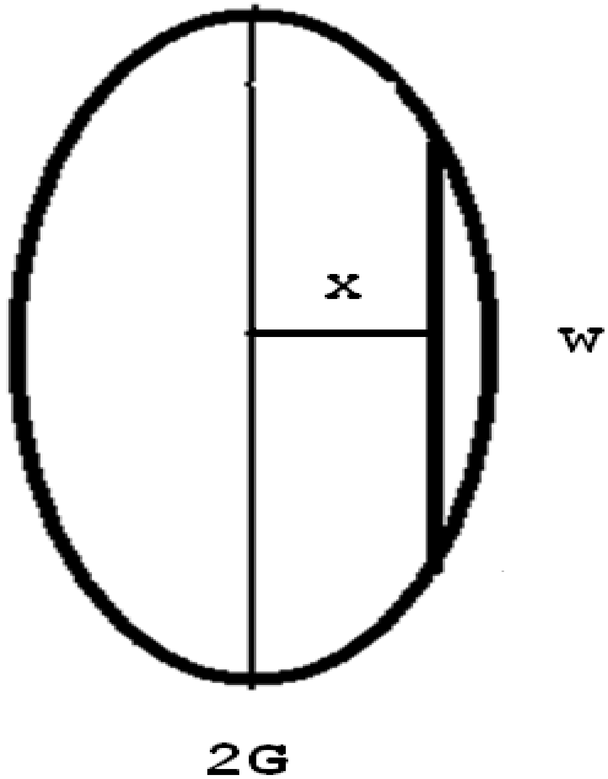
$$\int_{-G}^{+G} x\ x\theta\ w\ dx = K\,\theta$$

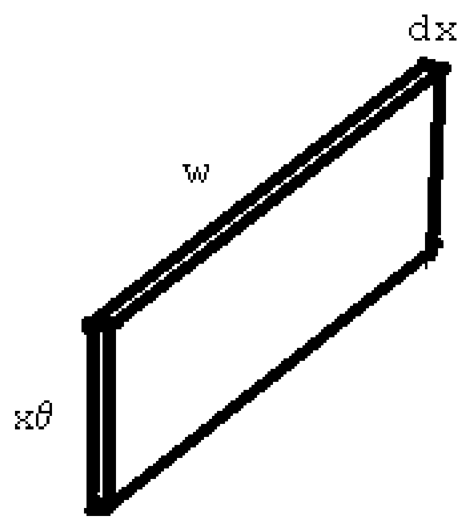
$$2 \int_{0}^{+G} x\ x\theta\ w\ dx = K\,\theta$$

This gives

$$K = \int_{-G}^{+G} x^2\ w\ dx = 2 \int_{0}^{+G} x^2\ w\ dx$$

$$K = \sum_{-G}^{+G} x^2\ w\ \Delta x = 2 \sum_{0}^{+G} x^2\ w\ \Delta x$$

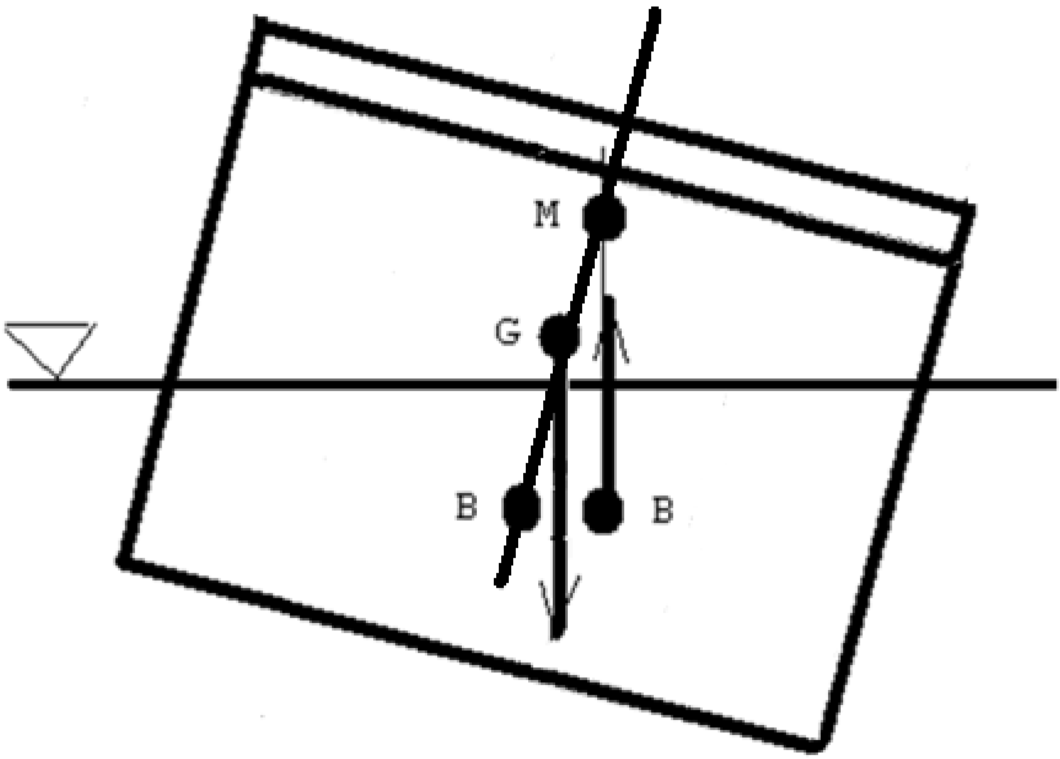

x
w
2G


dx
w
xθ

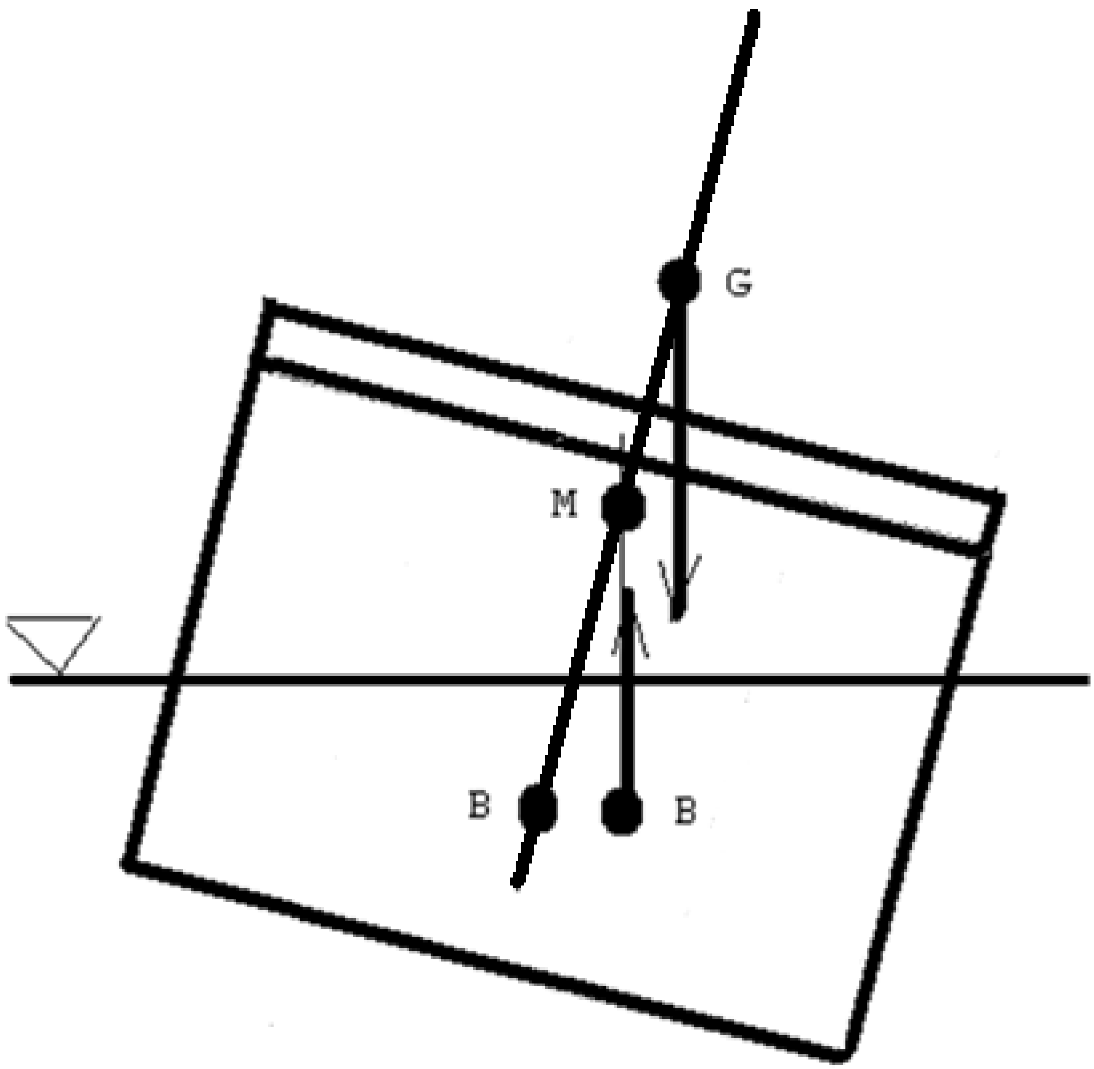
The metacenter M occurs at the intersection of two lines. One line passes through the center of gravity or G and the center of buoyancy or B of a floating body when it is not rotated: the other line is a vertical line through B when the body is rotated. Inspection of a sketch of these lines shows that, if M is above G, gravity and buoyancy generate a restoring moment, whereas if M is below G, gravity and buoyancy generate an overturning moment. One finds the location of M by finding the shift in the center of volume generated during rotation and noting that this shift could result from a rotation about an imaginary point which turns out to be the metacenter. The distance between B and the center of gravity G is BG. Geometry gives GM:

$$GM = BM - BG$$

If GM is positive, M is above G and the body is stable. If GM is negative, M is below G and the body is unstable.



STABLE



UNSTABLE

SINGLE HULL BODIES

$$S \ \rho g \ V = \int_{-G}^{+G} x \ \rho g \ x\Theta \ w \ dx$$

$$S \ \rho g \ V = 2 \int_{0}^{+G} x \ \rho g \ x\Theta \ w \ dx$$

Slice volume is: $dV = x\Theta \ w \ dx$

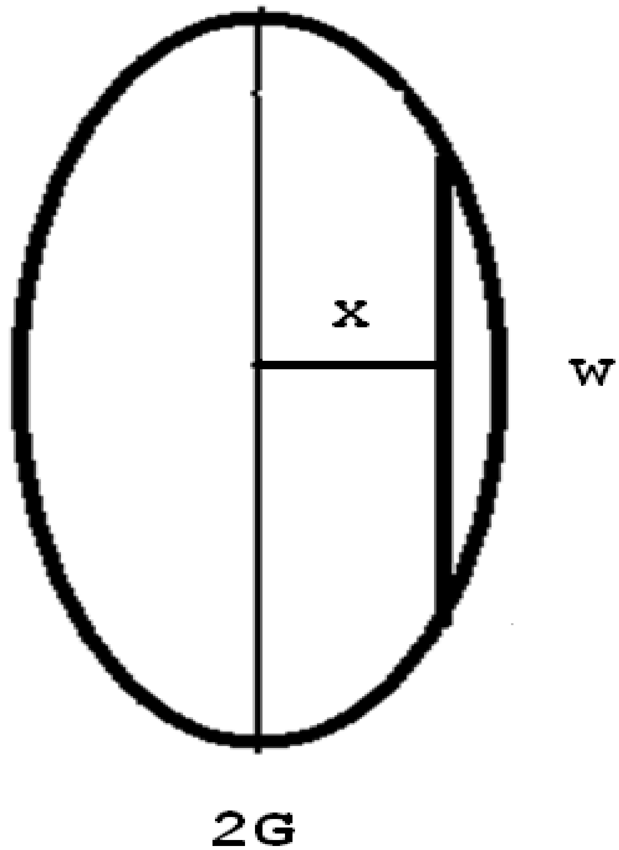
Slice Weight is: $dW = \rho g \ dV$

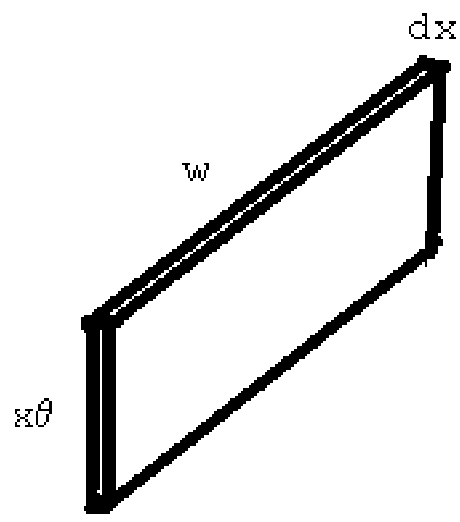
Slice Moment is: $x \ dW$

Integration gives: $\rho g \ K \ \Theta$

Manipulation gives: $S = K/V \ \Theta = R \ \Theta$

Metacentric Radius: R


x
w
2G


dx
w
xθ

SINGLE BOX RECTANGULAR BARGE

For roll of the barge the wedge factor is

$$K = 2 \int_{0}^{+G} x^2 \, w \, dx$$

$$= 2 * 2L * G^3/3$$

The volume of the barge is

$$V = 2L * 2G * h$$

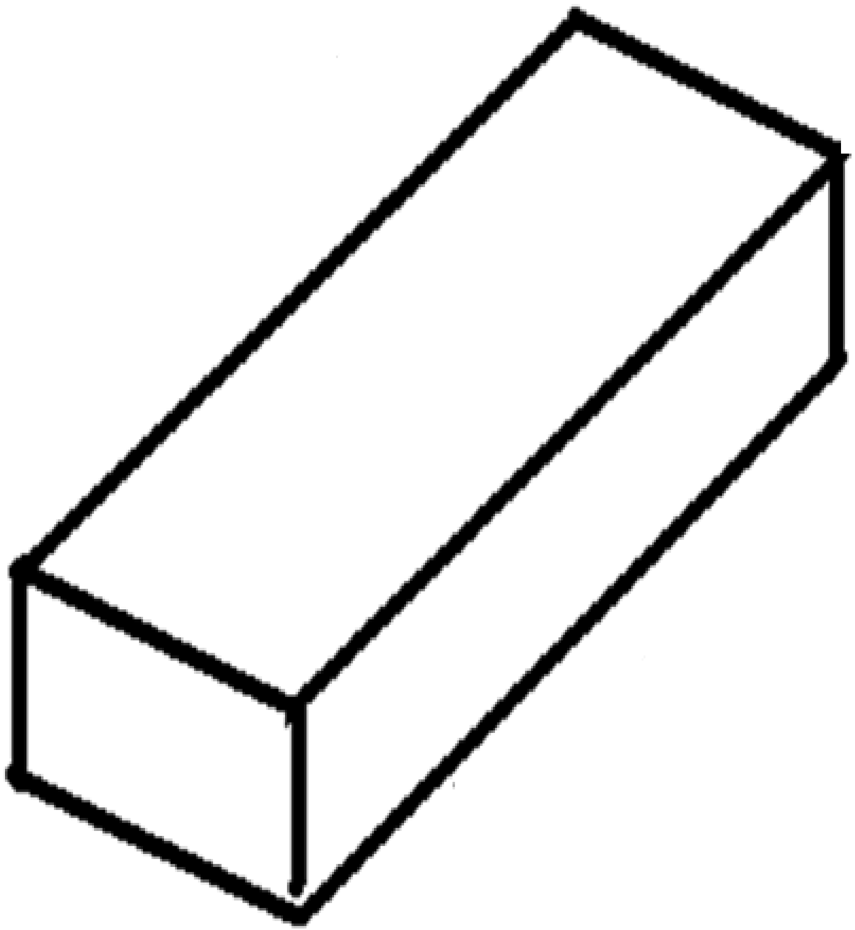
Manipulation gives

$$S = K/V \, \Theta = R \, \Theta$$

$$= G^2/[3h] \, \Theta$$

So the roll metacentric radius is

$$R = G^2/[3h]$$



GBS RIG

For roll of the GBS the wedge factor is

$$K = 2 \int_{0}^{+G} x^2 \, 2\sqrt{[G^2-x^2]} \, dx$$

$$= \pi G^4/4$$

The volume of the GBS rig is

$$V = \pi G^2 * h$$

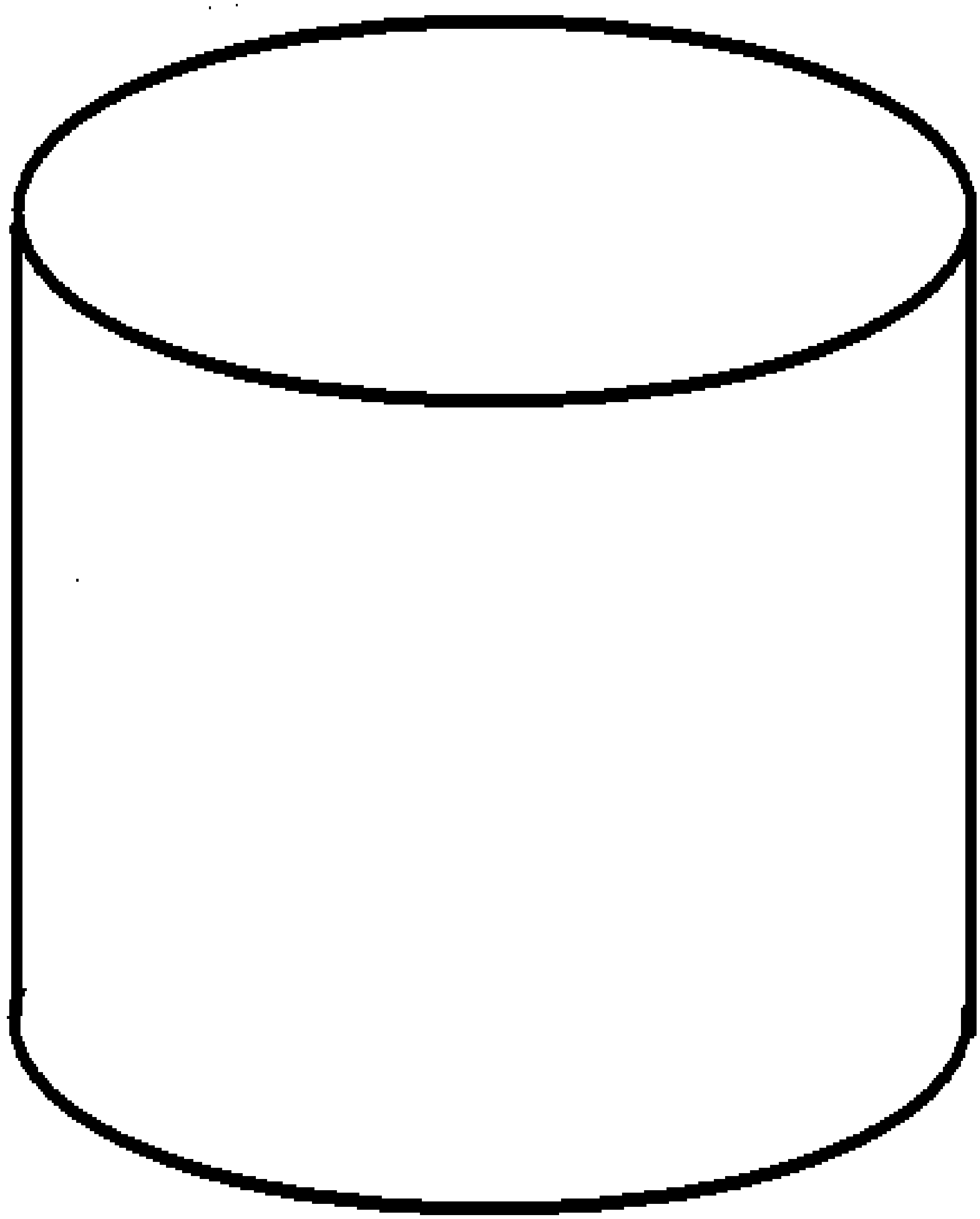
Manipulation gives

$$S = K/V \, \Theta = R \, \Theta$$

$$= G^2/4h \, \Theta$$

So the roll metacentric radius is

$$R = G^2/4h$$



DOUBLE HULL BODIES

$$S\ \rho g\ V = 2 \int_{H-G}^{H+G} x\ \rho g\ x\Theta\ w\ dx$$

$$S\ \rho g\ V = 2 \int_{-G}^{+G} [H+r]\ \rho g\ [H+r]\ \Theta\ w\ dr$$

Slice volume is: $dV = x\Theta\ w\ dx$

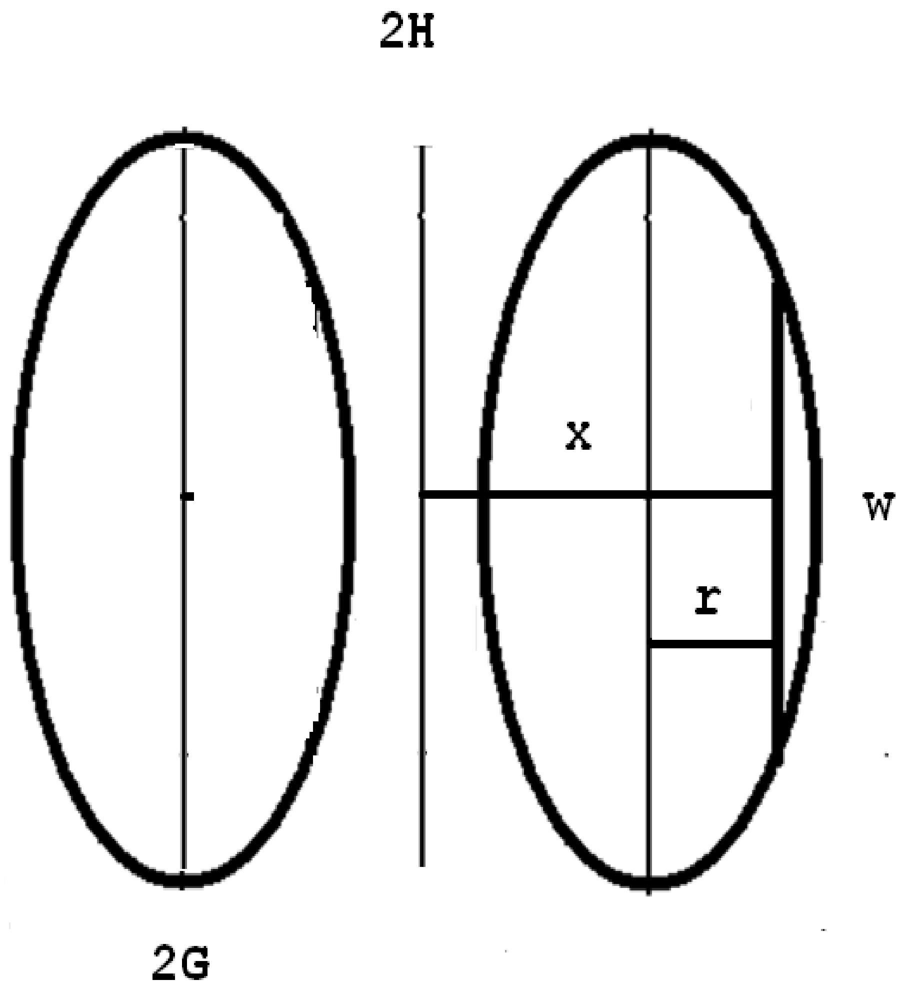
Slice Weight is: $dW = \rho g\ dV$

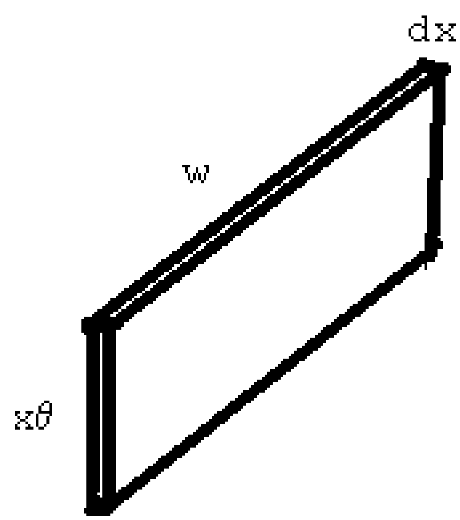
Slice Moment is: $x\ dW$

Integration gives: $\rho g\ K\ \Theta$

Manipulation gives: $S = K/V\ \Theta = R\ \Theta$

Metacentric Radius: R


2H
x
w
r
2G


dx
w
xθ

DOUBLE BOX RECTANGULAR BARGE

For roll of the barge the wedge factor is

$$K = 2 \int_{-G}^{+G} (H+r)^2 \, 2L \, dr$$

$$= 2L\,(4G^3/3 + 4H^2G)$$

The volume of the barge is

$$V = 2 * 2L * 2G * h$$

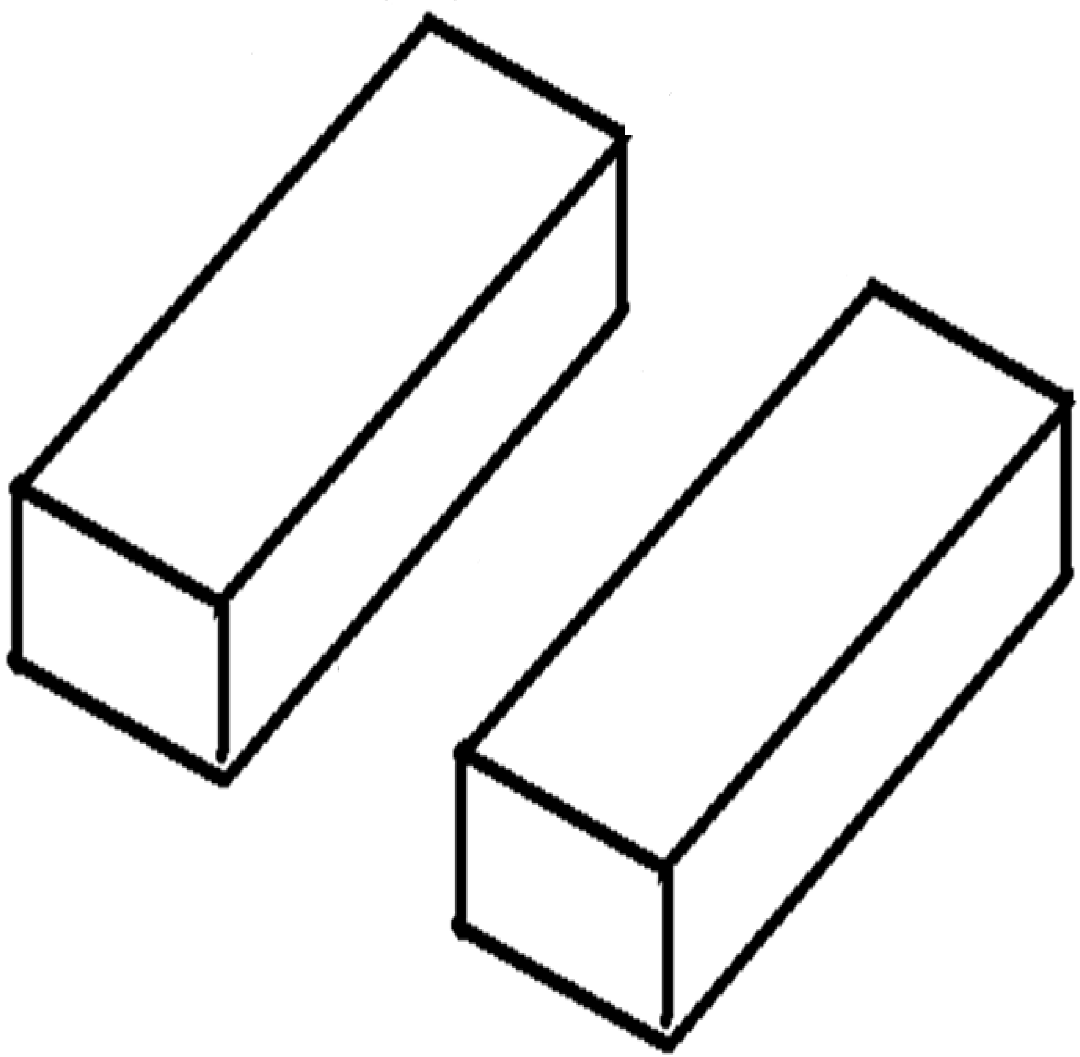
Manipulation gives

$$S = K/V\ \Theta = R\ \Theta$$

$$= (G^2/3h + H^2/h)\ \Theta$$

So the roll metacentric radius is

$$R = G^2/3h + H^2/h$$



OIL RIG

For roll of the rig the wedge factor is

$$K = 4 \int_{-G}^{+G} (H+r)^2 \; 2\sqrt{[G^2-r^2]} \;\; dr$$

$$= \pi G^4 + 4\pi G^2 H^2$$

The volume of the rig is

$$V = 4 * \pi G^2 * h$$

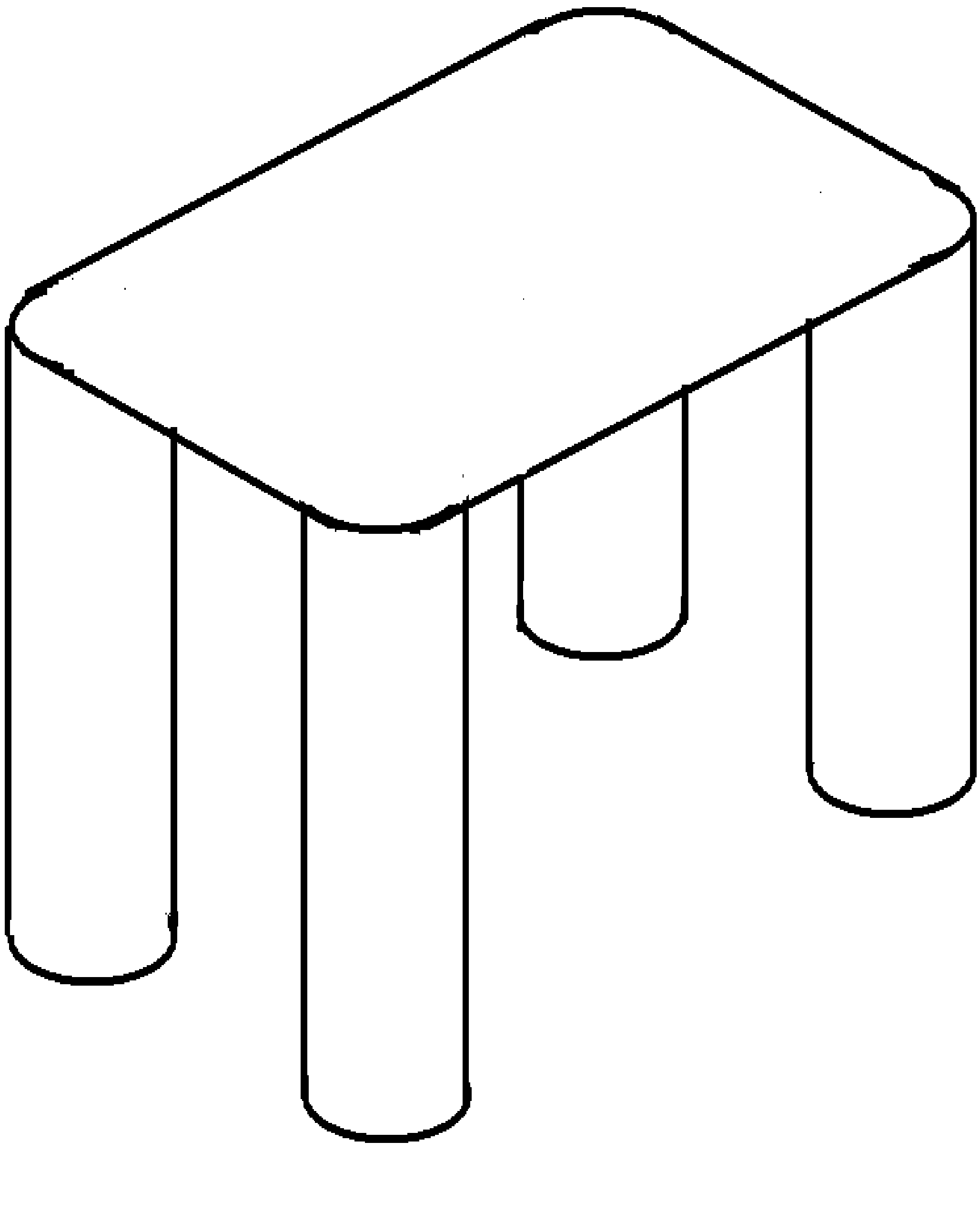
Manipulation gives

$$S = K/V \; \Theta \; = \; R \; \Theta$$

$$= (G^2/4h + H^2/h) \;\; \Theta$$

So the roll metacentric radius is

$$R \; = \; G^2/4h + H^2/h$$



When the spacing of the legs is large relative to the diameter of the legs, the wedge shaped volumes can be taken to be cylinders with total volume

$$4\ \pi G^2\ H\Theta$$

The moment of these volumes is

$$H\ 4\ \pi G^2\ H\Theta\ \ = K\ \Theta$$

Manipulation gives

$$S = K/V\ \Theta\ \ =\ \ R\ \Theta$$

$$= H^2/h\ \ \Theta$$

So the roll metacentric radius is

$$R\ \ =\ \ H^2/h$$

INTEGRALS

$$\int_{-G}^{+G} \sqrt{[G^2-r^2]}\ dr = r/2 \sqrt{[G^2-r^2]} + G^2/2\ \mathrm{Sin}^{-1}[r/G]$$

$$\int_{-G}^{+G} r \sqrt{[G^2-r^2]}\ dr = - [G^2-r^2]^{3/2} / 3$$

$$\int_{-G}^{+G} r^2 \sqrt{[G^2-r^2]}\ dr = - r [G^2-r^2]^{3/2} / 4$$

$$+ r G^2/8 \sqrt{[G^2-r^2]} + G^4/8\ \mathrm{Sin}^{-1} [r/G]$$

```
%
%   OIL RIG ROLL STABILITY
%
    PANELS=10000;PI=3.14159;
    RADIUS=5.0;DEPTH=5.0;
    CHANGE=2.0*RADIUS/PANELS;
    GRAVITY=9.81;DENSITY=1000.0;
    VOLUME=4.0*DEPTH*PI*RADIUS^2;
    CENTROID=10.0;
%
%   APPROXIMATE METACENTER
    BM=CENTROID^2/DEPTH
%   EXACT METACENTER
    BM=CENTROID^2/DEPTH ....
     +RADIUS^2/DEPTH/4.0
%
%   PANEL METHOD
    WEDGE=0.0;
    LOCATION=-RADIUS+CHANGE/2.0;
    for STEPS=1:PANELS
    ARM=CENTROID+LOCATION;
    WIDTH=2.0*sqrt(RADIUS^2-LOCATION^2);
    WEDGE=WEDGE+4.0*ARM^2*WIDTH*CHANGE;
    LOCATION=LOCATION+CHANGE;
    end
    BM=WEDGE/VOLUME
```

FLUIDS IN MOTION

CONCEPTS

OVERVIEW OF FLUID FLOWS

MOLECULAR NATURE OF LIQUIDS AND GASES

The molecules of a liquid are on average much closer together than those in a gas. This leads to significant intermolecular forces in a liquid and a high resistance to compression. Intermolecular forces in a gas are much less significant and the resistance to compression is much smaller. In a liquid, intermolecular forces give rise to wavy molecular trajectories whereas the lack of such forces in a gas causes trajectories to be basically straight. In a gas, pressure is due mainly to rebound forces associated with the high speed motion of its molecules. In a liquid, intermolecular forces also contribute. When pressure in a liquid, at 20^{o}C say, is lowered sufficiently, vapor bubbles form in it. When such bubbles collapse inside a pump, they can damage its blades: the phenomenon is known as cavitation. In both liquids and gases, temperature is basically a measure of the kinetic energy of molecular motion. In a gas, viscosity is due mainly due to the high speed random motion of its molecules. In a flow, these cause a lateral transfer of momentum. In a liquid, this transfer is due mainly to intermolecular forces.

TURBULENT FLOWS

At low speeds, fluid particles move along smooth streamline paths: motion has a laminar or layered structure. At high speeds, particles have superimposed onto their basic streamwise observable motion a random walk or chaotic motion. Particles move as groups in small spinning bodies known as eddies. The flow pattern is said to be turbulent. The small eddies in a turbulent flow diffuse momentum. This is basically what viscosity does in a laminar gas flow. So, to solve practical flow problems, engineers often try to model turbulence as an extra or eddy viscosity. Turbulent flows are too complex to deal with analytically: one must use CFD.

BOUNDARY LAYER FLOWS

When a body moves through a viscous fluid, the fluid at its surface moves with it. It does not slip over the surface. When a body moves at high speed, the transition between the surface and flow outside is known as a boundary layer. In a relative sense, it is a very thin layer. Within it, viscosity plays a dominant role because normal velocity gradients are very large. Gradients are responsible for skin friction drag on things like the wings and fuselage of aircraft. Boundary layers can separate from surfaces and radically alter the surrounding flow pattern. This is what happens when a wing stalls.

LOW REYNOLDS NUMBER FLOWS

When fluid moves through narrow spaces, the Reynolds Number of the flow is very low because the gap between the spaces, which is the characteristic dimension for such a flow, is very small. Low Reynolds Number means that viscous forces on the fluid are much greater than inertia forces. High pressures are needed to push fluid through such spaces. Hydrodynamic lubrication devices use the high pressures to support loads.

IDEAL OR POTENTIAL FLOWS

Well away from fluid boundaries, viscous forces are often small relative to inertia and pressure forces. Flows without viscosity are known as ideal or potential flows. The governing equations for such flows give a very accurate description of water waves. When they are applied to flow around a wing, they predict zero lift! Also, they give an unrealistic flow pattern around the wing. Something can be added to the formulation which mimics viscosity. When this is done, lift and flow patterns become realistic. So, without viscosity, planes could not fly!

COMPRESSIBLE FLOWS

When a fluid moves at around the local speed of sound, fluid compressibility becomes important. This is especially the case for high speed gas flows such as that in a rocket nozzle.

Subsonic flows have a local flow speed which is everywhere less than the local speed of sound. Most commercial jets fly at speed around 0.75 times the local speed of sound. Aircraft are said to fly at supersonic speeds when the local flow speed is everywhere greater than the local speed of sound. When one flies at supersonic speeds, the air ahead of it is unaware it is coming because disturbance waves generated by the craft are all swept downstream by the high speed flow: none can propagate upstream. Shock waves form near the craft: one is usually attached to its nose. Shock waves are very thin surfaces in a flow, usually only around 0.00025mm thick, across which there is a large increase in temperature and pressure. They cause very high drag.

UNSTEADY FLOWS IN PIPE NETWORKS

Unsteady flow in pipe networks can be caused by a number of factors. A turbomachine with blades can send pressure waves down a pipe. If the period of these waves matches a natural period of the pipe wave speed resonance develops. Sudden changes in valves or turbomachines cause pressure waves in pipe networks. These can cause pipes to explode or implode. In some cases interaction between pipes and devices is such that oscillations develop automatically. Examples include oscillations set up by leaky valves and those set up by slow turbomachine controllers.

FLOWS IN STREAM TUBES

CONSERVATION LAWS IN INTEGRAL FORM

Conservation of Mass states that the time rate of change of mass of a specific group of fluid particles in a flow is zero. Conservation of Momentum states that the time rate of change of momentum of a specific group must balance with the net load acting on it. Conservation of Energy states that the time rate of change of energy of a specific group must balance with heat and work interactions of the group with its surroundings. Mathematically one can write:

Conservation of Mass

$$\frac{D}{Dt}\int_V \rho \, dV = \int_V \partial\rho/\partial t \, dV + \int_S \rho \, \mathbf{v}.\mathbf{n} \, dS = 0$$

Conservation of Momentum

$$\frac{D}{Dt}\int_V [\rho\mathbf{v}] \, dV = \int_V \partial[\rho\mathbf{v}]/\partial t \, dV + \int_S [\rho\mathbf{v}] \, \mathbf{v}.\mathbf{n} \, dS$$

$$= \int_S \boldsymbol{\sigma} \, dS + \int_V \rho\mathbf{b} \, dV$$

Conservation of Energy

$$\frac{D}{Dt}\int_V [\rho e] \, dV = \int_V \partial[\rho e]/\partial t \, dV + \int_S [\rho e] \, \mathbf{v}.\mathbf{n} \, dS$$

$$= -\int_S \mathbf{q}.\mathbf{n} \, dS + \int_S \mathbf{v}.\boldsymbol{\sigma} \, dS$$

In these equations, V is fluid volume, S is fluid surface area, t is time, **n** is outward unit normal on S, **v** is velocity, ρ is density, **σ** denotes surface stresses such as pressure and viscous traction, **b** denotes body forces such as gravity, e is energy density and **q** denotes heat flux.

CONSERVATION LAWS IN STREAM TUBE FORM

Conservation of Mass for a stream tube is:

$$\sum[\rho CA]_{OUT} - \sum[\rho CA]_{IN} = 0$$

In this equation, ρ is density, C is flow speed and A is pipe area. Letting ρCA equal $\dot{M}$ allows one to rewrite mass as

$$\sum \dot{M}_{OUT} - \sum \dot{M}_{IN} = 0 \qquad \sum \dot{M}_{OUT} = \sum \dot{M}_{IN}$$

Conservation of Momentum for a stream tube is:

$$\sum[\rho \mathbf{v} CA]_{OUT} - \sum[\rho \mathbf{v} CA]_{IN}$$

$$= - \sum[PA\mathbf{n}]_{OUT} - \sum[PA\mathbf{n}]_{IN} + \mathbf{R}$$

Expansion gives

$$\sum [\dot{M} U]_{OUT} - \sum [\dot{M} U]_{IN} = - \sum PAn_x + R_x$$

$$\sum [\dot{M} V]_{OUT} - \sum [\dot{M} V]_{IN} = - \sum PAn_y + R_y$$

$$\sum [\dot{M} W]_{OUT} - \sum [\dot{M} W]_{IN} = - \sum PAn_z + R_z$$

In these equations, P is pressure, U V W are velocity components and **R** is the wall force on the fluid.

Conservation of Energy for a stream tube is

$$\sum [\dot{M}(C^2/2 + gz)]_{OUT} - \sum [\dot{M}(C^2/2 + gz)]_{IN} =$$

$$- \sum[PA\mathbf{C}]_{OUT} + \sum[PA\mathbf{C}]_{IN} + \sum\dot{T} - \sum\dot{L}$$

Manipulation gives

$$\sum [\dot{M}gh]_{OUT} - \sum [\dot{M}gh]_{IN} = + \sum\dot{T} - \sum\dot{L}$$

where h is known as head and is given by

$$h = C^2/2g + P/\rho g + z$$

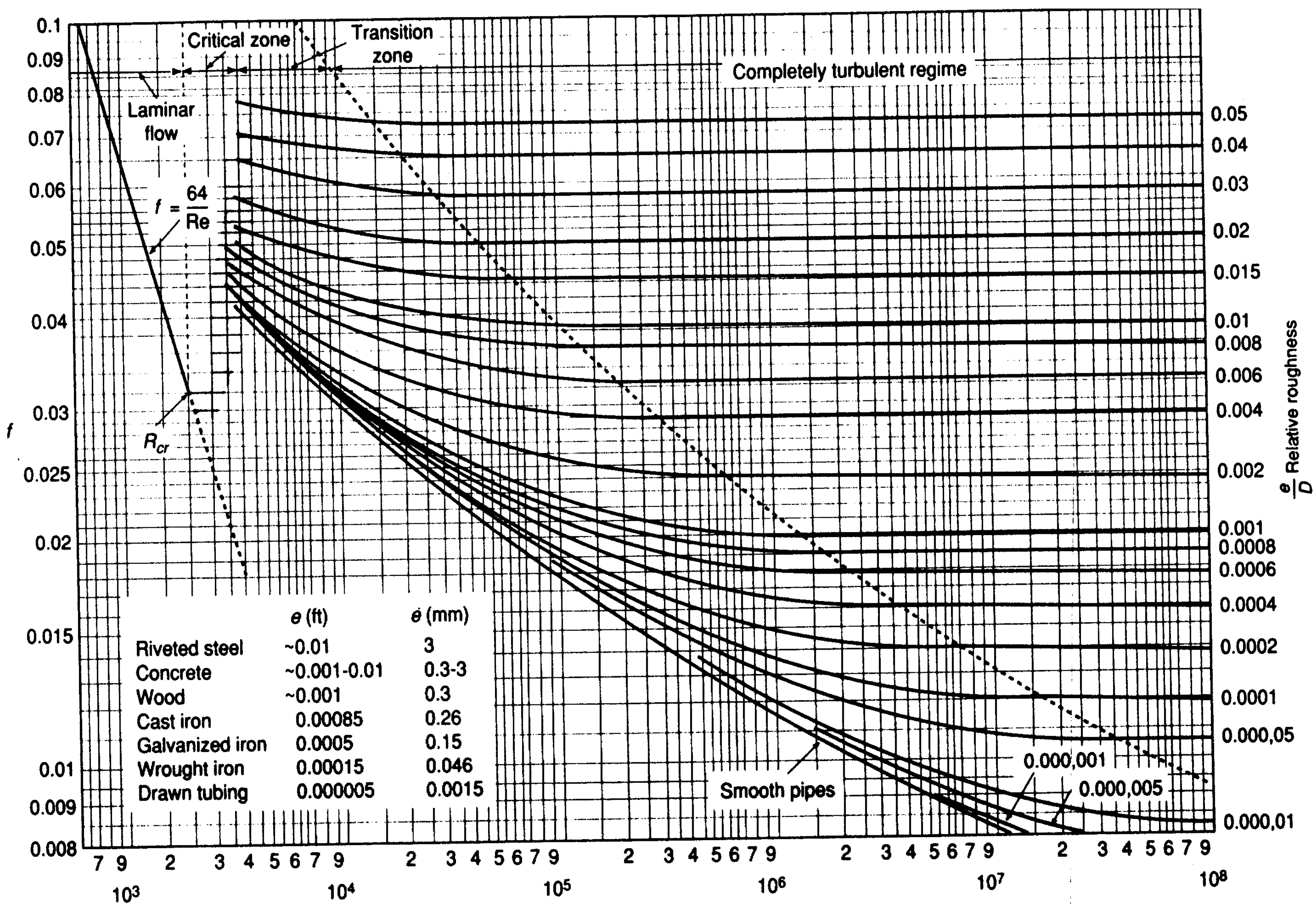
It represents each energy as an equivalent height of fluid. One can represent shaft power and lost power as

$$\dot{T} = \dot{M}gh_T \qquad \dot{L} = \dot{M}gh_L$$

The head loss is given by

$$h_L = (fL/D + \Sigma K)\ C^2/2g$$

where f is pipe friction factor, L is pipe length, D is pipe diameter and K accounts for losses at constrictions such as bends. The Moody Diagram gives f as a function of Reynolds Number $Re=CD/\upsilon$ and pipe relative roughness $\varepsilon=e/D$.



Moody diagram. (From L. F. Moody, *Trans. ASME*, Vol. 66, 1944.)

BERNOULLI EQUATION

When there is no shaft work and friction is insignificant, conservation of energy for a stream tube shows that h_{OUT} is equal to h_{IN}, which implies that h is constant:

$$C^2/2g + P/\rho g + z = K$$

This equation is known as the Bernoulli Equation. It can also be derived from conservation of momentum. For a short stream tube, a force balance gives:

$$\rho\, DC/Dt = \rho\, (\partial C/\partial t + C\partial C/\partial s) = - \partial P/\partial s - \rho g\, \partial z/\partial s$$

For steady flow this becomes

$$\rho\, CdC/ds = \rho\, d[C^2/2]/ds = - dP/ds - \rho g\, dz/ds$$

Integration of this gives the Bernoulli equation:

$$C^2/2 + P/\rho + gz = \kappa$$

This equation shows that, when pressure goes down in a flow, speed goes up and visa versa. From an energy perspective, flow work causes the speed changes. From a momentum perspective, it is due to pressure forces.

SYSTEM DEMAND

For a system where a pipe connects two reservoirs, the head H versus flow Q system demand equation has the form:

$$H = X + Y\,Q^2$$

$$X = \Delta\ [P/\rho g + z] \qquad Y = [fL/D + \Sigma K]/[2gA^2]$$

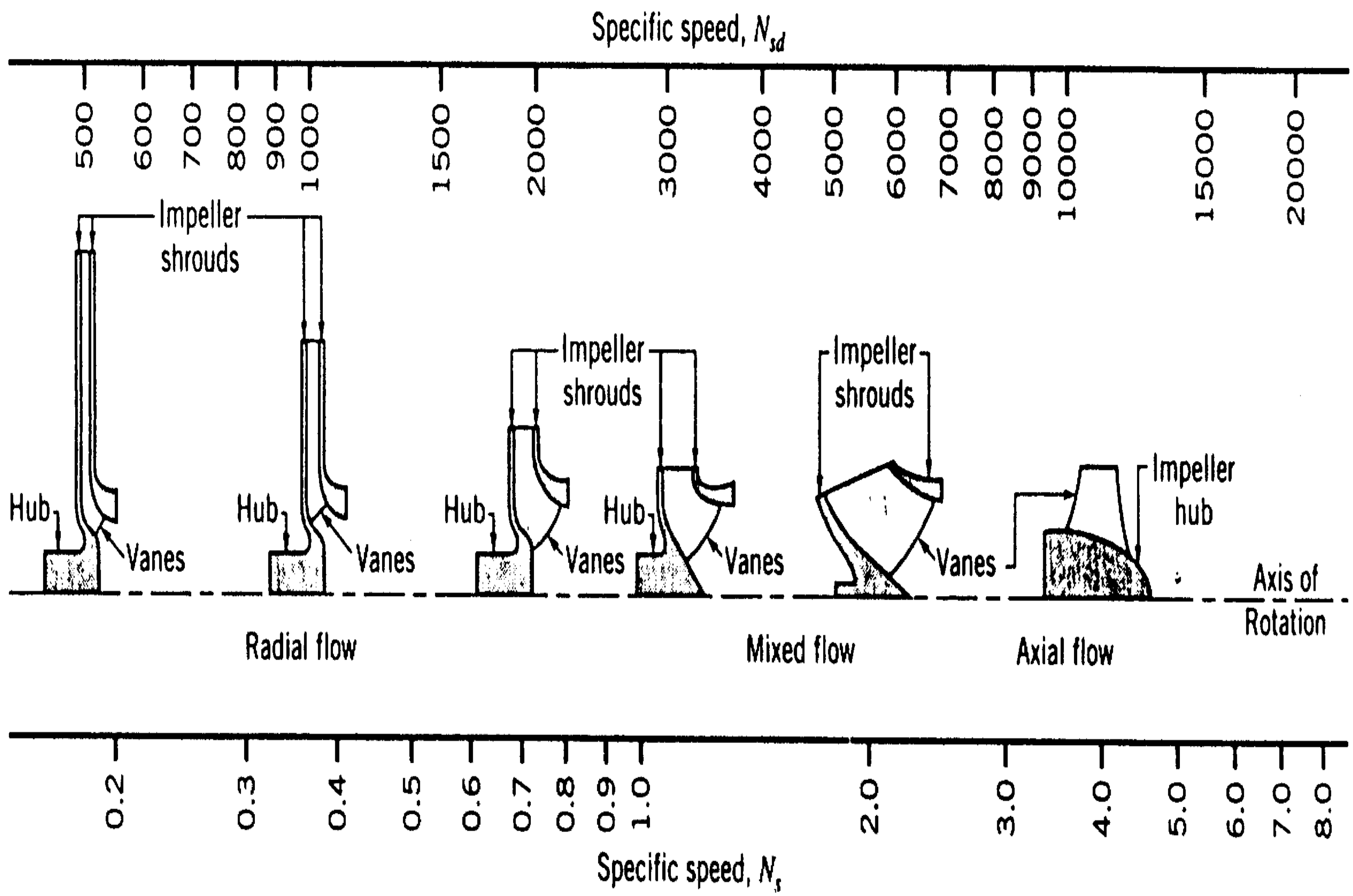
X accounts for pressure and height changes between the reservoirs and Y accounts for losses along the pipe.

PUMP SELECTION

To pick a pump, one first calculates the specific speed $\mathbf{N}$ based on the system operating point. This is a nondimensional number which does not have pump size in it:

$$\mathbf{N} = [N\ \sqrt{Q}]/[H^{3/4}]$$

This allows one to pick the appropriate type of pump. Axial pumps have high Q but low H which gives them high $\mathbf{N}$. Radial pumps have lower Q but higher H which gives them lower $\mathbf{N}$. Positive Displacement pumps have the lowest Q but highest H which gives them the lowest $\mathbf{N}$. Next one scans pump catalogs of the type indicated by specific speed and picks the size of pump that will meet the system demand, while it is operating at its best efficiency point (BEP) or best operating point (BOP). Finally, to prevent cavitation, the pump is located in the system at a point where it has the Net Positive Suction Head or NPSH recommended by the manufacturer:



Specific speed, N_sd
500
600
700
800
900
1000
1500
2000
3000
4000
5000
6000
7000
8000
9000
10000
15000
20000
Impeller shrouds
Hub
Vanes
Impeller shrouds
Hub
Vanes
Impeller shrouds
Hub
Vanes
Impeller hub
Axis of Rotation
Radial flow
Mixed flow
Axial flow
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
Specific speed, N_s

$$NPSH = P_s/\rho g + C_sC_s/2g - P_v/\rho g$$

In this equation, P_v is the absolute vapor pressure of the fluid being pumped, and P_s and C_s are the absolute pressure and speed at the pump inlet. For a system where a pipe connects a low reservoir to a high reservoir, conservation of energy from the low reservoir to the pump inlet gives:

$$P_o/\rho g \; - \; [P_s/\rho g + C_sC_s/2g + d] \; = \; h_L$$

where P_o is the absolute pressure of the air above the low reservoir and d is the height of the pump above the surface of the low reservoir. Manipulation gives

$$d \; = \; (P_o-P_v)/\rho g \; - \; h_L \; - \; NPSH$$

This shows that d might have to be negative.

ELECTRICAL ANALOGY

Electrical power **P** is V I where V is volts and I is current. By analogy, fluid power **P** is P Q where P is pressure and Q is volumetric flow rate. Note that power is force F times speed C. In a flow, force F is pressure P times area A. So power is P times A times C. Now volumetric flow rate Q is C times A. So power becomes P times Q. One can write pressure P in terms of head H as: $P=\rho gH$. Power becomes: $\mathbf{P}=\rho gHQ$. Voltage drop along a wire is $\Delta V=RI$ where R is the resistance of the wire. By analogy, the pressure drop along a pipe due to losses is $\Delta P=RQ^2$ where R is the resistance of the pipe.

FLUIDS IN MOTION

TURBOMACHINES

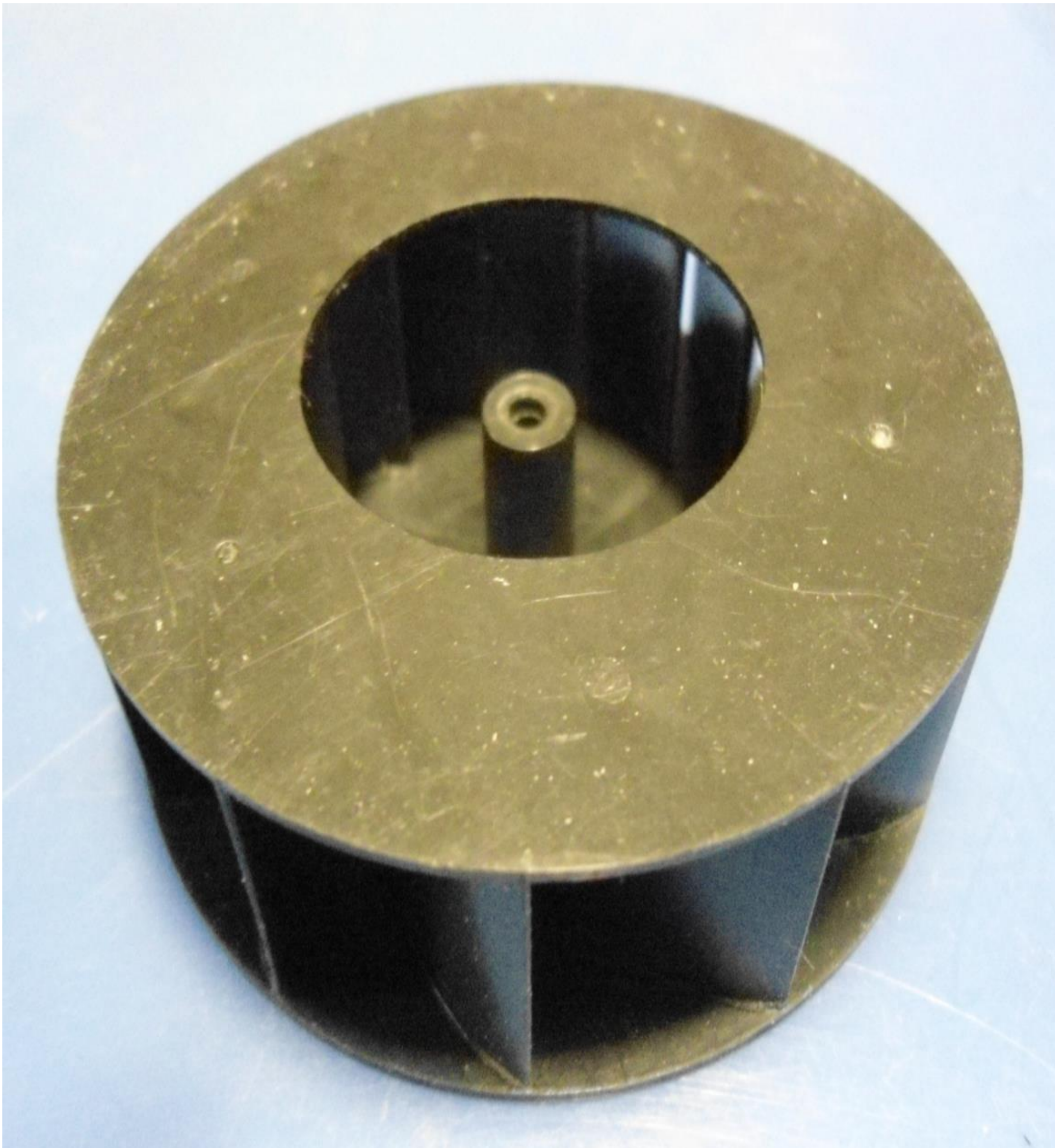
TURBOMACHINE POWER

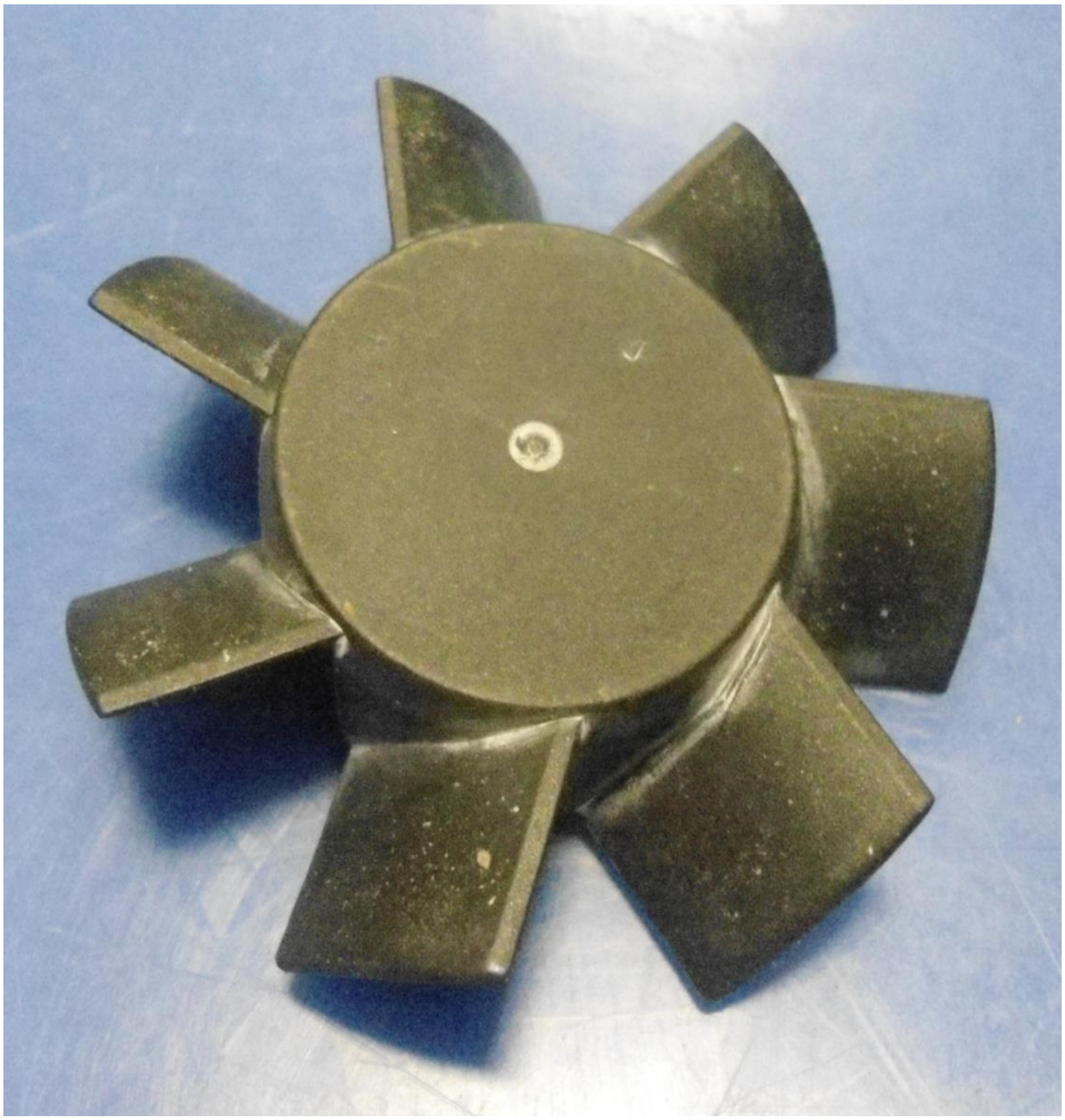
Swirl is the only component of fluid velocity that has a moment arm around the axis of rotation or shaft of a turbomachine. Because of this, it is the only one that can contribute to shaft power. The shaft power equation is:

$$\mathbf{P} = \Delta\ [T\ \omega] = \Delta\ [\rho Q\ V_T\ R\ \omega]$$

The swirl or tangential component of fluid velocity is V_T. The symbol Δ indicates we are looking at changes from inlet to outlet. The tangential momentum at an inlet or an outlet is $\rho Q\ V_T$. Multiplying momentum by moment arm R gives the torque T. Multiplying torque by the speed ω gives the power $\mathbf{P}$. The power equation is good for pumps and turbines. Power is absorbed at an inlet and expelled at an outlet. If the outlet power is greater than the inlet power, then the machine is a pump. If the outlet power is less than the inlet power, then the machine is a turbine. Geometry can be used to connect V_T to the flow rate Q and the rotor speed ω.

Theoretical analysis of turbomachines makes use of a number of velocities. These are: the tangential flow velocity V_T; the normal flow velocity V_N; the blade or bucket velocity V_B; the relative velocity V_R; the jet velocity V_J.





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TURBOMACHINE SCALING LAWS

Scaling laws allow us to predict prototype behavior from model data. Generally the model and prototype must look the same. This is known as geometric similitude. The flow patterns at both scales must also look the same. This is known as kinematic or motion similitude. Finally, certain force ratios in the flow must be the same at both scales. This is known as kinetic or dynamic similitude. Sometimes getting all force ratios the same is impossible and one must use engineering judgement to resolve the issue.

SCALING LAWS FOR TURBINES

For turbines, we are interested mainly in the power of the device as a function of its rotational speed. The simplest way to develop a nondimensional power is to divide power $\mathbf{P}$ by something which has the units of power. The power in a flow is equal to its dynamic pressure P times its volumetric flow rate Q:

$$P\ Q$$

So, we can define a power coefficient $C_{\mathbf{P}}$:

$$C_{\mathbf{P}} = \mathbf{P} \,/\, [P\ Q]$$

To develop a nondimensional version of the rotational speed of the turbine, we can divide the tip speed of the blades $R\omega$ by the flow speed U, which is usually equal to a jet speed V_J. So, we can define a speed coefficient $C_{\mathbf{S}}$:

$$C_{\mathbf{S}} = R\omega \,/\, V_J$$

SCALING LAWS FOR PUMPS

For a pump, it is customary to let N be the rotor RPM and D be the rotor diameter. All flow speeds U scale as ND and all areas A scale as D^2. Pressures are set by the dynamic pressure $\rho U^2/2$. Ignoring constants, one can define a reference pressure $[\rho N^2 D^2]$ and a reference flow $[ND^3]$. Since fluid power is just pressure times flow, one can also define a reference power $[\rho N^3 D^5]$. Dividing dimensional quantities by reference quantities gives the scaling laws:

$$\text{Pressure Coefficient} \quad C_P = P / [\rho N^2 D^2]$$

$$\text{Flow Coefficient} \quad C_Q = Q / [ND^3]$$

$$\text{Power Coefficient} \quad C_{\mathbf{P}} = \mathbf{P} / [\rho N^3 D^5]$$

On the pressure versus flow characteristic of a pump, there is a best efficiency point (BEP) or best operating point (BOP). For geometrically similar pumps that have the same operating point on the C_P versus C_Q curve, the coefficients show that if D is doubled, P increases 4 fold, Q increases 8 fold and $\mathbf{P}$ increases 32 fold, whereas if N is doubled, P increases 4 fold, Q doubles and $\mathbf{P}$ increases 8 fold.

PELTON WHEEL TURBINE THEORY

The power output of the turbine is: $\mathbf{P} = T\ \omega$ where T is the torque on the rotor and ω is the rotational speed of the rotor. The torque is:

$$T = \Delta\ (\rho Q\ V_T\ R)$$

The tangential flow velocities at inlet and outlet are:

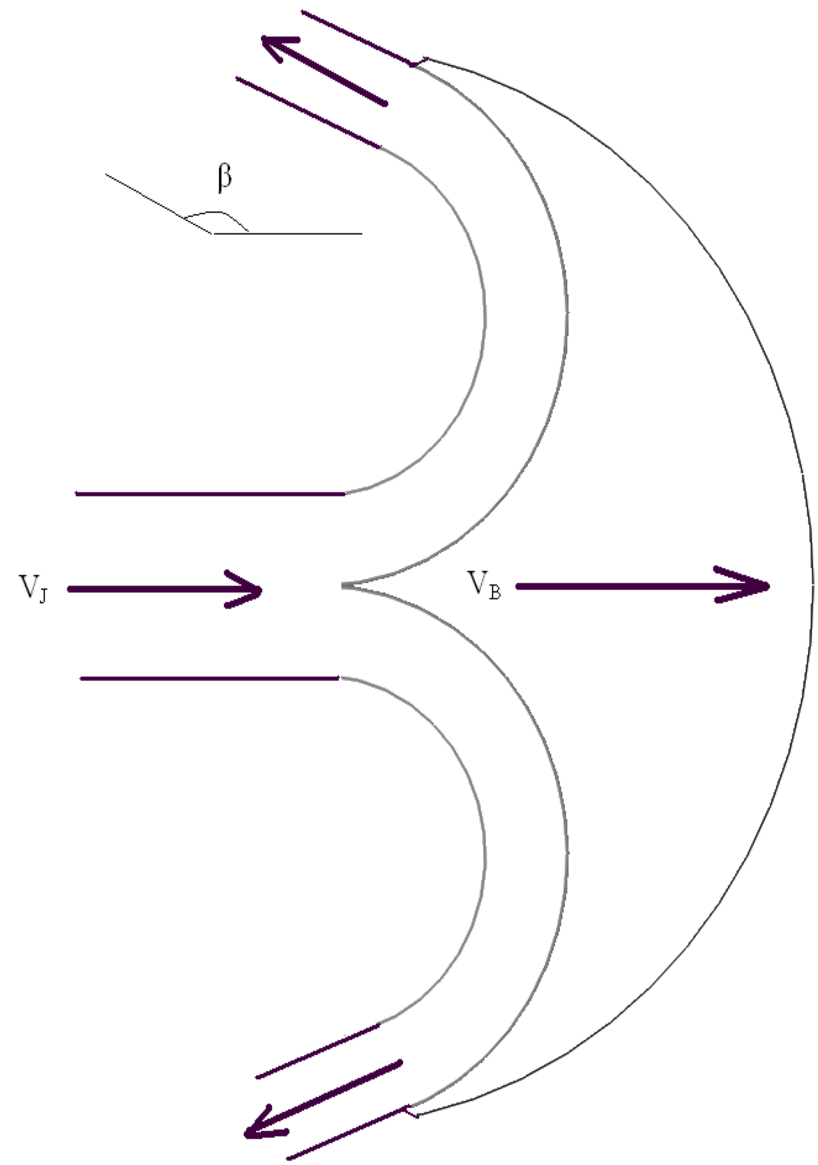
$$V_{IN} = V_J \qquad V_{OUT} = (V_J - V_B)\ K\ \cos\beta + V_B$$

where β is the bucket outlet angle and K is a loss factor. The blade and jet velocities are:

$$V_B = R\ \omega \qquad V_J = k\ \sqrt{[2P/\rho]}$$

So power becomes:

$$\mathbf{P} = \rho Q\ (V_J - V_B)\ (1 - K\ \cos\beta)\ V_B$$


β
V_J
V_B

FRANCIS TURBINE THEORY

The power output of the turbine is: $\mathbf{P} = T\ \omega$ where T is the torque on the rotor and ω is the rotational speed of the rotor. The torque is:

$$T = \Delta\ (\rho Q\ V_T\ R)$$

The tangential flow velocities at inlet and outlet are:

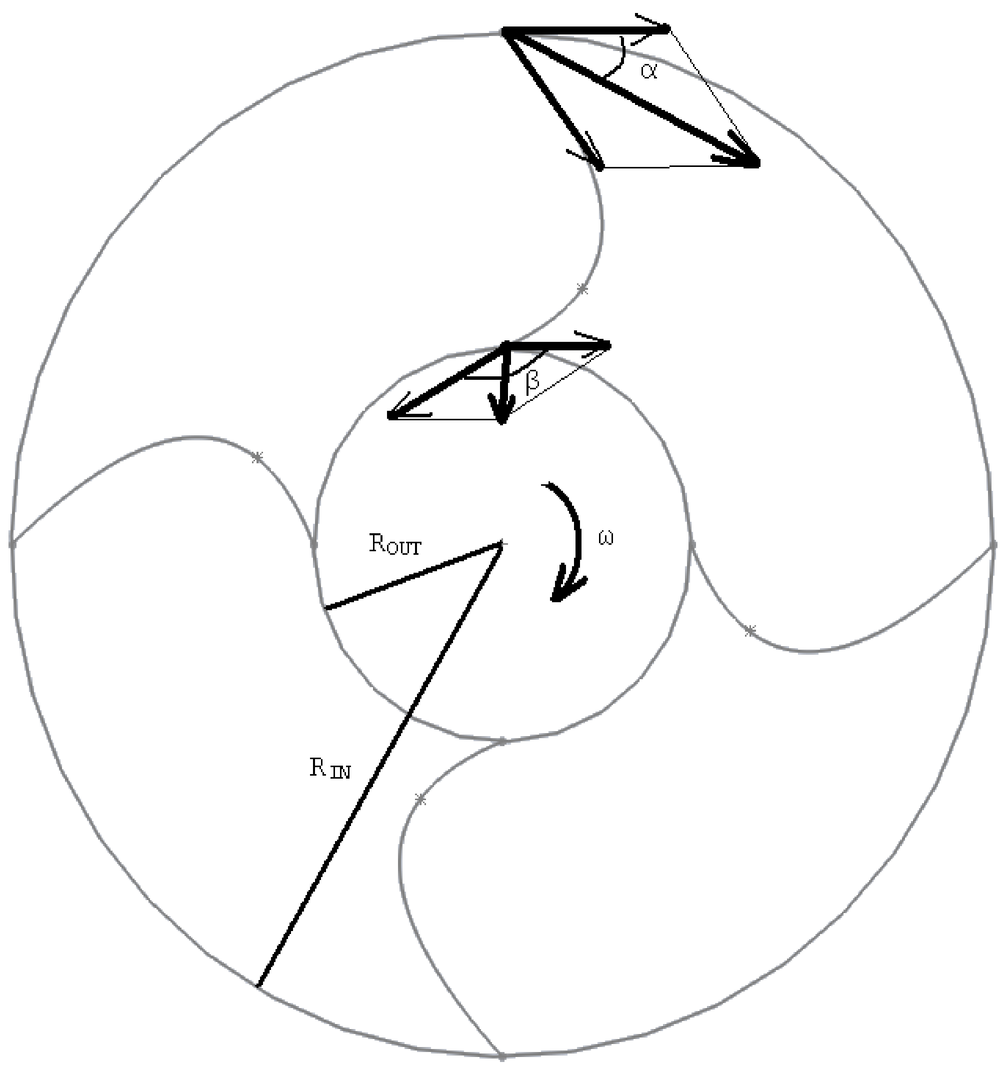
$$V_{IN} = V_N\ \mathrm{Cot}[\alpha] \qquad V_{OUT} = V_B + V_N\ \mathrm{Cot}[\beta]$$

where α is the inlet guide vane angle and β is the blade outlet angle. The blade and normal velocities are:

$$V_B = R\ \omega \qquad V_N = Q\ /\ [\pi\ 2R\ h]$$

where h is the depth of the rotor. So power becomes:

$$\mathbf{P} = \rho Q\ (V_{IN}\ R_{IN} - V_{OUT}\ R_{OUT})\ \omega$$
$$= \rho Q\ (\ [V_T V_B]_{IN} - [V_T V_B]_{OUT}\ )$$


α
β
ω
R_{OUT}
R_{IN}

KAPLAN TURBINE THEORY

The power output of the turbine is: $\mathbf{P} = T\ \omega$ where T is the torque on the rotor and ω is the rotational speed of the rotor. The torque is:

$$T = \Delta\ (\rho Q\ V_T\ R)$$

The tangential flow velocities at inlet and outlet are:

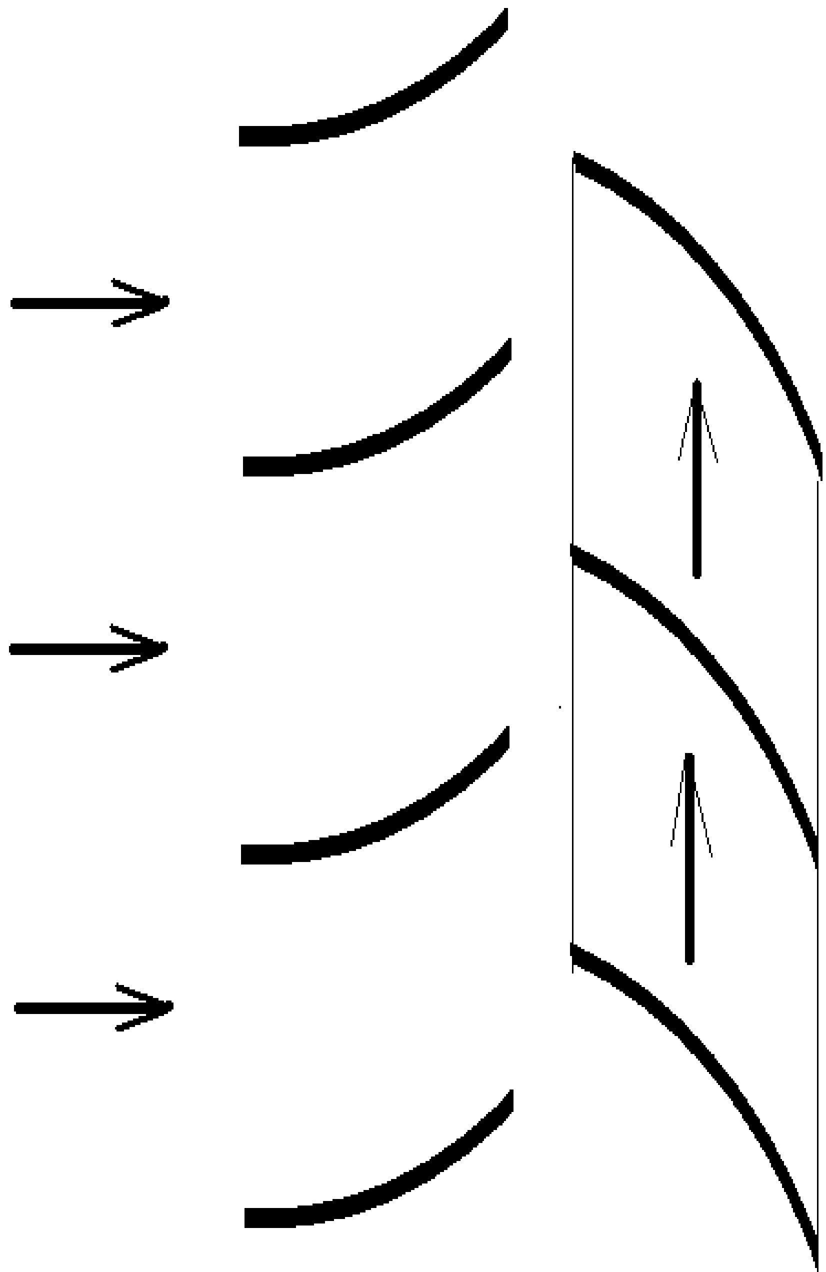
$$V_{IN} = V_N\ \mathrm{Cot}[\alpha] \qquad V_{OUT} = V_B + V_N\ \mathrm{Cot}[\beta]$$

where α is the inlet guide vane angle and β is the blade outlet angle. The blade and normal velocities are:

$$V_B = (R_O+R_I)/2\ \omega \qquad V_N = Q\ /\ [\pi\ (R_O\ R_O - R_I\ R_I)]$$

where R_O and R_I are outer radius and inner radius of the blade respectively. So power becomes:

$$\mathbf{P} = \rho Q\ (V_{IN}\ R_{IN} - V_{OUT}\ R_{OUT})\ \omega$$
$$= \rho Q\ (\ [V_T V_B]_{IN} - [V_T V_B]_{OUT}\ )$$



CENTRIFUGAL PUMP THEORY

The power output of the pump is:

$$\mathbf{P} = T\ \omega\ = \Delta\ (\rho Q\ V_T\ R)\ \omega$$

The tangential flow velocities at inlet and outlet are:

$$V_{IN} = V_N\ \mathrm{Cot}[\alpha] \qquad V_{OUT} = V_B + V_N\ \mathrm{Cot}[\beta]$$

The blade and normal velocities are:

$$V_B = R\ \omega \qquad V_N = Q\ /\ [\pi\ 2R\ h]$$

Power output is also

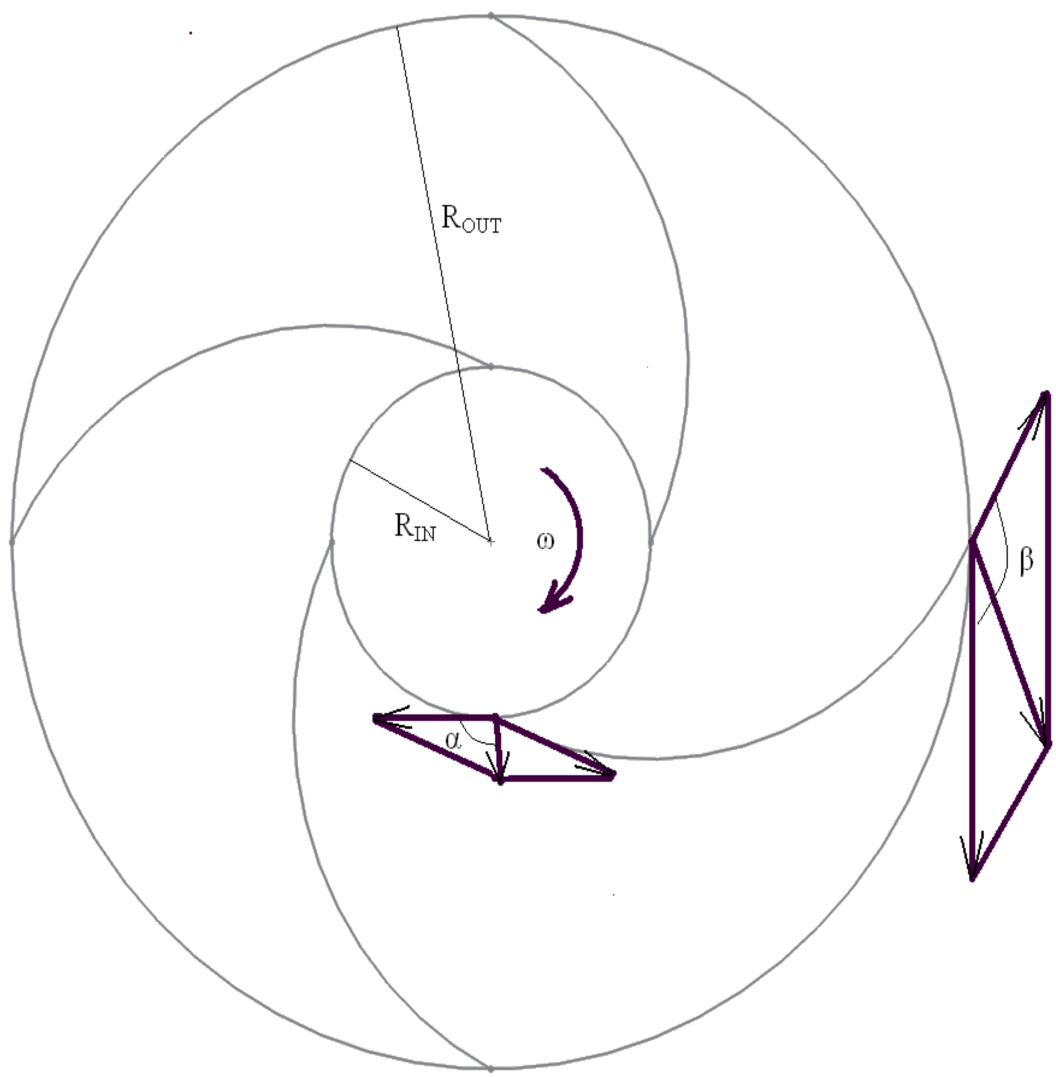
$$\mathbf{P} = P\ Q$$

Manipulation gives

$$P = \mathbf{P}\ /\ Q\ = \Delta\ (\rho\ V_T\ R)\ \omega$$

$$= \rho\ (V_{OUT}\ R_{OUT} - V_{IN}\ R_{IN})\ \omega$$

$$= \rho\ (\ [V_T V_B]_{OUT} - [V_T V_B]_{IN}\ )$$



R_{OUT}
R_{IN}
ω
α
β

PROPELLOR PUMP THEORY

The power output of the pump is:

$$\mathbf{P} = T\ \omega = \Delta\ (\rho Q\ V_T\ R)\ \omega$$

The tangential flow velocities at inlet and outlet are:

$$V_{IN} = V_N\ \mathrm{Cot}[\alpha] \qquad V_{OUT} = V_B + V_N\ \mathrm{Cot}[\beta]$$

The blade and normal velocities are:

$$V_B = (R_O+R_I)/2\ \omega \qquad V_N = Q\ /\ [\pi\ (R_O\ R_O - R_I\ R_I)]$$

Power output is also

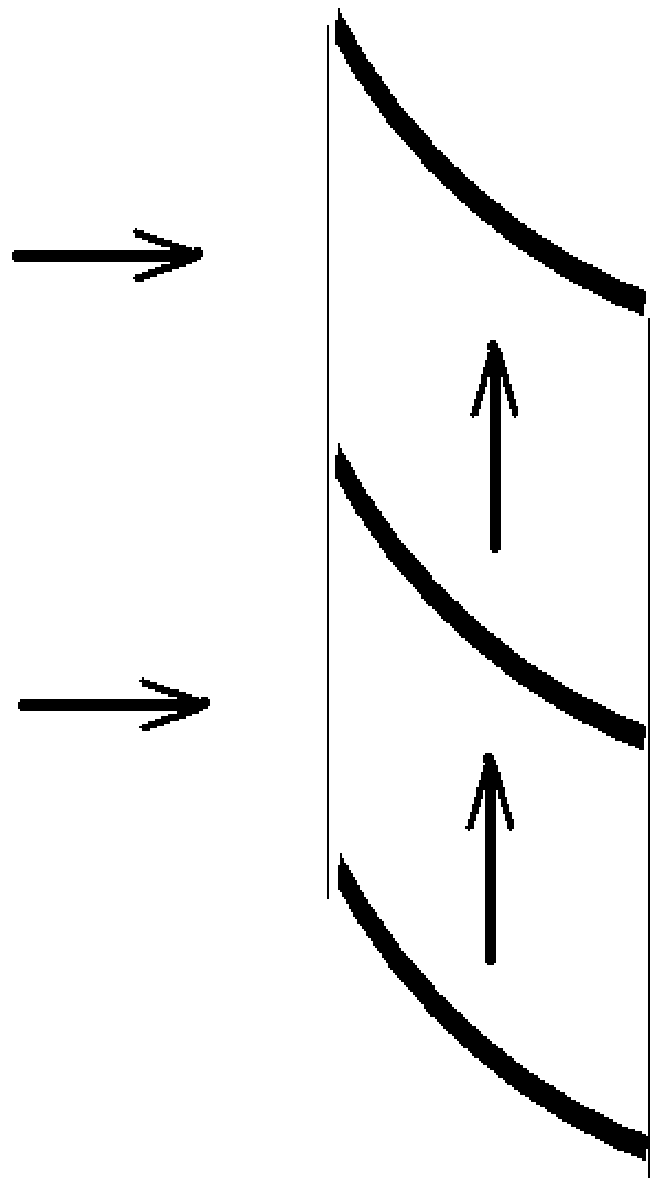
$$\mathbf{P} = P\ Q$$

Manipulation gives

$$P = \mathbf{P}\ /\ Q\ = \Delta\ (\rho\ V_T\ R)\ \omega$$

$$= \rho\ (V_{OUT}\ R_{OUT} - V_{IN}\ R_{IN})\ \omega$$

$$= \rho\ (\ [V_T V_B]_{OUT} - [V_T V_B]_{IN}\ )$$



FLUIDS IN MOTION

SCALING

LAWS

PREAMBLE

Scaling laws allow us to predict prototype behavior from model data. Generally the model and prototype must look the same. This is known as geometric similitude. The flow patterns at both scales must also look the same. This is known as kinematic or motion similitude. Finally, certain force ratios in the flow must be the same at both scales. This is known as kinetic or dynamic similitude. Sometimes getting all force ratios the same is impossible and one must use engineering judgement to resolve the issue.

The simplest way to derive scaling laws is to use common sense. If you need to develop a nondimensional power coefficient, you need to divide power by a reference power. The reference power could be based on things like the properties of the fluid and conditions imposed by the surroundings. One could also derive the scaling laws using a more formal procedure known as the Method of Indices. Most fluids texts call this the Buckingham π Theorem. For this, the variables and parameters of interest are divided into primary and secondary categories. When using the Buckingham π Theorem, each nondimensional coefficient is known as a π.

FLUID FORCE RATIOS

Many flow situations depend on the strength of one type of fluid force relative to another type of fluid force. The ratio of the forces is used to define the flow. Every flow situation will have a characteristic dimension which defines the size of the geometry. Let this dimension be D. All areas in the flow scale as D^2 and all volumes scale as D^3. Every flow situation will have a characteristic flow speed. Let this speed be C. Inertia forces in a flow scale as dynamic pressure times area: $\rho C^2/2\ D^2$. Viscous forces scale as shear stress times area: $\mu\ C/D\ D^2$. Gravity forces scale as weight density times volume: $\rho g\ D^3$. Surface tension forces scale as σ D where σ is surface tension. Fluid elastic or compressibility forces scale as $K\ D^2$ or $\rho\ a^2\ D^2$.

Ignoring the constant 2, the ratio of pressure forces to inertia forces gives the Euler Number: $\Delta P/[\rho C^2]$.

Ignoring the constant 2, the ratio of inertia forces to viscous forces gives the Reynolds Number: $\rho CD/\mu$.

Ignoring the constant 2, the ratio of inertia forces to gravity forces gives the Froude Number: $C^2/[gD]$ or $C/\sqrt{[gD]}$.

Ignoring the constant 2, the ratio of inertia forces to surface tension forces gives the Weber Number: $\rho C^2 D/\sigma$.

The ratio of inertia forces to elastic or compressibility forces gives the Mach Number: C^2/a^2 or C/a.

ILLUSTRATION : WAKE DRAG ON BODIES

For a body moving through a fluid, the wake drag on it can be represented nondimenionally as a drag coefficient:

$$C_D = \mathbf{D} / [[\rho U^2/2]\ A]$$

The reference drag is the dynamic pressure associated with the motion of the body times its profile area as seen from upstream. Usually C_D is a function of Reynolds Number:

$$Re = UD/\nu$$

This is inertia forces divided by viscous forces.

ILLUSTRATION : LIFT ON WINGS

The lift force on a wing can be represented nondimenionally as a lift coefficient:

$$C_L = \mathbf{L} / [[\rho U^2/2]\ A]$$

The reference lift is the dynamic pressure associated with the motion of the wing times its planform area as seen from above. Below stall, C_L is a weak function of Reynolds Number. It is a strong function of the wing angle of attack.

ILLUSTRATION : WAVE DRAG ON SHIPS

The drag on a ship due to wave generation can be represented nondimensionally as a drag coefficient:

$$C_D = \mathbf{D} / [[\rho U^2/2] A]$$

In this case C_D is a function of Froude Number:

$$Fr = U/\sqrt{[gL]}$$

This is inertia forces divided by gravity forces.

ILLUSTRATION : OSCILLATORY MOTION

Sometimes flows are oscillatory. In this case we need to nondimensionalize the flow period $\mathbf{T}$ with a reference period T. For a body with characteristic dimension D in a flow with speed U, the reference period is the transit time:

$$T = D/U$$

So the nondimensional period is:

$$C_T = \mathbf{T}/T$$

ILLUSTRATION : VORTEX SHEDDING

Vortices are often shed from structures in an asymmetric pattern. The Strouhal Number for such flows is the transit time T divided by the vortex shedding period T: St = T/T. For a circular cylinder St is around 0.2 so T is around 5 times T. One can form a period ratio

$$C_T = \mathsf{T} / \mathsf{T}$$

where T is a natural period of vibration of the structure. Resonance would occur when C_T is equal to unity.

ILLUSTRATION : HARBOR RESONANCE

Consider a harbor with a surface area S and a neck with area A and length L. The motion of the water in the neck causes the water level in the harbor to rise or fall. The hydrostatic force due to this level moves the water in the neck. This is basically a mass on a spring. Analysis shows that the natural period of vibration is

$$T = 2\pi/\omega = 2\pi \sqrt{([SL]/[gA])}$$

Dividing this by the tide period T gives

$$C_T = T / \mathsf{T}$$

Resonance would occur when C_T is equal to unity.

ILLUSTRATION : HEAD LOSS IN PIPES

The pressure drop ΔP due to friction for flow along a pipe is a function of the pipe diameter D, the pipe length L, the roughness size e, the density of the fluid ρ, the viscosity of the fluid μ and the speed of the flow C. Manipulation of the variables gives the nondimensional coefficients

Pressure Coefficient	$\Delta P / [\rho C^2/2]$
Reynolds Number	$\rho C D / \mu$
Length to Diameter Ratio	L / D
Roughness to Diameter Ratio	e / D

An equation for pressure drop is

$$\Delta P = f \; L/D \;\; \rho C^2/2$$

This can be written as a head loss

$$\Delta h = f \; L/D \;\; C^2/2g$$

The friction factor f would be a function of the Reynolds Number and the Relative Roughness Ratio.

ILLUSTRATION: PUMPS

For a pump, it is customary to let N be the rotor RPM and D be the rotor diameter. All flow speeds U scale as ND and all areas A scale as D^2. Pressures are set by the dynamic pressure $\rho U^2/2$. Ignoring constants, one can define a reference pressure $[\rho N^2 D^2]$ and a reference flow $[ND^3]$. Since fluid power is just pressure times flow, one can also define a reference power $[\rho N^3 D^5]$. Dividing dimensional quantities by reference quantities gives the scaling laws:

$$\text{Pressure Coefficient} \quad C_P = P / [\rho N^2 D^2]$$

$$\text{Flow Coefficient} \quad C_Q = Q / [ND^3]$$

$$\text{Power Coefficient} \quad C_{\mathbf{P}} = \mathbf{P} / [\rho N^3 D^5]$$

On the pressure versus flow characteristic of a pump, there is a best efficiency point (BEP) or best operating point (BOP). For geometrically similar pumps that have the same operating point on the C_P versus C_Q curve, the coefficients show that if D is doubled, P increases 4 fold, Q increases 8 fold and **P** increases 32 fold, whereas if N is doubled, P increases 4 fold, Q doubles and **P** increases 8 fold.

ILLUSTRATION: TURBINES

For a turbine, we are interested mainly in the power output of the device as a function of its rotational speed. The simplest way to develop a nondimensional power is to divide power **P** by something which has the units of power. The power in a flow is its dynamic pressure P times volumetric flow rate Q. For a flow, the dynamic pressure P is

$$P = \rho V^2/2$$

where ρ denotes the density of fluid and V is the speed of the flow. Volumetric flow Q is the speed of the flow V times its flow area A. So, a reference power is

$$\rho V^2/2 \quad VA$$

So, we can define a power coefficient $C_{\mathbf{P}}$

$$C_{\mathbf{P}} = \mathbf{P} \quad / \quad [\rho \; V^3/2 \quad A]$$

To develop a nondimensional version of the rotational speed, we can divide the tip speed of the blades Rω by the flow speed V. So, we can define a speed coefficient $C_{\mathbf{S}}$

$$C_{\mathbf{S}} = R\omega \; / \; V$$

One could derive the power and rotor speed coefficients using the Buckingham π Theorem. Power and speed would be primary variables. The flow speed and area and the density of the fluid would be secondary variables. For power, the goal is to find π_P where

$$\pi_P = \mathbf{P} \quad V^a \quad \rho^b \quad A^c$$

We need to find the a b c that make the right hand side dimensionless. In terms of the basic units of mass M and length L and time T, one can write

$$M^0L^0T^0 = M\ L/T^2\ \ L\ /T \quad [L/T]^a \quad [M/L^3]^b \quad [L^2]^c$$

Inspection shows that

$$a=-3 \qquad b=-1 \qquad c=-1$$

With this, π_P becomes

$$\pi_P = \mathbf{P} \ / \ [\rho V^3 \ A]$$

Similarly, for rotor speed, the goal is to find π_S where

$$\pi_S = \omega \quad V^a \quad \rho^b \quad R^c$$

Manipulation shows that

$$a=-1 \qquad b=0 \qquad c=+1$$

With this, π_S becomes

$$\pi_S = R\omega \ / \ V$$

As can be seen, each π is basically the same as a C.