FLUID STRUCTURE INTERACTIONS

These notes cover three different types of fluid structure interactions. The first is Flow Induced Vibration of Structures. The second is Unsteady Flow in Pipe Networks. The third is Water Wave Interaction with Structures.

FLUID STRUCTURE INTERACTIONS

FLOW INDUCED VIBRATIONS OF STRUCTURES

PREAMBLE

There are two types of vibrations: resonance and instability. Resonance occurs when a structure is excited at a natural frequency. When damping is low, the structure is able to absorb energy each oscillation cycle and dangerous amplitudes can build up. There are two types of instability: static and dynamic. Static instability occurs when a negative fluid stiffness overcomes a positive structural stiffness. Usually, because of nonlinearity, this instability is oscillatory: oscillations are often referred to as relaxation oscillations. Examples are wing stall flutter and gate valve vibration. Dynamic instability occurs when a negative fluid damping overcomes a positive structural damping. Examples include galloping of slender structures and tube bundle vibrations. In many cases, a system oscillates at a structural natural frequency. In these cases, frequency is a parameter in a semi empirical critical speed equation. Natural frequencies depend on the inertia of the structure and its stiffness. Usually the damping of the structure is ignored. It usually has only a small influence on periods. If the structure has a heavy fluid surrounding it, some of the fluid mass must be considered part of the structure. The structure appears more massive than it really is. For a simple discrete mass stiffness system, there is only one natural period. For distributed mass/stiffness systems, like wires and beams, there are an infinite number of natural periods. For each period, there is a mode shape. This shows the level of vibration at points along the structure. Structural frequencies can be obtained analytically for discrete mass/spring systems and for uniform wires and beams. For complex structures, they can be obtained using approximate procedures like the Galerkin Method of Weighted Residuals. In some cases, the fluid structure interaction is so complex that vibration frequencies depend on both the structure and the fluid. Examples include flutter of wings and panels and pipe whip due to internal flow.

These notes start with a description of some flow induced vibrations of slender structures. Next vibration of lifting bodies like wings and propellors is considered. Then vibration of panels exposed to flow is discussed. Finally, vibration in pipe networks is considered.

FLOW INDUCED VIBRATION OF SLENDER STRUCTURES

VORTEX INDUCED RESONANCE

Vortices shed from most slender structures in an asymmetric pattern. The shedding causes a lateral vibration of the structure. When the vortex shedding frequency is close to a natural frequency of the structure, the structure undergoes resonance. Once the structure begins to oscillate, it causes a phenomenon known as lock in. The vortices shed at the natural frequency. In other words, the structure motion controls the vortex shedding. It also increases the correlation length along the span. This means that vortex shedding along the span occurs at the same time. This gives rise to greater lateral loads. So, once shedding starts, it quickly amplifies motion.

VORTEX INDUCED INSTABILITY

Beyond a certain critical flow speed, a shear layer that has separated from a structure can reattach and create a very strong attached vortex. This occurs only for certain shapes. When such a shape is moving laterally in a flow, the attached vortex pulls it even more laterally! The phenomenon is known as galloping. The structure moves until its stiffness stops it. The vortex disappears and the structure starts moving back the other way. As it does so, the vortex appears on the other side of the structure which pulls it the other way. Another type of galloping is known as wake galloping. This is an oval shaped orbit motion of a cylindrical structure in the outer wake of another structure which is just upstream.

WAKE BREATHING OF A CYLINDER IN A FLOW

There are two modes of wake breathing. In the first mode, the Reynolds Number is near the point where the boundary layer becomes turbulent and the wake becomes smaller. When the cylinder moves upstream into such a flow, its drag drops, whereas when it moves downstream away from such a flow, its drag rises. This promotes a streamwise vibration of the cylinder. In the second mode of wake breathing, when the cylinder moves into a wake, added mass phenomena cause the wake to grow, whereas when the cylinder moves away from the wake, it causes it to shrink. This promotes a streamwise vibration of the cylinder.

FLOW INDUCED VIBRATIONS OF TUBE BUNDLES

There are three mechanisms that can cause tube bundles in a flow to vibrate. One is known as the displacement mechanism. As tubes move relative to each other, some passageways narrow while others widen. Fluid speeds up in narrowed passageways and slows down in widened passageways. Bernoulli shows that in the narrowed passageways pressure decreases while in the widened passageways it increases. Common sense would suggest that if tube stiffness and damping are low, at some point as flow increases, tubes must flutter or vibrate. The displacement mechanism has one serious drawback. It predicts that a single flexible tube in an otherwise rigid bundle cannot flutter but it can undergo a nonlinear oscillation called divergence. It is known from experiments that a single flexible tube in an otherwise rigid bundle can flutter. Another mechanism known as the velocity mechanism does predict flutter in the single flexible tube case. This mechanism is based on the idea that, when a tube is moving, the fluid force on it due its motion lags behind the motion because the upstream flow which influences the force needs time to redistribute. This time lag introduces a negative damping which can overcome the positive damping due to structural and viscous phenomena. The time lag is roughly the tube spacing divided by the flow speed within the bundle. Details of this model are beyond the scope of this note. The third mechanism for tube vibration involves vortex shedding and turbulence within the bundle.

CRITICAL SPEED EQUATIONS

For a slender structure, the Strouhal Number S is the transit time T divided by the vortex shedding period ᴛ: S=T/ᴛ. The transit time T is D/U. Solving for flow speed U gives: U = D/[Sᴛ]. During resonance, ᴛ=**T** where **T** is the structural period. So the critical flow speed is:

$$U = D/[S\ \mathbf{T}]$$

For the lateral vibration of a slender structure known as galloping, the critical flow speed U is

$$U = U_{\circ}\ M/M_{\circ}\ \zeta\ \mathbf{a} \qquad U_{\circ} = D/\mathbf{T} \qquad M_{\circ} = \rho D^2$$

The factor ζ accounts for damping: it is typically in the range 0.01 to 0.1. The parameter **a** accounts for the shape of the structure. For a square cross section structure **a** is 8 while for a circular cross section structure **a** is ∞.

For tube bundle vibration, the critical flow speed is

$$U = \beta/\mathbf{T}\ \sqrt{[M\delta/\rho]} \qquad U = \beta U_{\circ}\ \sqrt{[\delta M/M_{\circ}]}$$

The factor δ accounts for damping, and the parameter β accounts for the bundle geometry. Typically δ is in the range 0.05 to 0.25 while β is in the range 2.5 to 6.0.

VIBRATION MODES OF SIMPLE WIRES AND BEAMS

For a wire under tension free to undergo lateral motion, the governing equation is:

$$\partial/\partial x\ (T\partial Y/\partial x) = M\ \partial^2 Y/\partial t^2$$

where Y is the lateral deflection, T is the tension in the wire, M is its mass per unit length, x is position along the wire and t is time. For a uniform wire with constant M and T, this can be written as the wave equation:

$$a^2\ \partial^2 Y/\partial x^2 = \partial^2 Y/\partial t^2 \qquad a^2 = T/M$$

where a is the wave speed. During steady free vibration of a wire, one can write for each point on the wire:

$$Y = \mathbf{Y}\ \text{Sin}\ \omega t$$

Substitution into the governing equation gives:

$$a^2\ d^2\mathbf{Y}/dx^2 = -\ \omega^2\ \mathbf{Y}$$

$$d^2\mathbf{Y}/dx^2 = -\ \beta^2\ \mathbf{Y} \qquad \beta^2 = \omega^2/a^2$$

A general solution is

$$\mathbf{Y} = \mathbf{Y}_o \sin \beta x$$

For a wire held at both ends, **Y** is zero at both ends. This implies that β must be $n\pi/L$, where n is any positive integer and L is the length of the wire. Substitution into the β^2 equation gives the natural frequencies:

$$\omega_n = n\pi a/L = n\pi/L \sqrt{[T/M]}$$

The corresponding natural periods are:

$$\mathbf{T}_n = 2L/n \sqrt{[M/T]}$$

The natural mode shapes are:

$$\sin [n\pi x/L]$$

For a beam free to undergo lateral motion, the governing equation is

$$- \partial^2/\partial x^2 \; (EI \; \partial^2 Y/\partial x^2) = M \; \partial^2 Y/\partial t^2$$

where E is the beam material Elastic Modulus and I is the section area moment of inertia.

During steady free vibration of a beam, one can write for each point on the beam:

$$Y = \mathbf{Y} \sin \omega t$$

Substitution into the equation of motion gives:

$$d^2/dx^2 \ (EI \ d^2\mathbf{Y}/dx^2) = \omega^2 \ M \ \mathbf{Y}$$

For a uniform beam with constant M and EI, this becomes:

$$d^4\mathbf{Y}/dx^4 = \beta^4 \ \mathbf{Y} \qquad \beta^4 = \omega^2 \ M/[EI]$$

The general solution is:

$$\mathbf{Y} = A \ \mathrm{Sin}[\beta x] + B \ \mathrm{Cos}[\beta x] + C \ \mathrm{Sinh}[\beta x] + D \ \mathrm{Cosh}[\beta x]$$

where A and B and C and D are constants of integration. These are determined by the boundary conditions.

For a beam with pivot supports, the boundary conditions are zero deflection and zero bending moment at each end. This implies that at each end:

$$\mathbf{Y} = 0 \qquad d^2\mathbf{Y}/dx^2 = 0$$

In this case, the general solution reduces to:

$$\mathbf{Y} = \mathbf{Y}_o \ \mathrm{Sin} \ \beta x$$

As for the wire, β must be $n\pi/L$, where n is any positive integer and L is the length of the beam. Substitution into the β^4 equation gives the natural frequencies:

$$\omega_n = [n\pi/L]^2 \sqrt{[EI]/M}$$

The corresponding natural periods are:

$$\mathbf{T}_n = [L/n]^2 \, 2/\pi \, \sqrt{M/[EI]}$$

The natural mode shapes are:

$$\mathrm{Sin}\ [n\pi x/L]$$

For a cantilever beam, the boundary conditions at the wall are zero deflection and zero slope. This implies that

$$\mathbf{Y} = 0 \qquad d\mathbf{Y}/dx = 0$$

Application of these conditions shows that:

$$C = -A \qquad D = -B$$

At the free end of the beam, the bending moment and shear are both zero. This implies that

$$d^2\mathbf{Y}/dx^2 = 0 \qquad d^3\mathbf{Y}/dx^3 = 0$$

Application of these conditions gives

$$[\sin\beta L+\sinh\beta L]\ A + [\cos\beta L+\cosh\beta L]\ B = 0$$

$$[\cos\beta L+\cosh\beta L]\ A - [\sin\beta L-\sinh\beta L]\ B = 0$$

Manipulation of these equations gives the β condition:

$$\cos\beta_n L\ \cosh\beta_n L + 1 = 0$$

This gives the natural frequencies of the beam. For each frequency, one gets the natural mode shape:

$$(\sin[\beta_n L] - \sinh[\beta_n L])\ (\sin[\beta_n x] - \sinh[\beta_n x])$$

$$+$$

$$(\cos[\beta_n L] + \cosh[\beta_n L])\ (\cos[\beta_n x] - \cosh[\beta_n x])$$

The first 3 natural frequencies are:

$$\omega_1 = 3.52/L^2\ \sqrt{[EI]/M}$$
$$\omega_2 = 22.03/L^2\ \sqrt{[EI]/M}$$
$$\omega_3 = 61.70/L^2\ \sqrt{[EI]/M}$$

The corresponding natural periods are:

$$\mathbf{T}_1 = 2\pi L^2/3.52\ \sqrt{M/[EI]}$$
$$\mathbf{T}_2 = 2\pi L^2/22.03\ \sqrt{M/[EI]}$$
$$\mathbf{T}_3 = 2\pi L^2/61.70\ \sqrt{M/[EI]}$$

VIBRATION MODES OF COMPLEX WIRES

The equation governing the lateral motion of a wire is:

$$- \partial/\partial x \ (T\partial Y/\partial x) \ + \ M \ \partial^2 Y/\partial t^2 \ = \ 0$$

In this equation, Y is deflection of the wire from its neutral position, T is its tension, x is location along the wire, M is the mass of the wire and t is time. During steady free vibration of a wire:

$$Y = \mathbf{Y} \ \text{Sin} \ \omega t$$

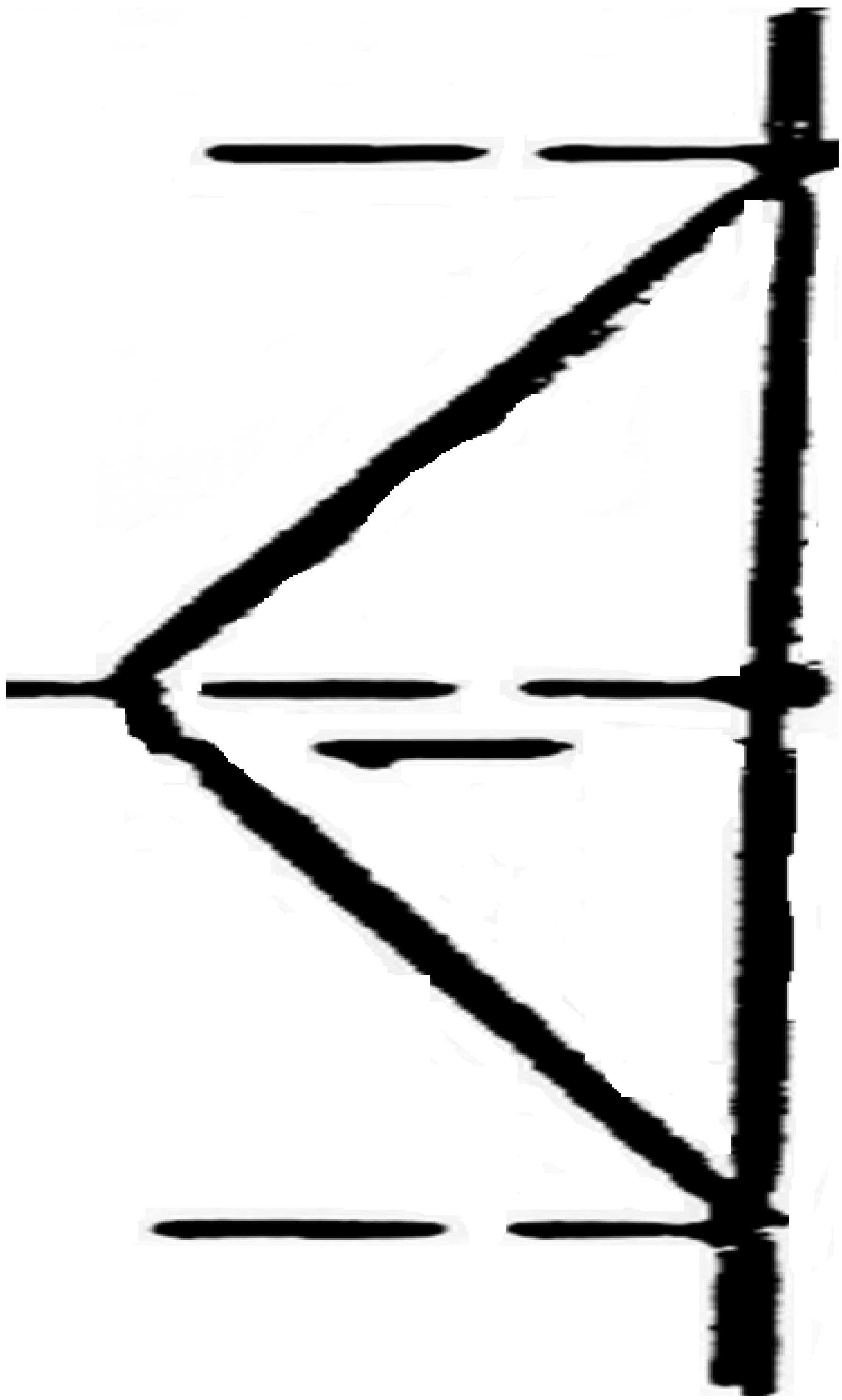
Substitution into the equation of motion gives

$$- d/dx \ (T \ d\mathbf{Y}/dx) \ - \ \omega^2 \ M \ \mathbf{Y} = \ 0$$

For a Galerkin finite element analysis, we assume that deflection along the wire can be given as a sum of scaled shape functions:

$$\mathbf{Y} \ = \ \Sigma \ A \ n$$

where n is deflection at a node and A is a shape function. For shape functions, we use piecewise linear polynomials. The sketch on the next page shows one for a typical node.



Substitution of the assumed form for **Y** into the governing equation gives a residual. In a Galerkin analysis, weighted averages of this residual along the wire are set to zero. After some manipulation, one gets

$$\int_0^L [dW/dx\ T\ d\mathbf{Y}/dx - W\ \omega^2\ M\ \mathbf{Y}]\ dx = 0$$

where L is the length of the wire and W is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon \qquad A_R = 1-\varepsilon$$

where ε is a local coordinate. The subscripts L and R indicate elements immediately to the left and right of the node. Notice the integration by parts of the space derivative term in the integral. This introduces slope end boundary conditions into the formulation. Such boundary conditions are not needed for a wire held at both ends. Application of vibration theory gives the vibration modes of the wire. A computer program was written to do this. For a uniform wire with L=10 and M=10 and T=100, theory gives ω_1=0.993. With 10 elements, Galerkin gives ω_1=0.998.

VIBRATION MODES OF COMPLEX BEAMS

The equation governing the lateral motion of a beam is:

$$\partial^2/\partial x^2 \ (EI\partial^2 Y/\partial x^2) \ + \ M \ \partial^2 Y/\partial t^2 \ = \ 0$$

In this equation, Y is deflection of the beam from its neutral position, EI is its flexural rigidity, x is location along the beam, M is the mass of the beam and t is time. During steady free vibration of a beam:

$$Y = \mathbf{Y} \text{ Sin } \omega t$$

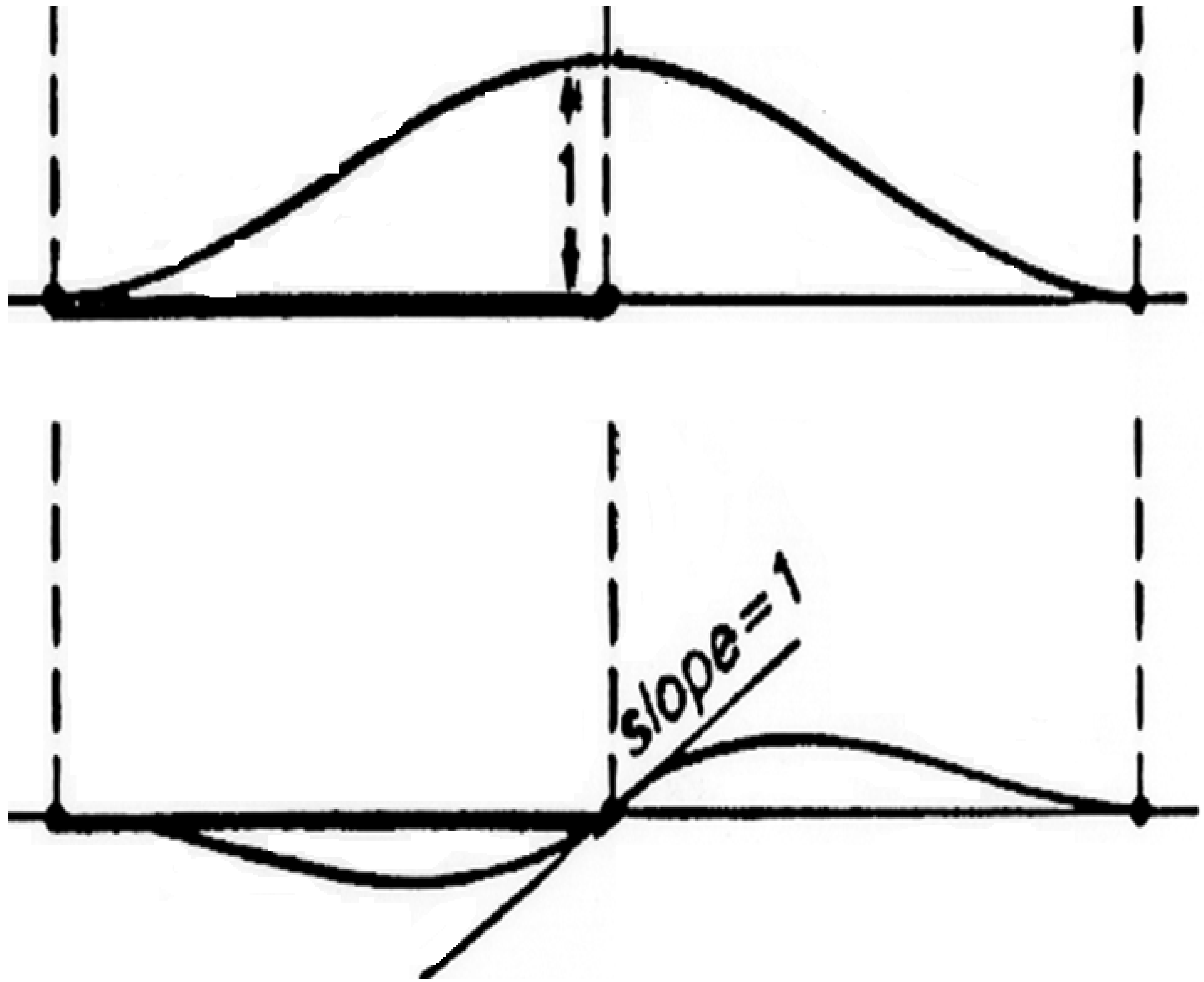
Substitution into the equation of motion gives

$$d^2/dx^2 \ (EI \ d^2\mathbf{Y}/dx^2) \ - \ \omega^2 \ M \ \mathbf{Y} \ = \ 0$$

For a Galerkin finite element analysis, we assume that deflection can be given as a sum of scaled shape functions:

$$\mathbf{Y} \ = \ \Sigma \ [A \ n \ + \ B \ m]$$

where n is the deflection at a node and m is the slope at the node. A and B are shape functions. Theory shows that these must be Hermite polynomials. Such polynomials must be used because the stiffness term is 4^{th} order. The sketch on the next page shows what they look like for a typical node.


1
slope=1

Substitution of the assumed form for **Y** into the governing equation gives a residual. In a Galerkin analysis, weighted averages of this residual along the beam are set to zero. After some manipulation, one gets

$$\int_0^L [d^2W/dx^2 \; EI \; d^2\mathbf{Y}/dx^2 - W \, \omega^2 \, M \, \mathbf{Y}] \, dx = 0$$

where L is the length of the beam and W is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of the node. Notice the double integration by parts of the space derivative term in the integral. This introduces tip shear and tip bending moment boundary conditions into the formulation. These are both zero for a cantilever beam. Application of vibration theory gives the vibration modes of the beam. A computer program was written to do this. For a uniform beam with L=1 and M=10 and EI=8.33, theory gives ω_1=3.213. With 10 elements, Galerkin gives ω_1=3.210.

GOVERNING EQUATIONS FOR WIRES AND BEAMS

Sketch A shows a wire under tension. A force balance on a small segment of the wire gives:

$$- T\partial Y/\partial x \quad + \quad [T\partial Y/\partial x + \partial/\partial x \;(T\partial Y/\partial x)\;\Delta x] \quad = \quad M\;\Delta x\;\partial^2 Y/\partial t^2$$

Manipulation gives the equation of motion:

$$\partial/\partial x\;(T\partial Y/\partial x) \quad = \quad M\;\partial^2 Y/\partial t^2$$

Sketches B and C show a beam undergoing bending. A force balance on a small segment of the beam gives:

$$- \mathbf{Q} \quad + \quad (\mathbf{Q} + \partial \mathbf{Q}/\partial x\;\Delta x) \quad = \quad M\;\Delta x\;\partial^2 Y/\partial t^2$$

Manipulation gives:

$$\partial \mathbf{Q}/\partial x \quad = \quad M\;\partial^2 Y/\partial t^2$$

A moment balance on the beam segment gives:

$$- \mathbf{M} \quad + \quad (\mathbf{M} + \partial \mathbf{M}/\partial x\;\Delta x) \quad + \quad (\mathbf{Q} + \partial \mathbf{Q}/\partial x\;\Delta x)\;\Delta x \quad = \quad 0$$

Manipulation gives :

$$\mathbf{Q} \quad = \quad - \partial \mathbf{M}/\partial x$$

Sketch D shows how a beam is strained when bent. Inspection of the sketch shows that the strain is:

$$\varepsilon = Y/R$$

The stress is:

$$\sigma = E\ \varepsilon$$

where E is the Elastic Modulus. Geometry gives

$$R\partial\Theta = \partial s \qquad \partial\Theta/\partial s = 1/R$$

$$\partial s = \partial x \qquad \Theta = \partial Y/\partial x$$

Manipulation gives:

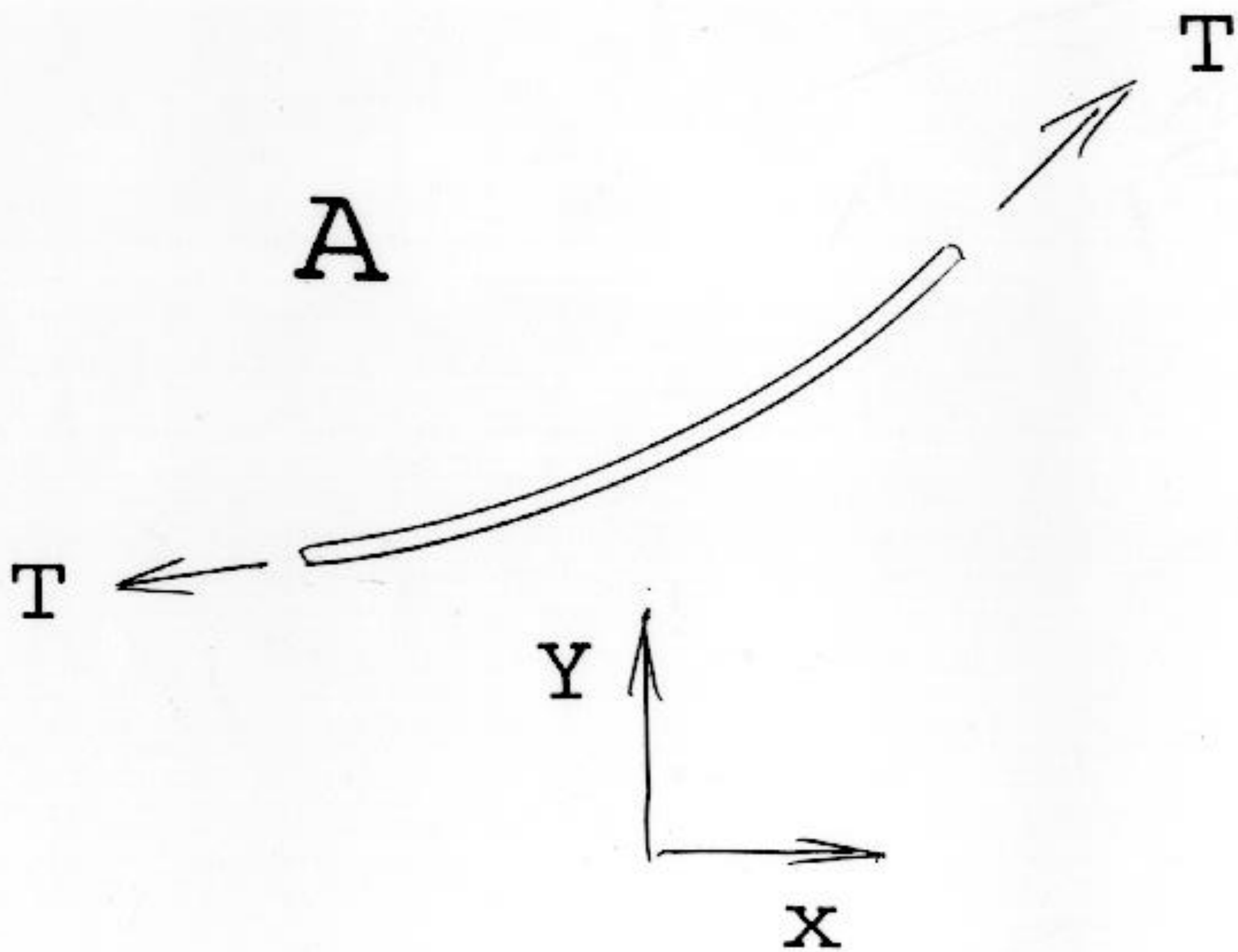
$$\partial^2 Y/\partial x^2 = 1/R$$

Moment considerations give:

$$\mathbf{M} = \int \sigma Y\ dA = E/R \int Y^2\ dA = EI/R = EI\ \partial^2 Y/\partial x^2$$

So, the equation of motion becomes

$$-\ \partial^2/\partial x^2\ \mathbf{M} = -\ \partial^2/\partial x^2\ (EI\ \partial^2 Y/\partial x^2) = M\ \partial^2 Y/\partial t^2$$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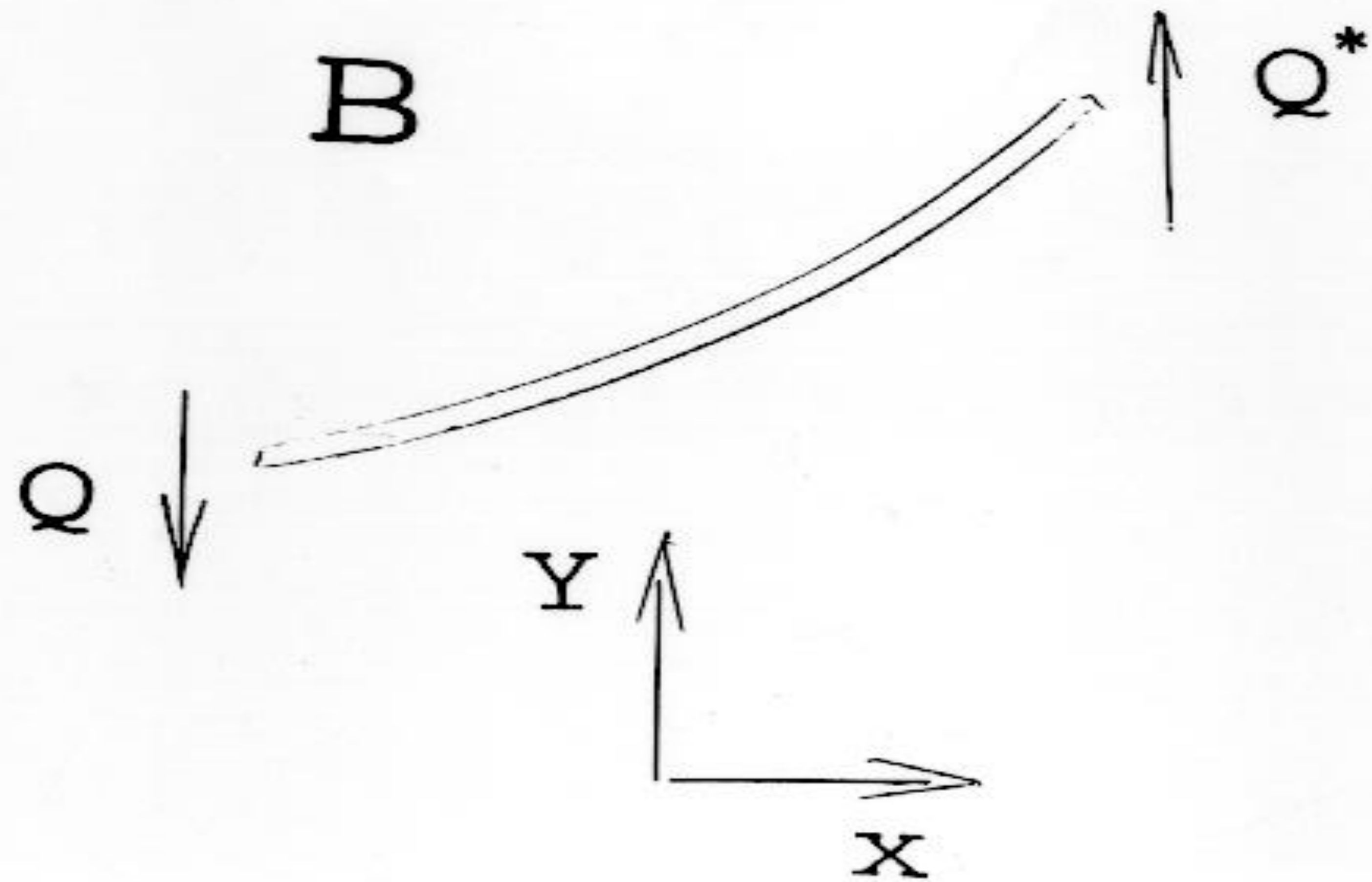
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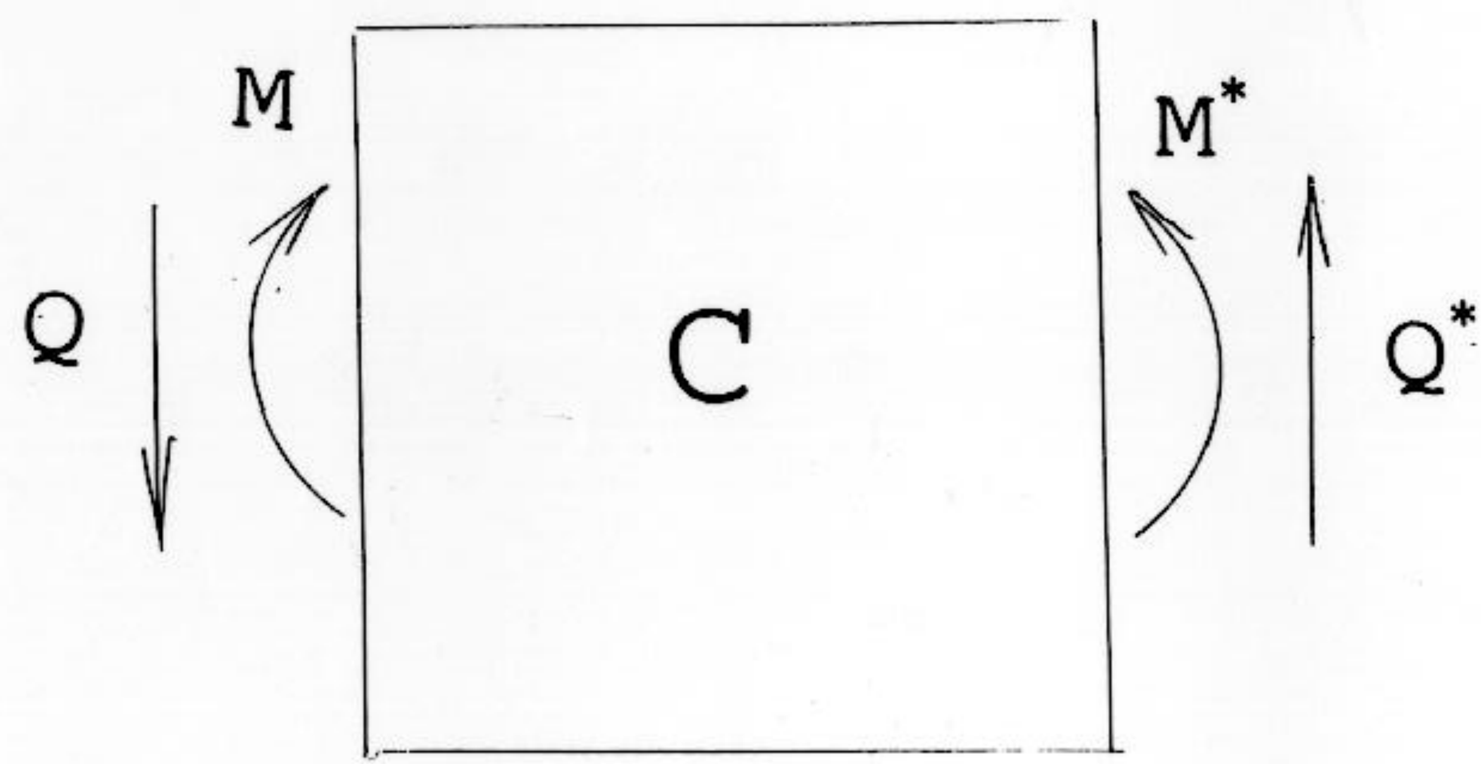
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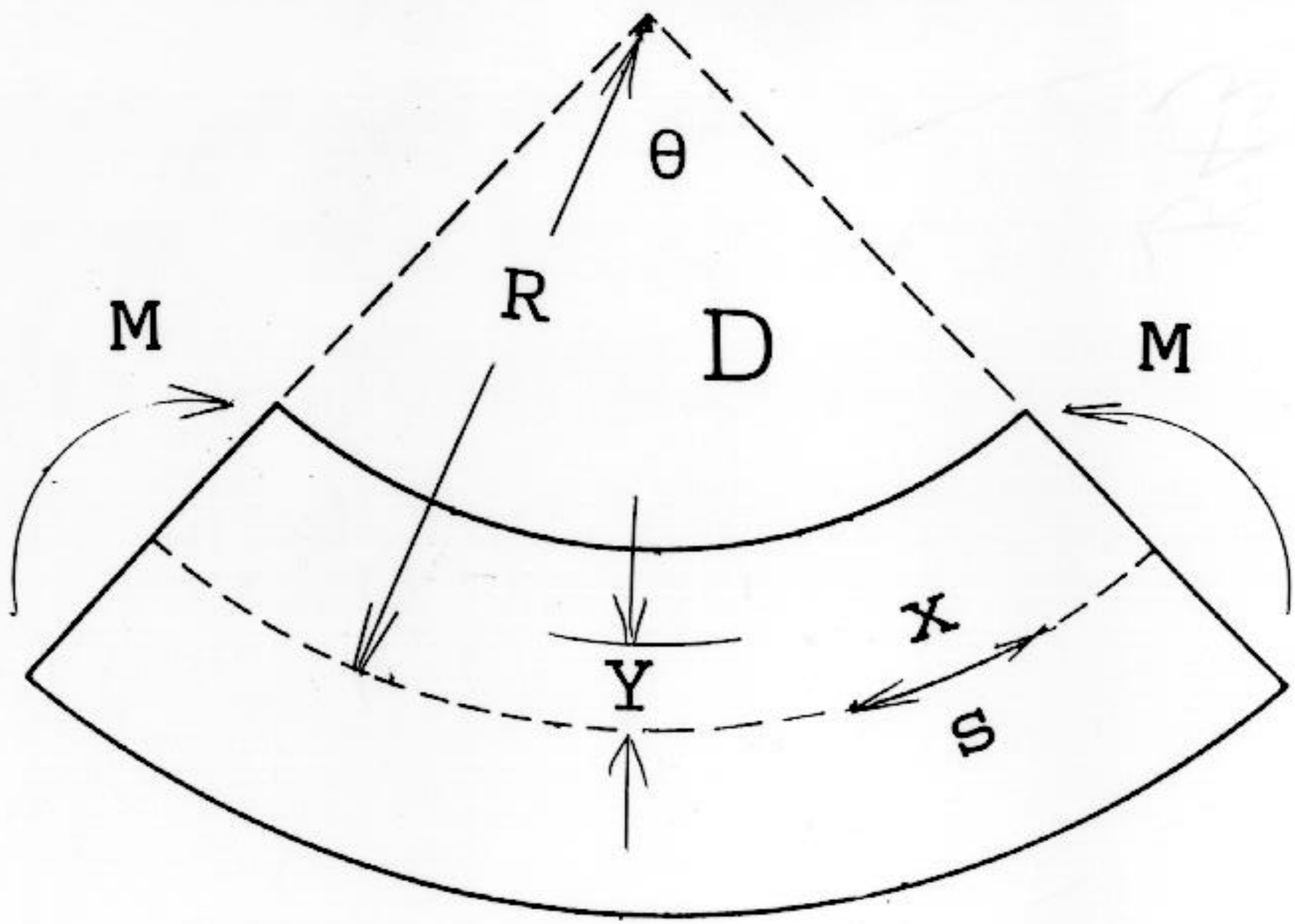
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FEA FIRST MODE VIBRATION OF A SIMPLE WIRE

The equation governing lateral motion of a simple wire is

$$- \partial/\partial x \ (T \ \partial Y/\partial x) \ + \ M \ \partial^2 Y/\partial t^2 \ = \ 0$$

For free vibration, Y = **Y** Sin[ωt], where **Y** is the deflection shape. Substitution into the governing equation gives

$$- d/dx \ (T \ d\mathbf{Y}/dx) \ - \ M \ \omega^2 \ \mathbf{Y} = \ 0$$

For a two element, Galerkin Method of Weighted Residuals, Finite Element Analysis, the deflection shape has the form

$$\mathbf{Y} \ = \ A \ n \qquad\qquad A_L=\varepsilon \qquad A_R=1-\varepsilon$$

where n is the deflection of the node at the middle of the wire, A is a shape function, A_L is the part of A to the left of the node, A_R is the part of A to the right of the node and ε is a local element coordinate. Substitution into the governing equation gives a residual. Weighting this residual by the shape function W and integrating along the wire gives

$$\int_0^L [\ - \ W \ T \ d^2\mathbf{Y}/dx^2 \ - \ W \ M \ \omega^2 \ \mathbf{Y} \ ] \ dx \ = \ 0$$

Integration by parts of the space derivative term gives

$$\int_0^L [\ dW/dx \ T \ d\mathbf{Y}/dx \ - \ W \ M \ \omega^2 \ \mathbf{Y} \ ] \ dx \ = \ 0$$

This has left element and right element contributions

$$\int_0^S [\ dW/dx \ T \ d\mathbf{Y}/dx - W \ M \ \omega^2 \ \mathbf{Y} \ ] \ dx$$

$$\int_S^L [\, dW/dx \; T \; d\mathbf{Y}/dx - W \; M \; \omega^2 \; \mathbf{Y} \,] \; dx$$

where S=L/2 is the element span. In terms of the local element coordinate, each contribution becomes

$$\int_0^1 [\, dW/d\varepsilon \; T/[S^2] \; d\mathbf{Y}/d\varepsilon - W \; M \; \omega^2 \; \mathbf{Y} \,] \; S \; d\varepsilon$$

Setting the weighting function W equal to A gives

$$\int_0^1 [\, dA/d\varepsilon \; T/[S^2] \; d\mathbf{Y}/d\varepsilon - A \; M \; \omega^2 \; \mathbf{Y} \,] \; S \; d\varepsilon$$

The MWR integral contains the following integrals

$$\int [\, dA/d\varepsilon \; dA/d\varepsilon \,] \; d\varepsilon = 1 \qquad \int [\, A \; A \,] \; d\varepsilon = 1/3$$

Substitution into the MWR integrals gives

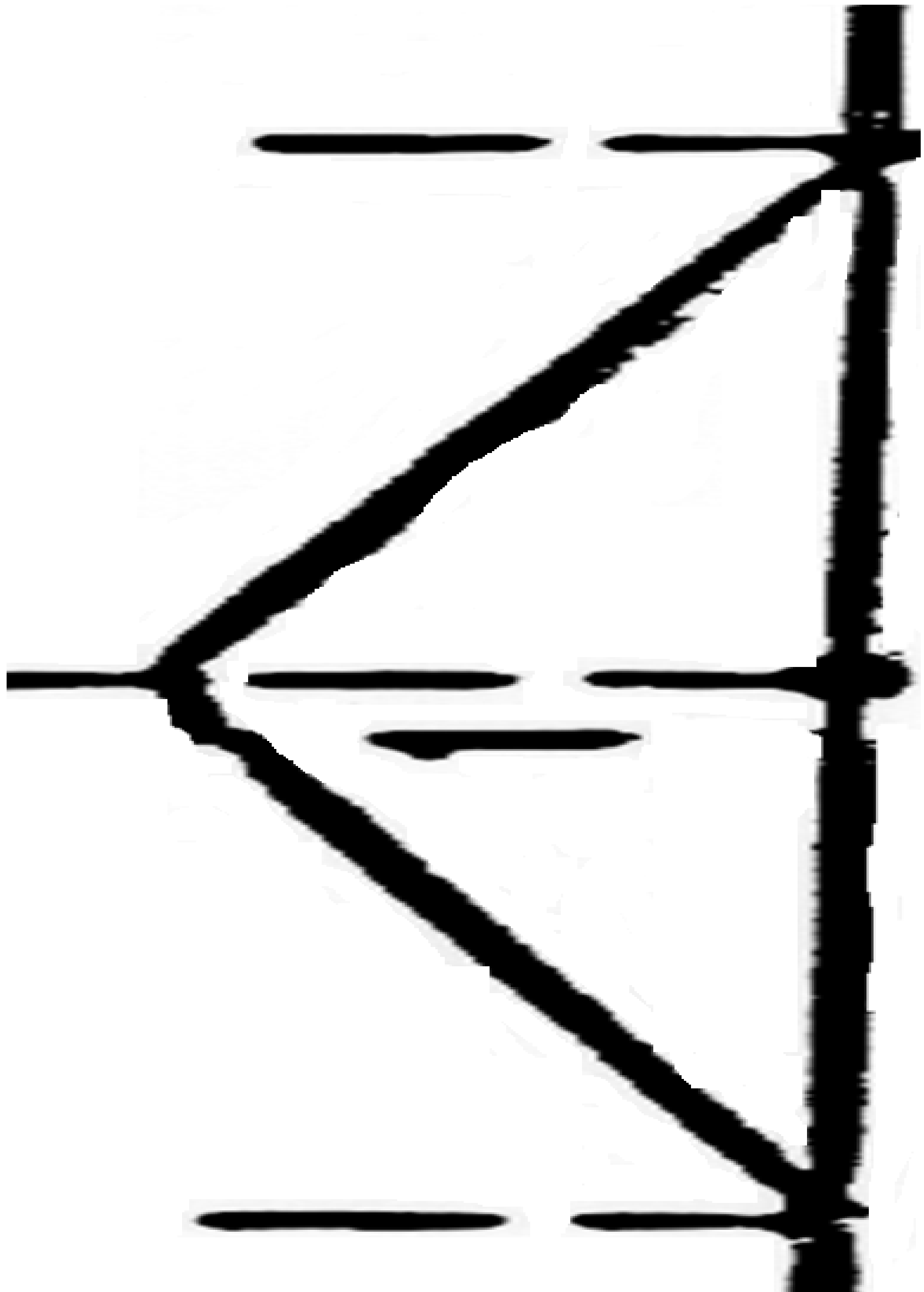
$$[\, T/S^2 - [M/3] \; \omega^2] \; S + [\, T/S^2 - [M/3] \; \omega^2] \; S = 0$$

Manipulation gives

$$\omega = \sqrt{3}/S \; \sqrt{[T/M]} = 2\sqrt{3}/L \; \sqrt{[T/M]} = 3.46/L \; \sqrt{[T/M]}$$

The simple wire theoretical frequency equation is

$$\omega = \pi/L \; \sqrt{[T/M]} = 3.14/L \; \sqrt{[T/M]}$$



FEA FIRST MODE VIBRATION OF A SIMPLE CANTILEVER BEAM

The equation governing lateral motion of a simple beam is

$$\partial^2/\partial x^2 \; (EI \; \partial^2 Y/\partial x^2) \; + \; M \; \partial^2 Y/\partial t^2 \; = \; 0$$

For free vibration, Y = **Y** Sin[ωt], where **Y** is the deflection shape. Substitution into the governing equation gives

$$d^2/dx^2 \; (EI \; d^2\mathbf{Y}/dx^2) \; - \; M \; \omega^2 \; \mathbf{Y} = \; 0$$

For a single element, Galerkin Method of Weighted Residuals, Finite Element Analysis, the deflection shape has the form

$$\mathbf{Y} \; = \; A \; n + B \; m$$

$$A = \varepsilon^2(3-2\varepsilon) \qquad\qquad B = L \; \varepsilon^2(\varepsilon-1)$$

where A and B are shape functions, ε is a local element coordinate, n is the deflection at the tip of the beam, m is the slope at the tip and L is the beam length. Substitution into the governing equation gives a residual. Weighting this residual by each of the shape functions and integrating along the length of the beam gives

$$\int_0^L \; [\; W \; EI \; d^4\mathbf{Y}/dx^4 \; - \; W \; M \; \omega^2 \; \mathbf{Y} \;] \; dx \; = \; 0$$

Integration by parts of the space derivative term gives

$$\int_0^L \; [\; d^2W/dx^2 \; EI \; d^2\mathbf{Y}/dx^2 \; - \; W \; M \; \omega^2 \; \mathbf{Y} \;] \; dx \; = \; 0$$

In terms of the local element coordinate, this becomes

$$\int_0^1 \; [\; d^2W/d\varepsilon^2 \; EI/[L^4] \; d^2\mathbf{Y}/d\varepsilon^2 \; - \; W \; M \; \omega^2 \; \mathbf{Y} \;] \; L \; d\varepsilon = 0$$

In the MWR integrals

$$d^2\mathbf{Y}/d\varepsilon^2 \; = d^2A/d\varepsilon^2 \; n + d^2B/d\varepsilon^2 \; m \; = \; [-12\varepsilon+6]n + [6L\varepsilon-2L]m$$

For a beam, there are two weighting functions. These are the shape functions A and B. The resulting MWR integrals contain the following integrals

$$\int [d^2A/d\varepsilon^2 \; d^2A/d\varepsilon^2]d\varepsilon = +12 \qquad \int [A\;A]d\varepsilon = +78/210$$

$$\int [d^2B/d\varepsilon^2 \; d^2B/d\varepsilon^2]d\varepsilon = +4L^2 \qquad \int [B\;B]d\varepsilon = +2/210\;L^2$$

$$\int [d^2A/d\varepsilon^2 \; d^2B/d\varepsilon^2]d\varepsilon = -6L \qquad \int [A\;B]d\varepsilon = -11/210\;L$$

Substitution into the MWR integrals gives

$$L(+12n - 6Lm)[EI]/[L^4] - L\;M\;\omega^2(+78/210n - 11L/210m) = 0$$

$$L(-6Ln + 4L^2m)[EI]/[L^4] - L\;M\;\omega^2(-11L/210n + 2L^2/210m) = 0$$

Manipulation gives

$$[+12[EI]/[L^4]-[78/210]M\omega^2]n+[-6L[EI]/[L^4]+[11L/210]M\omega^2]m=0$$

$$[-6L[EI]/[L^4]+[11L/210]M\omega^2]n+[+4L^2[EI]/[L^4]-[2L^2/210]M\omega^2]m=0$$

One can put these equations in matrix form. Setting the determinant of the square matrix to zero gives

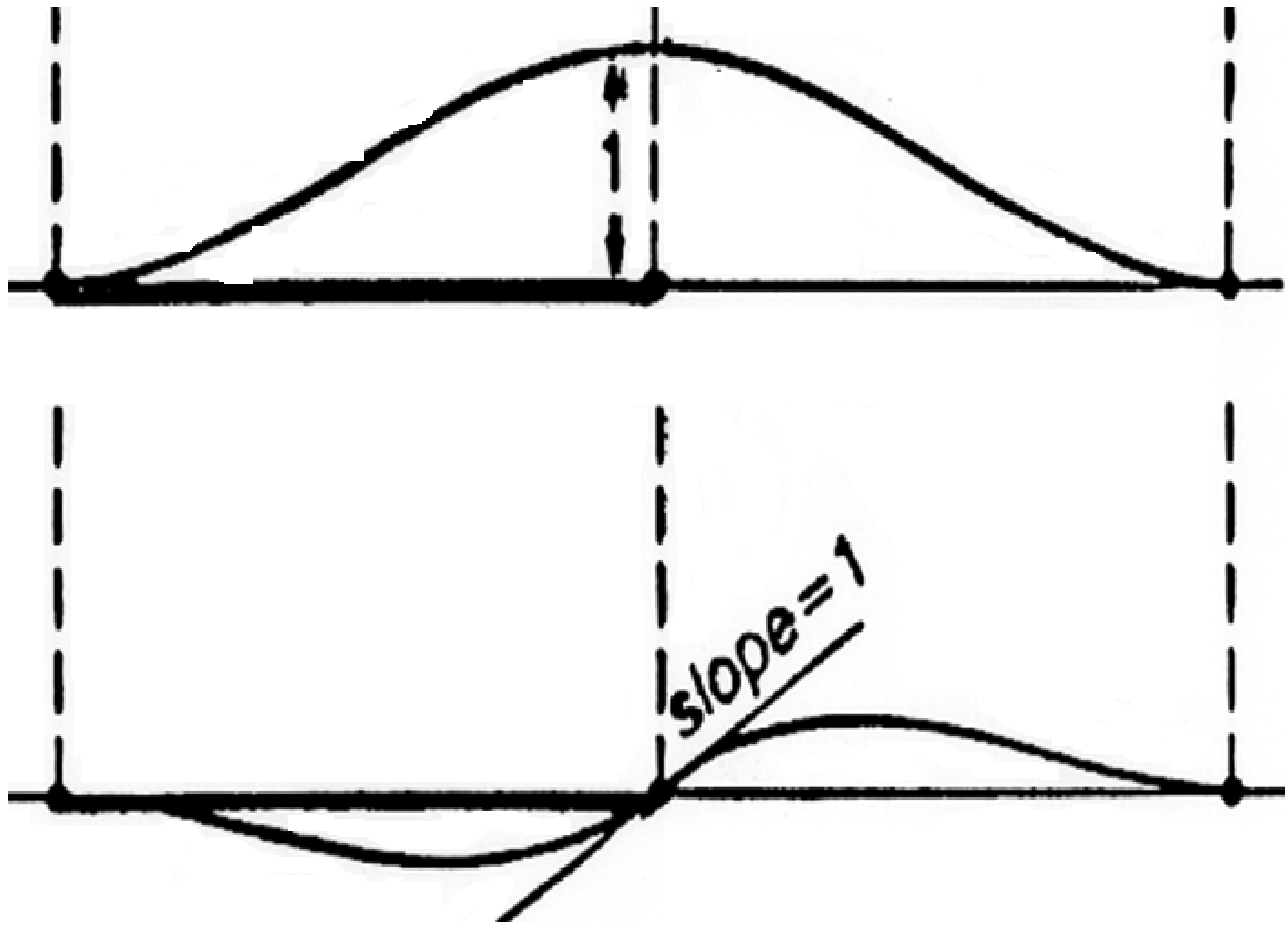
$$[+12[EI]/[L^4]-[78/210]M\omega^2]\;[+4L^2[EI]/[L^4]-[2L^2/210]M\omega^2]$$
$$-\;[-6L[EI]/[L^4]+[11L/210]M\omega^2]\;[-6L[EI]/[L^4]+[11L/210]M\omega^2] = 0$$

Manipulation gives a quadratic for ω^2. It gives

$$\omega = 3.53/L^2\;\sqrt{[EI]/M}$$

The simple beam theoretical solution is

$$\omega = 3.52/L^2\;\sqrt{[EI]/M}$$


1
slope = 1

LIFTING BODY INSTABILITIES

Flutter is a dynamic instability of a lifting body. When it occurs, the heave and pitch motions of the body are 90° out of phase. The passing stream does work on the body over an oscillation cycle. Divergence is a static instability. It occurs when the pitch moment due to fluid dynamics overcomes the moment due to the structural pitch stiffness of the body.

FLUTTER AND DIVERGENCE OF FOILS

A foil is a section of a lifting body. Here quasi steady fluid dynamics theory is used to get the loads on the foil. This ignores the fact that, when a foil is heaving and pitching, vortices are shed behind it because its circulation keeps changing. These vortices influence the loads on the foil. The equations governing motions of a foil are:

$$K\,h + i\,dh/dt + M\,d^2h/dt^2 + Ma\,d^2\alpha/dt^2 + L = H$$

$$k\,\alpha + j\,d\alpha/dt + I\,d^2\alpha/dt^2 + Ma\,d^2h/dt^2 + T = P$$

where h is the downward heave displacement of the foil, α is its upward pitch displacement, M is the mass of the foil, I is its rotary inertia, K is the heave stiffness of the foil, k is its pitch stiffness, i is the heave damping coefficient of the foil, j is its pitch damping coefficient, L is the lift on the foil, T is the pitch moment and H and P are disturbance loads. Quasi steady fluid dynamics theory gives for the fluid dynamic loads L and T:

$$L = \rho U^2/2 \; C C_P \; \beta \qquad T = \rho U^2/2 \; C^2 \; \kappa$$

where

$$\beta = \alpha + (dh/dt)/U + (3C/4-b)/U \; (d\alpha/dt)$$

$$\kappa = (C/4-b)/C \; C_P \; \beta + C\pi/[8U] \; (d\alpha/dt)$$

where U is the speed of the foil, C is its chord length, a indicates how far the center of gravity is behind the elastic axis, b is the distance between the elastic axis and the leading edge of the foil and C_P is a constant given by fluid dynamics theory: it is approximately 2π.

Note that the parameter β is the instantaneous angle of attack of the foil 3C/4 back from its leading edge. It is made up of three components. The first component is the pitch angle α. The second component is due to the change in flow direction caused by the heave rate dh/dt. The third component is due to the change in flow direction caused by the pitch rate dα/dt at the 3C/4 location. The 3C/4 location is suggested by flat plate foil theory. Theory shows that the center of pressure on a foil is at C/4 back from the leading edge. This gives rise to the first term in the pitch moment parameter κ. The second term is due to the distribution of pressure over the foil.

One can Laplace Transform the governing equations and manipulate to get a characteristic equation. Stability is dependent on the roots of this equation. One can get the roots numerically and plot them in a Root Locus Plot as a function of foil speed. This would give the critical speed corresponding to the onset of instability.

FLUTTER AND DIVERGENCE OF WINGS

Here strip theory is used to get the loads on a wing. The wing is broken into strips spanwise and quasi steady fluid dynamics theory is used to get the loads on each strip. This ignores the fact that, when a wing is heaving and pitching, vortices are shed behind it because its circulation keeps changing. These vortices influence the loads on the wing. It also ignores the fact that for a finite span wing vortices are shed along its span but mainly at its tips. These vortices create a downwash on the wing. This reduces the lift on the wing because it lowers its apparent angle of attack. It also tilts the load on the wing backwards and this gives rise to a drag. The equations governing heave and pitch motions of a wing are:

$$\partial^2/\partial y^2 \ (\mathbf{EI}\partial^2 h/\partial y^2) \ + \ M\ \partial^2 h/\partial t^2 \ + \ Ma\ \partial^2\alpha/\partial t^2 \ + \ \rho U^2/2\ CC_P\ \beta \ = \ H$$

$$- \partial/\partial y \ (\mathbf{GJ}\partial\alpha/\partial y) \ + \ I \ \partial^2\alpha/\partial t^2 \ + \ Ma \ \partial^2 h/\partial t^2$$
$$+ \ \rho U^2/2 \ C^2 \ \kappa \ = \ P$$

In these equations, h is the downward heave displacement of the wing and α is the upward pitch displacement of the wing. **EI** and **GJ** account for the stiffness of the wing per unit span. M and I are its inertias per unit span. The chord of the wing is C and its span is Q. The speed of the wing is U. The distance from the elastic axis to the center of gravity is a. The distance from the leading edge to the elastic axis is b. H and P are disturbance loads.

Fluid dynamic loads per unit span acting on the wing are determined by the β and κ parameters. These are:

$$\beta = \alpha + (\partial h/\partial t)/U + (3C/4-b)/U \ (\partial\alpha/\partial t)$$
$$\kappa = (C/4-b)/C \ C_P \ \beta + C\pi/8/U \ (\partial\alpha/\partial t)$$

For a Galerkin finite element analysis, we let h and α each be a sum of scaled shape functions as follows:

$$h = \Sigma \ [A \ n + B \ m] \qquad \alpha = \Sigma \ D \ p$$

A and B and D are the shape functions. In the equation for heave, n is the heave at a node while m is the heave slope at a node. In the equation for pitch, p is the pitch at a node. For a typical node, the shape functions are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

$$D_L = \varepsilon \qquad D_R = 1-\varepsilon$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of a node. The polynomials used for heave are known as Hermite polynomials. They must be used because the stiffness term in the heave governing equation is 4th order. They are not needed for pitch because its stiffness term is only 2nd order: linear shape functions are adequate for it.

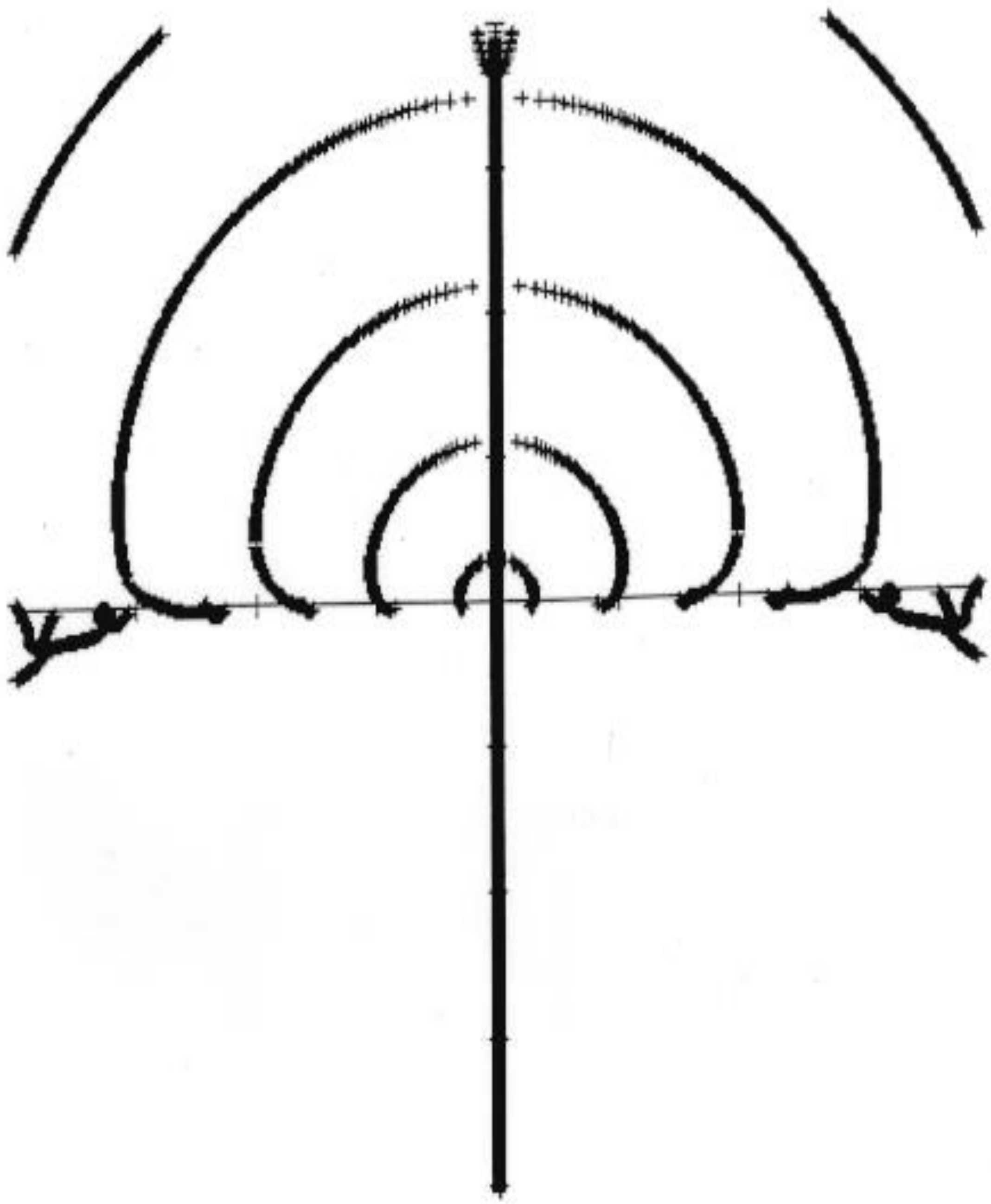
Substitution of the assumed forms for h and α into the governing equations gives residuals. In a Galerkin analysis, weighted averages of these residuals along the span of the wing are set to zero. After some manipulation, one gets

$$\int [\partial^2 W/\partial y^2 \ \mathbf{EI} \ \partial^2 h/\partial y^2 + WM \ \partial^2 h/\partial t^2 + WMa \ \partial^2 \alpha/\partial t^2 + W\rho U^2/2 \ CC_P \ \beta - WH] \ dy = 0$$

$$\int [\partial \mathbf{W}/\partial y \ \mathbf{GJ} \ \partial \alpha/\partial y + \mathbf{W} I \ \partial^2 \alpha/\partial t^2 + \mathbf{W} Ma \ \partial^2 h/\partial t^2 + \mathbf{W}\rho U^2/2 \ C^2 \ \kappa - \mathbf{W} P] \ dy = 0$$

where W and **W** are weighting functions. For a Galerkin analysis, these are just the shape functions used to define h and α. In other words, W is A and B for each node while **W** is D for each node. Notice the double integration by parts of the space derivative term in the heave integral. This introduces tip shear and tip bending moment boundary conditions into the formulation. Both of these are zero for a wing. Notice the single integration by parts of the space derivative term in the pitch integral. This introduces tip torsion into the formulation. Again this is zero for a wing.

After performing the integrations numerically using Gaussian Quadrature, one gets a set of Ordinary Differential Equations or ODEs in time. One can Laplace Transform these and manipulate to get a characteristic equation. Stability is dependent on the roots of this equation. Instead of using Laplace Transform approach, one can put the ODEs in a matrix form and use matrix manipulation to get the roots of the characteristic equation. One can plot them in a Root Locus Plot as a function of wing speed. This would give the critical speed corresponding to the onset of instability.



KELVIN HELMHOLTZ INSTABILITIES

Consider the flexible panel shown in Figure A. A fluid flowing over such a panel can cause it to flutter. The simplest analysis of this assumes the panel to be an infinitely long thin plate. It also assumes that the flow above and below the panel is potential flow. Conservation of mass considerations give:

$$\nabla^2\varphi_T = 0 \qquad \nabla^2\varphi_B = 0$$

where T indicates the top flow and B indicates the bottom flow. The kinematic constraints at walls are:

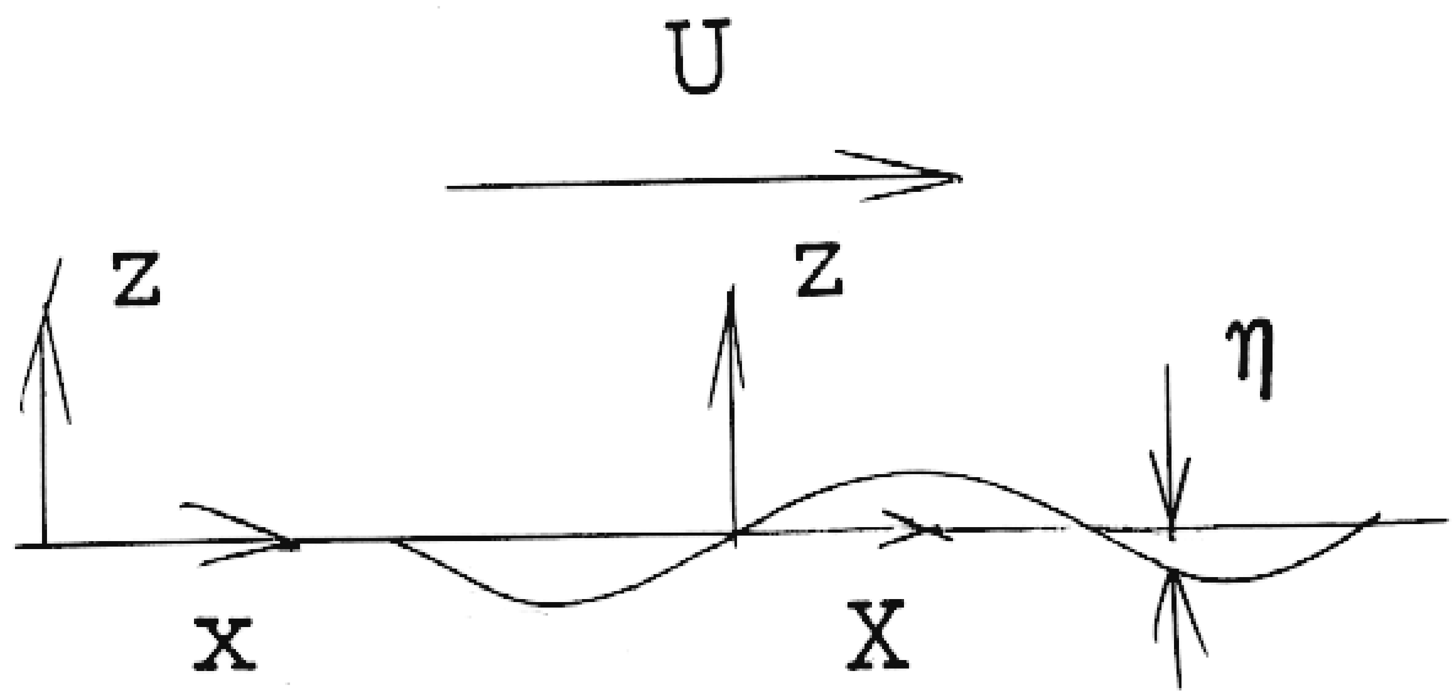
$$\partial\varphi_T/\partial z = 0 \qquad \text{at } z = +d_T$$

$$\partial\varphi_B/\partial z = 0 \qquad \text{at } z = -d_B$$

The panel kinematic constraints are based on:

$$D\eta/Dt = Dz/Dt$$

where η is the vertical deflection of the panel from its rest state. The η for a point on the panel must follow the z for



A

that point. The constraint gives for the top and bottom of the panel:

$$\partial\eta/\partial t + U\,\partial\eta/\partial x = \partial\varphi_T/\partial z \quad \text{at} \quad z = 0$$

$$\partial\eta/\partial t = \partial\varphi_B/\partial z \quad \text{at} \quad z = 0$$

The panel dynamic constraints are:

$$\partial\varphi_T/\partial t + U\,\partial\varphi_T/\partial x + P_T/\rho_T + g\eta = 0 \quad \text{at} \quad z = 0$$

$$\partial\varphi_B/\partial t + P_B/\rho_B + g\eta = 0 \quad \text{at} \quad z = 0$$

Finally, the equation of motion of the panel is:

$$\sigma\, w\, \partial^2\eta/\partial t^2 = (P_B - P_T)\,w - K\,\eta + T\,w\,\partial^2\eta/\partial x^2 - D\,w\,\partial^4\eta/\partial x^4$$

where σ is the sheet density of the panel, w is the panel width, K accounts for side support forces, T is the tension in the panel and D=EI is its flexural rigidity.

The dynamic constraints give:

$$P_T = -\rho_T\,(\partial\varphi_T/\partial t + U\partial\varphi_T/\partial x) - \rho_T\,g\eta \quad \text{at} \quad z = 0$$

$$P_B = -\rho_B\,(\partial\varphi_B/\partial t) - \rho_B\,g\eta \quad \text{at} \quad z = 0$$

Substitution into the panel equation of motion gives:

$$\sigma\,\partial^2\eta/\partial t^2 = -\rho_B\,(\partial\varphi_B/\partial t) + \rho_T\,(\partial\varphi_T/\partial t + U\partial\varphi_T/\partial x)$$

$$-\rho_B g\,\eta + \rho_T g\,\eta - K/w\,\eta + T\,\partial^2\eta/\partial x^2 - D\,\partial^4\eta/\partial x^4$$

Consider the general solution forms:

$$\varphi_T = [G \; Sinh[kz] + H \; Cosh[kz] \;] \quad e^{jkX}$$

$$\varphi_B = [I \; Sinh[kz] + J \; Cosh[kz] \;] \quad e^{jkX}$$

$$\eta = \eta_O \; e^{jkX}$$

where $kX = k(x - C_P t) = kx - \omega t$ where X is the horizontal coordinate of a wave fixed frame, x is the horizontal coordinate of an inertial frame, C_P is the wave phase speed, k is the wave number and ω is the wave frequency. The wall constraints give

$$\varphi_T = \varphi_{TO} \quad Cosh[k(d_T - z)]/Cosh[kd_T] \quad e^{jkX}$$

$$\varphi_B = \varphi_{BO} \quad Cosh[k(d_B + z)]/Cosh[kd_B] \quad e^{jkX}$$

$$\eta = \eta_O \; e^{jkX}$$

These satisfy everything except the panel kinematic constraints and the panel equation of motion. Substitution into the panel equations gives, after common terms are cancelled away:

$$-j\omega \; \eta_O \; + \; Ujk \; \eta_O \; = \; -k \; \varphi_{TO} \; Tanh[kd_T]$$

$$-j\omega \; \eta_O \; = \; +k \; \varphi_{BO} \; Tanh[kd_B]$$

$$\rho_T \; [-j\omega + Ujk \;] \; \varphi_{TO} - \; \rho_B \; [-j\omega] \; \varphi_{BO} + \; \rho_T g \; \eta_O \; - \; \rho_B g \; \eta_O$$

$$- \; Tk^2 \; \eta_O \; - \; Dk^4 \; \eta_O \; - \; K/w \; \eta_O \; - \; \sigma[-j\omega]^2 \; \eta_O \; = \; 0$$

Substitution into the last equation gives:

$$\rho_T[-j\omega+Ujk][+j\omega\eta_O-Ujk\eta_O]/[kTanh[kd_T]]$$
$$-\ \rho_B[-j\omega][-j\omega\eta_O]/[kTanh[kd_B]]\ +\ \rho_T g\ \eta_O$$
$$-\ \rho_B g\ \eta_O\ -\ Tk^2\ \eta_O\ -\ Dk^4\ \eta_O\ -\ K/w\ \eta_O\ -\ \sigma[-j\omega]^2\ \eta_O\ =\ 0$$

Manipulation of this gives an equation of the form:

$$A\ \omega^2\ +\ B\ \omega\ +\ C\ =\ 0$$

$$A\ =\ \rho_T/[kTanh[kd_T]]\ +\ \rho_B/[kTanh[kd_B]]\ +\ \sigma$$

$$B\ =\ -\ 2U\rho_T/Tanh[kd_T]$$

$$C\ =\ -\ S\ +\ U^2k\rho_T/Tanh[kd_T]$$

$$S\ =\ +\ Tk^2\ +\ Dk^4\ +\ K/w\ -\ \rho_T g\ +\ \rho_B g$$

When B^2 - 4AC is negative, the roots of the quadratic for ω form a complex conjugate pair:

$$\omega_1\ =\ \alpha\ +\ \beta j \qquad \omega_2\ =\ \alpha\ -\ \beta j$$
$$\alpha\ =\ -B/2A \qquad \beta\ =\ \sqrt{[4AC-B^2]}/2A$$

Substitution of ω_1 into the wave profile equation gives:

$$\eta_O\ e^{jkX} = (\Delta_R+\Delta_I j)\ e^{j[kx-(\alpha+\beta j)t]}$$

$$= (\Delta_R+\Delta_I j)\ e^{j[kx-\alpha t]}\ e^{j[-\beta j t]} = (\Delta_R+\Delta_I j)\ e^{+\beta t}\ e^{j[kx-\alpha t]}$$

$$= (\Delta_R+\Delta_I j)\ e^{+\beta t}\ [\mathrm{Cos}(kx-\alpha t)+j\mathrm{Sin}(kx-\alpha t)]$$

The real part of this is:

$$[\ \Delta_R\ \mathrm{Cos}(kx-\alpha t) - \Delta_I\ \mathrm{Sin}(kx-\alpha t)\ ]\ e^{+\beta t}$$

$$= \Delta\ \ e^{+\beta t}\ \mathrm{Sin}[(kx-\alpha t) + \varepsilon]$$

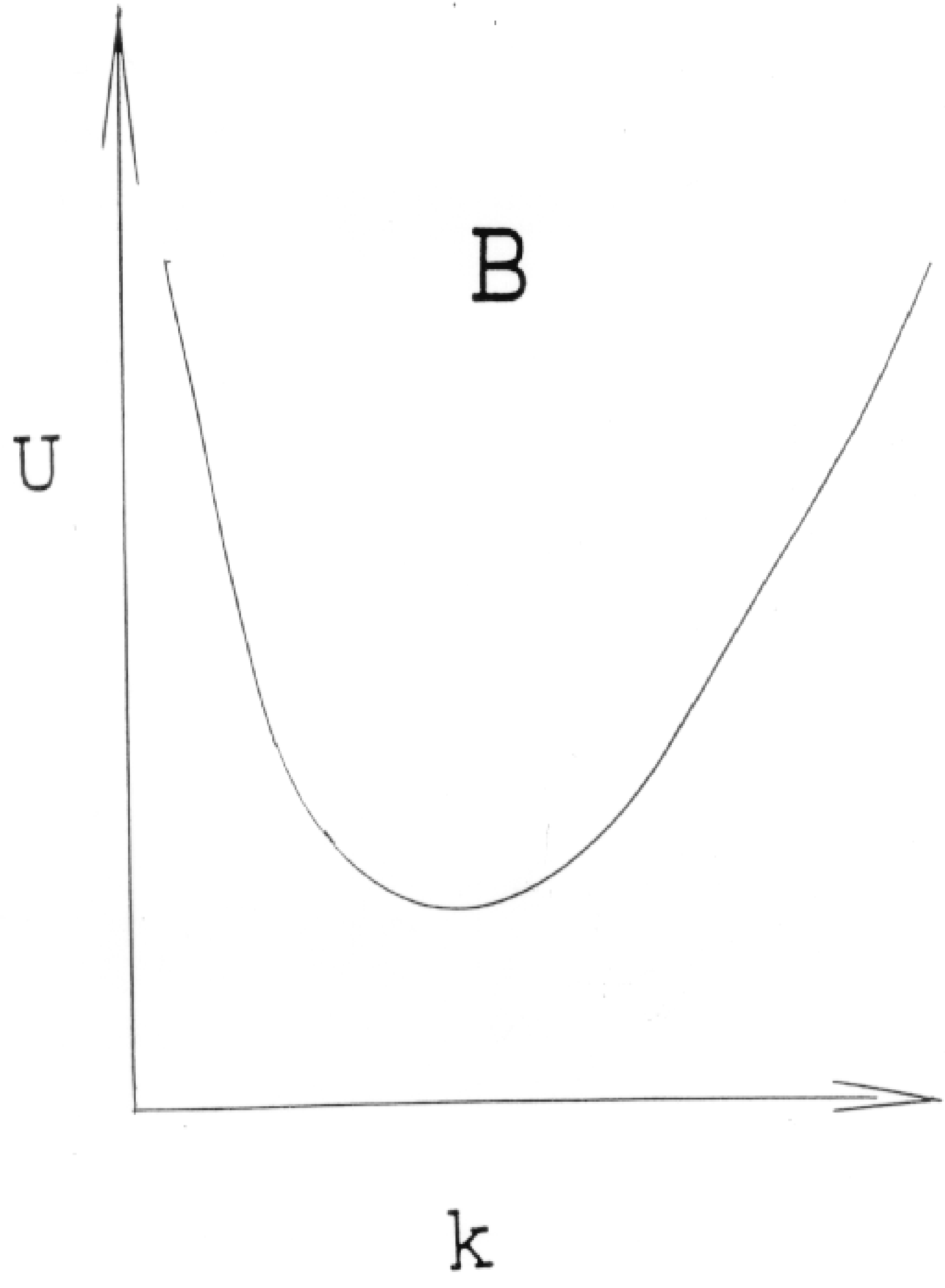
This shows that, when B^2-4AC is negative, the ω_1 wave grows. Similarly, one can show that the ω_2 wave decays. Substitution into $B^2 - 4AC = 0$ gives the critical speed:

$$U^2 = S\ V/W$$

$$V = \rho_T/[k\mathrm{Tanh}[kd_T]] + \rho_B/[k\mathrm{Tanh}[kd_B]] + \sigma$$

$$W = \rho_B\rho_T/[\mathrm{Tanh}[kd_T]\mathrm{Tanh}[kd_B]] + k\sigma\rho_{T/}\mathrm{Tanh}[kd_T]$$

This is sketched in Figure B. The plot shows that, if U is below a certain level, the panel does not flutter. For U beyond this level, it flutters for a range of k.


B
U
k

For a membrane under uniform pressure load

$$\mathbf{T}\ d^2\Delta/dx^2 = P$$

Integration shows that the mean deflection is:

$$\boldsymbol{\Delta} = P\ w^2 / [12\ \mathbf{T}]$$

This gives the side support stiffness

$$K^* = [12\ \mathbf{T}] / w^2$$

For a beam under uniform pressure load

$$\mathbf{EI}\ d^4\Delta/dx^4 = P$$

Integration shows that the mean deflection is:

$$\boldsymbol{\Delta} = P\ w^4 / [120\ \mathbf{EI}]$$

This gives the side support stiffness

$$K^* = [120\ \mathbf{EI}] / w^4$$

PIPE INSTABILITIES DUE TO INTERNAL FLOW

The equation governing the lateral vibration of a pipe containing an internal flow is

$$M \, \partial^2 Y/\partial t^2 \;=\; - \, \partial^2/\partial x^2 \, (EI \, \partial^2 Y/\partial x^2) \; + \; T \, \partial^2 Y/\partial x^2$$

$$- \; PA \, \partial^2 Y/\partial x^2 \; - \; \rho A U^2 \, \partial^2 Y/\partial x^2 \; - \; 2\rho A U \, \partial^2 Y/\partial x \partial t$$

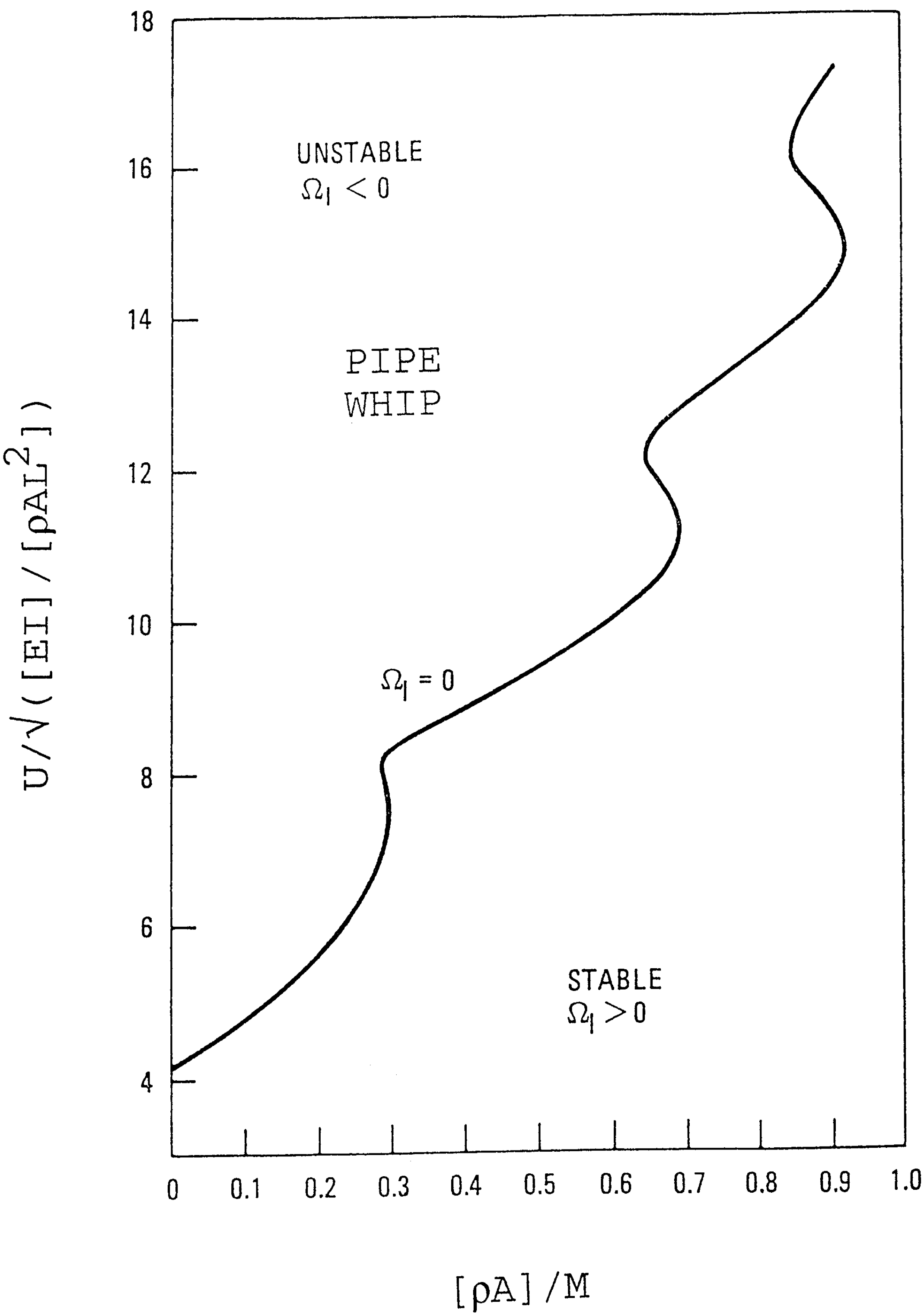
For a pipe pivoted at both ends, a static force balance shows that centrifugal forces generated by fluid motion can cause buckling when U is greater than

$$U^2 = [\; EI/[\rho A] \; \pi^2/L^2 + T/[\rho A] - P/\rho \;]$$

For a pipe clamped at one end and open and free at the other end, a stability analysis shows that the pipe can undergo a flutter like phenomenon known as pipe whip. The critical speed U can be obtained from the sketch on the next page. A straight line fit to the wavy curve there is

$$U = [4 + 14 \, M_o/M] \; U_o$$

$$U_o = \sqrt{[EI]/[M_o L^2]} \qquad\qquad M_o = \rho A$$



18
16
14
12
10
8
6
4
UNSTABLE
$\Omega_I < 0$
PIPE
WHIP
$\Omega_I = 0$
STABLE
$\Omega_I > 0$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$U/\sqrt{([EI]/[\rho AL^2])}$
$[\rho A]/M$

PIPE WHIP INSTABILTY

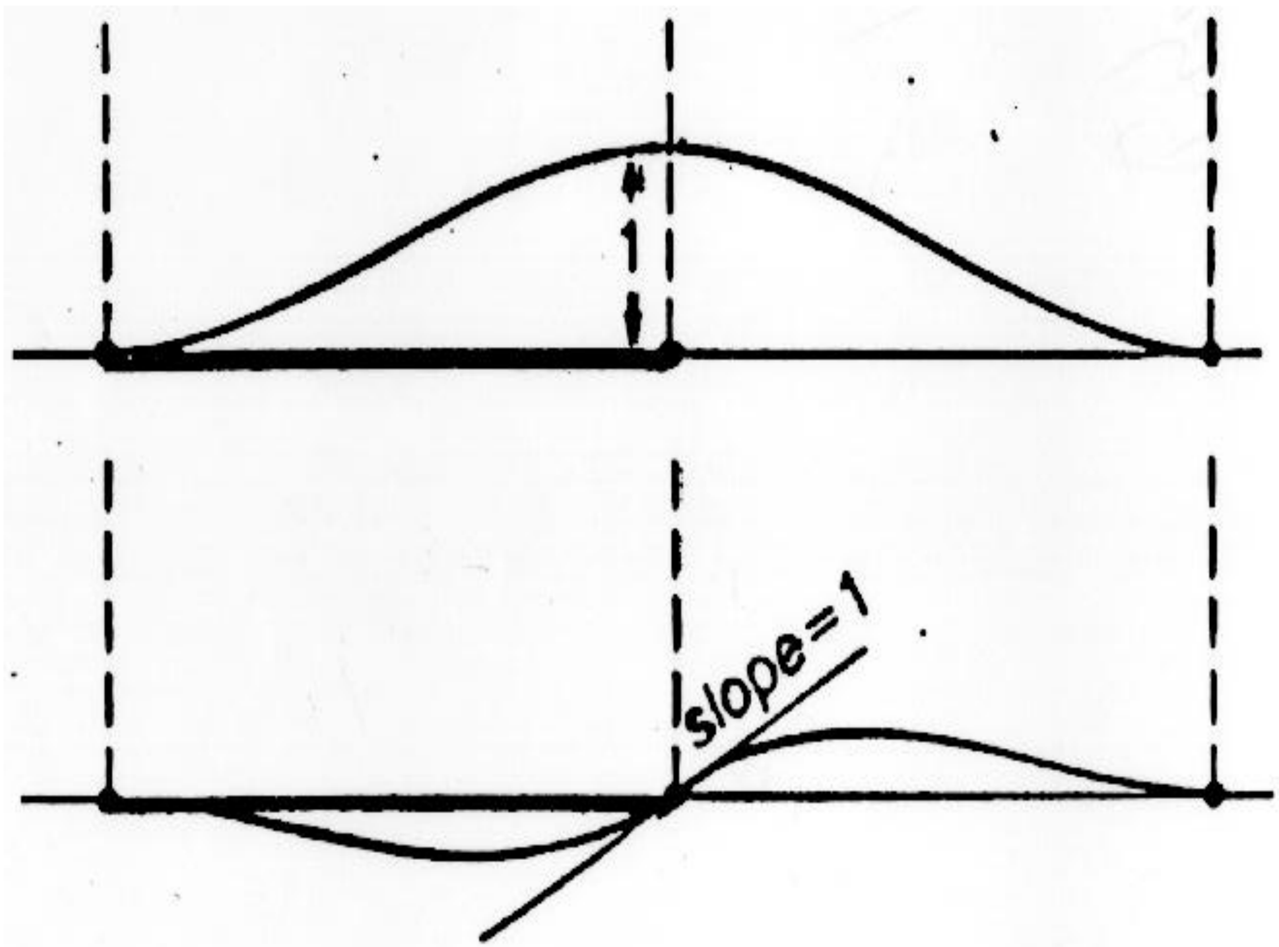
The equation governing the lateral vibration of a pipe containing an internal flow is

$$0 = M\,\partial^2 Y/\partial t^2 + \partial^2/\partial x^2\,(EI\,\partial^2 Y/\partial x^2) - T\,\partial^2 Y/\partial x^2$$
$$+ PA\,\partial^2 Y/\partial x^2 + \rho A U^2\,\partial^2 Y/\partial x^2 + 2\rho A U\,\partial^2 Y/\partial x \partial t$$

In this equation, Y is the lateral deflection of the pipe from its neutral position, M is the total mass of the pipe per unit length, EI is its flexural rigidity, T is tension, P is pressure, U is flow speed, A is pipe area, x is location along the pipe and t is time. For a Galerkin finite element analysis, we assume that the deflection of the pipe can be given as a sum of scaled shape functions:

$$\mathbf{Y} = \Sigma\,[A\,n + B\,m]$$

where n is the deflection at a node and m is the slope at the node. A and B are shape functions. Theory shows that these must be Hermite polynomials. Such polynomials must be used because the EI term is 4th order. The sketch on the next page shows what they look like for a typical node.


1
slope=1

Substitution of the assumed form for **Y** into the governing equation gives a residual **R**. In a Galerkin analysis, weighted averages of this residual along the pipe are set to zero:

$$\int_0^L \mathbf{W}\,\mathbf{R}\,dx = 0$$

where L is the length of the pipe and **W** is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of the node. After performing the integrations and applying boundary conditions, one gets a set of ODEs in time. One can put them in a matrix form and use matrix manipulation to get the roots $\boldsymbol{\lambda}$ of the system characteristic equation.

$$[GI]\ |d\Phi/dt| + [GS]\ |\Phi| = |0|$$

$$[GI]\ \boldsymbol{\lambda}\ |\Phi_o| + [GS]\ |\Phi_o| = |0|$$

One can plot the roots in a Root Locus Plot to get the critical speed corresponding to the onset of instability.

FLUID STRUCTURE INTERACTIONS

UNSTEADY FLOW

IN PIPE NETWORKS

PREAMBLE

Unsteady flow in pipe networks can be caused by a number of factors. A turbomachine with blades can send pressure waves down a pipe. If the period of these waves matches a natural period of the pipe wave speed resonance develops. A piston pump can send similar waves down a pipe. Waves on the surface of a water reservoir can also excite resonance of inlet pipes. One way to avoid resonance is to change the wave speed of the pipes in the network. For liquids, one can do this by adding a gas such as air. This can be bled into the network at critical locations or it can be held in a flexible tube which runs inside the pipes. One could also use a flexible pipe to change the wave speed. Sudden valve or turbomachine changes can send waves up and down pipes. These can cause the pipes to explode or implode. In some cases interaction between pipes and devices is such that oscillations develop automatically. Examples include oscillations set up by leaky valves and those set up by slow turbomachine controllers. To lessen the severity of transients in a hydraulic network, one can use gas accumulators. Hydro plants use surge pipes. Another way to lessen the severity of transients is use of relief valves. These are spring loaded valves which open when the pressure reaches a preset level. This can be high or low. For high pressure liquids, they create a pathway back to a sump. For low pressure liquids, they allow a gas such as air to enter the pipe. Bypass valves and check valves can be used to isolate turbomachines when they fail.

There are three procedures that can be used to study unsteady flow in pipe networks. The most complex of these is the Method of Characteristics. This finds directions in space and time along which the partial differential equations of mass and momentum reduce to an ordinary differential equation in time. Computational Fluid Dynamics codes have been developed based on this method that can handle extremely complex pipe networks. A second procedure is known as Graphical Waterhammer. It is a graphical form of a procedure known as Algebraic Waterhammer. It makes extensive use of PU plots. A third procedure is known as the Impedance Method. This makes use of Laplace Transforms. It employs something called the Impedance Transfer Function. It resembles closely a method used to study Electrical Transmission Lines.

These notes start with a physical description of how pressure waves propagate along a pipe. This is followed by a derivation of the basic wave equations. Then, wave speeds for waves in flexible tubes and mixtures are given. Next, an outline of Algebraic/Graphical Waterhammer is given. Finally, the Method of Characteristics is presented.

WAVE PROPAGATION IN PIPES

Consider flow in a rigid pipe with a valve at its downstream end and a reservoir at its upstream end. Assume that there are no friction losses. This implies that the pressure and flow speed are the same everywhere along the pipe.

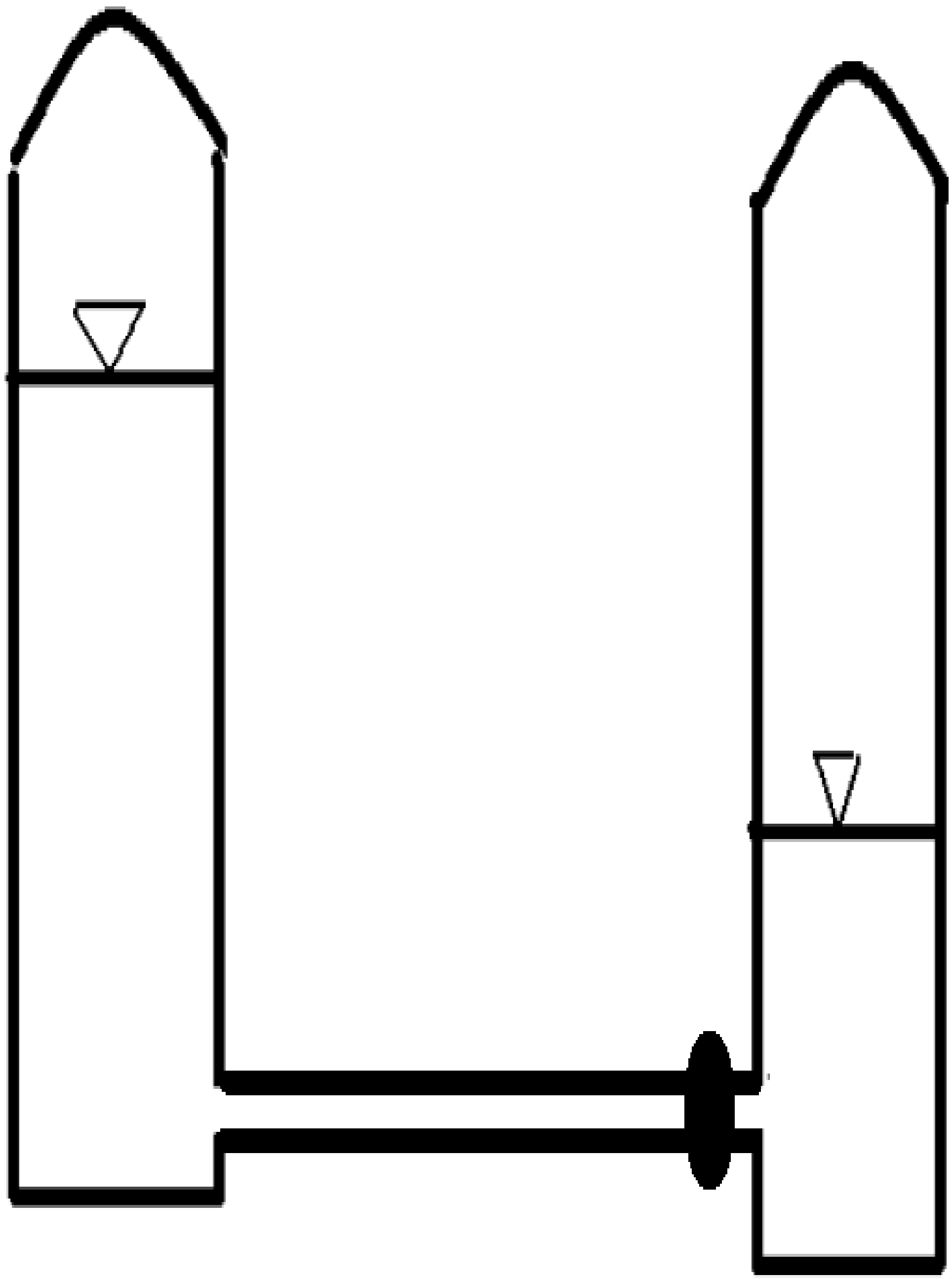
Imagine now that the valve is suddenly closed. This causes a high pressure or surge wave to propagate up the pipe. As it does so, it brings the fluid to rest. The fluid immediately next to the valve is stopped first. The valve is like a wall. Fluid enters an infinitesimal layer next to this wall and pressurizes it and stops. This layer becomes like a wall for an infinitesimal layer just upstream. Fluid then enters that layer and pressurizes it and stops. As the surge wave propagates up the pipe, it causes an infinite number of these pressurizations. When it reaches the reservoir, all of the inflow has been stopped, and pressure is high everywhere along the pipe. The pipe resembles a compressed spring.

When the surge wave reaches the reservoir, it creates a pressure imbalance. The layer of fluid just inside the pipe has high pressure fluid downstream of it and reservoir pressure upstream. Fluid exits the layer on its upstream side and depressurizes it. The pressure drops back to the reservoir level. A backflow wave is created. The speed of the backflow is exactly the same as the speed of the original inflow. The pressure that was generated by taking the original inflow away is exactly what is available to generate the backflow. The backflow wave propagates down the pipe restoring pressure everywhere to its original level.

When the backflow wave reaches the valve, it creates a flow imbalance. This causes a low pressure or suction wave to propagate up the pipe. As it does so, it brings the fluid to rest. Again, the valve is like a wall. Because of backflow, fluid exits an infinitesimal layer next to this wall and depressurizes it and stops. The pressure drops below the reservoir level by exactly the amount it was above the reservoir level in the surge wave.

When the suction wave reaches the reservoir, all of the backflow has been stopped, and pressure is low everywhere along the pipe. The pipe resembles a stretched spring. At the reservoir, the suction wave creates a pressure imbalance. An inflow wave is created. The speed of the inflow is exactly the same as the speed of the backflow. The inflow wave travels down the pipe restoring pressure to its original level. Conditions in the pipe become what they were just before the valve was closed.

During one cycle of vibration, there are 4 transits of the pipe by pressure waves. This means that the natural period of the pipe is 4 times the length of the pipe divided by the wave speed. Without friction, the vibration cycle repeats over and over. With friction, it gradually dies away.



BASIC WAVE EQUATIONS

Consider a wave travelling up a rigid pipe. In a reference frame moving with the wave, mass considerations give

$$\rho \ A \ (U+a) = (\rho+\Delta\rho) \ A \ (U+\Delta U+a)$$

where ρ is density, A is pipe area, U is flow velocity and a is wave speed. When a >> U, this reduces to

$$\rho \ \Delta U = - \ a \ \Delta\rho$$

Momentum considerations give

$$[(\rho+\Delta\rho)A(U+\Delta U+a) \ (U+\Delta U+a) - \rho A(U+a) \ (U+a)] = [P - [P+\Delta P]] \ A$$

$$\rho A(U+a) \ [(U+\Delta U+a) - (U+a)] = - \ \Delta P \ A$$

where P is pressure. When a >> U, this reduces to

$$\rho \ a \ \Delta U = - \ \Delta P$$

Manipulations give

$$a = \sqrt{\ [\Delta P/\Delta\rho]}$$

For a gas such as air moving down a pipe, one can assume ideal gas behavior for which:

$$P/\rho = R \ T$$

R is the ideal gas constant and T is the absolute temperature of the gas. For a wave propagating through a gas, one can assume processes are isentropic: in other words, adiabatic and frictionless. The wave moves so fast through the gas that there is no time for heat transfer or friction. The isentropic equation of state is:

$$P = K \rho^{k}$$

where K is another constant and k is the ratio of specific heats. Differentiation of this equation gives

$$\Delta P/\Delta\rho = K\ k\ \rho^{k-1} = K\ k\ \rho^{k} / \rho$$
$$= k/\rho\ \ K\ \rho^{k} = k\ P/\rho$$

The ideal gas law into this gives

$$\Delta P/\Delta\rho = k\ R\ T$$

So wave speed for a gas becomes

$$a = \sqrt{[k\ R\ T]}$$

For a liquid, fluid mechanics shows that

$$\Delta P = -K\ \Delta V/V$$

where K is the bulk modulus of the liquid. It is a measure of its compressibility. For a bit of fluid mass

$$\Delta M = \Delta [\rho V] = V \Delta\rho + \rho \Delta V = 0$$

This implies that

$$\Delta P = K \Delta\rho/\rho \qquad \Delta P/\Delta\rho = K/\rho$$

So wave speed for a liquid becomes

$$a = \sqrt{[K/\rho]}$$

The bulk modulus of a gas follows from

$$a = \sqrt{[k R T]} = \sqrt{[K/\rho]}$$

$$K/\rho = k R T \qquad K = k \rho R T$$

$$K = k P$$

For a flexible pipe

$$a = \sqrt{[\mathbf{K}/\rho]}$$

$$\mathbf{K} = K / [1 + [DK]/[Ee]]$$

where E is the Elastic Modulus of the pipe wall material, e is the wall thickness and D is the pipe diameter.

WAVES IN FLEXIBLE TUBES

Conservation of Mass for a flexible tube is

$$\rho\ A\ (U+a) = (\rho+\Delta\rho)\ (A+\Delta A)\ (U+\Delta U+a)$$

Manipulation of this equation gives when U<<a

$$\rho A\ \Delta U\ +\ (U+a) A\ \Delta\rho + \rho(U+a)\ \Delta A = 0$$

$$\Delta U/a\ + \Delta\rho/\rho + \Delta A/A = 0$$

Conservation of Momentum for a flexible tube is

$$[\rho A(U+a)]\ [(U+\Delta U+a)-(U+a)] =$$

$$PA + [P+\Delta P]\ \Delta A - [P+\Delta P][A+\Delta A]$$

Manipulation of this equation gives when U<<a

$$\rho A(U+a)\ \Delta U\ + A\ \Delta P\ =\ 0$$

$$\rho a\ \Delta U\ + \Delta P\ =\ 0$$

More manipulation gives

$$\Delta U = -\ \Delta P/[\rho a] \qquad\qquad \Delta U/a = -\Delta P/[\rho a^2]$$

Experiments show that

$$\Delta P = K\ \Delta\rho/\rho \qquad\qquad \Delta\rho/\rho = \Delta P/K$$

For a thin wall pipe, the hoop stress follows from

$$[2e]\ \sigma = \Delta P\ D \qquad\qquad \sigma = \Delta P\ D/[2e]$$

The hoop strain is

$$\varepsilon = [\pi\Delta D]/[\pi D] = \Delta D/D$$

Substitution into the stress strain connection gives

$$\sigma = E\ \varepsilon \qquad\qquad \Delta P\ D/[2e] = E\ \Delta D/D$$

Geometry gives

$$A = \pi\ D^2/4 \qquad\qquad \Delta A = \pi\ 2D/4\ \Delta D$$

$$\Delta A/A = 2\ \Delta D/D = \Delta P\ D/[Ee]$$

With this Conservation of Mass becomes

$$-\ \Delta P/[\rho a^2]\ +\ \Delta P/K\ +\ \Delta P\ D/[Ee] = 0$$

Manipulation of Conservation of Mass gives

$$a = \sqrt{\ [\mathbf{K}/\rho]}$$

$$\mathbf{K}\ =\ K\ /\ [\ 1 + [DK]/[Ee]\ ]$$

WAVES IN MIXTURES

For a mixture the wave speed is:

$$a_M = \sqrt{[K_M/\rho_M]}$$

The mixture density follows from:

$$M_M = \Sigma M_C \qquad \rho_M V_M = \Sigma \rho_C V_C$$

$$\rho_M = \Sigma[\rho_C V_C]/V_M$$

Experiments show that

$$\Delta P = - K_M [\Delta V_M/V_M]$$

Manipulation gives the bulk modulus

$$K_M = - \Delta P / [\Delta V_M/V_M] \qquad V_M = \Sigma V_C \qquad \Delta V_M = \Sigma \Delta V_C$$

For each component in the mixture:

$$\Delta P = - K_C [\Delta V_C/V_C] \qquad \Delta V_C = - [V_C/K_C] \Delta P$$

The mixture bulk modulus becomes:

$$K_M = \Sigma V_C / \Sigma[V_C/K_C]$$

The mixture analysis is also valid for mixtures of small solid particles and a fluid, such as a dusty gas.

ALGEBRAIC/GRAPHICAL WATERHAMMER

Waterhammer analysis allows one to connect unknown pressure and flow velocity at one end of a pipe to known pressure and velocity at the other end of the pipe one transit time back in time. The derivation of the waterhammer equations starts with the conservation of momentum and mass equations for unsteady flow in a pipe. These are:

$$\rho\ \partial U/\partial t + \rho U\ \partial U/\partial x + \partial P/\partial x - \rho g\ \mathrm{Sin}\alpha + f/D\ \rho U|U|/2 = 0$$

$$\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x = 0$$

where P is pressure and U is velocity. For the case where gravity and friction are insignificant and the mean flow speed is approximately zero, these reduce to:

$$\rho\ \partial U/\partial t + \partial P/\partial x = 0$$

$$\partial P/\partial t + \rho a^2\ \partial U/\partial x = 0$$

Manipulation gives the wave equations:

$$\partial^2 P/\partial t^2 = a^2\ \partial^2 P/\partial x^2$$

$$\partial^2 U/\partial t^2 = a^2\ \partial^2 U/\partial x^2$$

The general solution consists of two waves: one wave which travels up the pipe known as the F wave and the other which travels down the pipe known as the f wave.

In terms of these waves, pressure and velocity are:

$$P - P_o = f(N) + F(M)$$

$$U - U_o = [f(N) - F(M)] / [\rho a]$$

where N and M are wave fixed frames given by:

$$N = x - a\ t \qquad M = x + a\ t$$

For a given point N on the f wave, the N equation shows that x must increase as time increases, which means the wave must be moving down the pipe. For a given point M on the F wave, the M equation shows that x must decrease as time increases, which means the wave must be moving up the pipe. Substitution of the general solution into mass or momentum or the wave equations shows that they are valid solutions.

Multiplying U by ρa and subtracting it from P gives:

$$[P-P_o] - \rho a[U-U_o] = 2F(M)$$

Let the F wave travel from the downstream end of the pipe to the upstream end. For a point on the wave, the value of F would be the same. This implies

$$\Delta P = + \rho a \ \Delta U$$

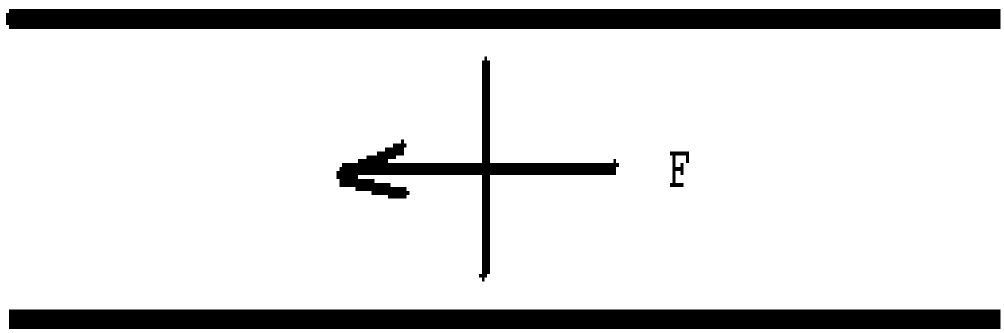
Multiplying U by ρa and adding it to P gives:

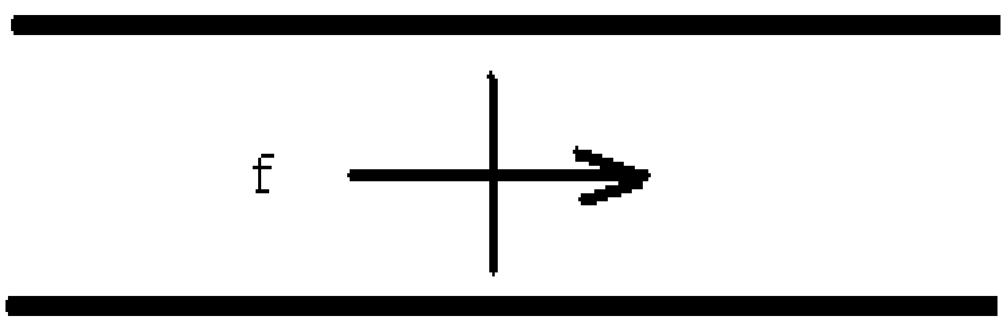
$$[P-P_o] + \rho a[U-U_o] = 2f(N)$$

Let the f wave travel from the upstream end of the pipe to the downstream end. For a point on the wave, the value of f would be the same. This implies

$$\Delta P = - \rho a \ \Delta U$$

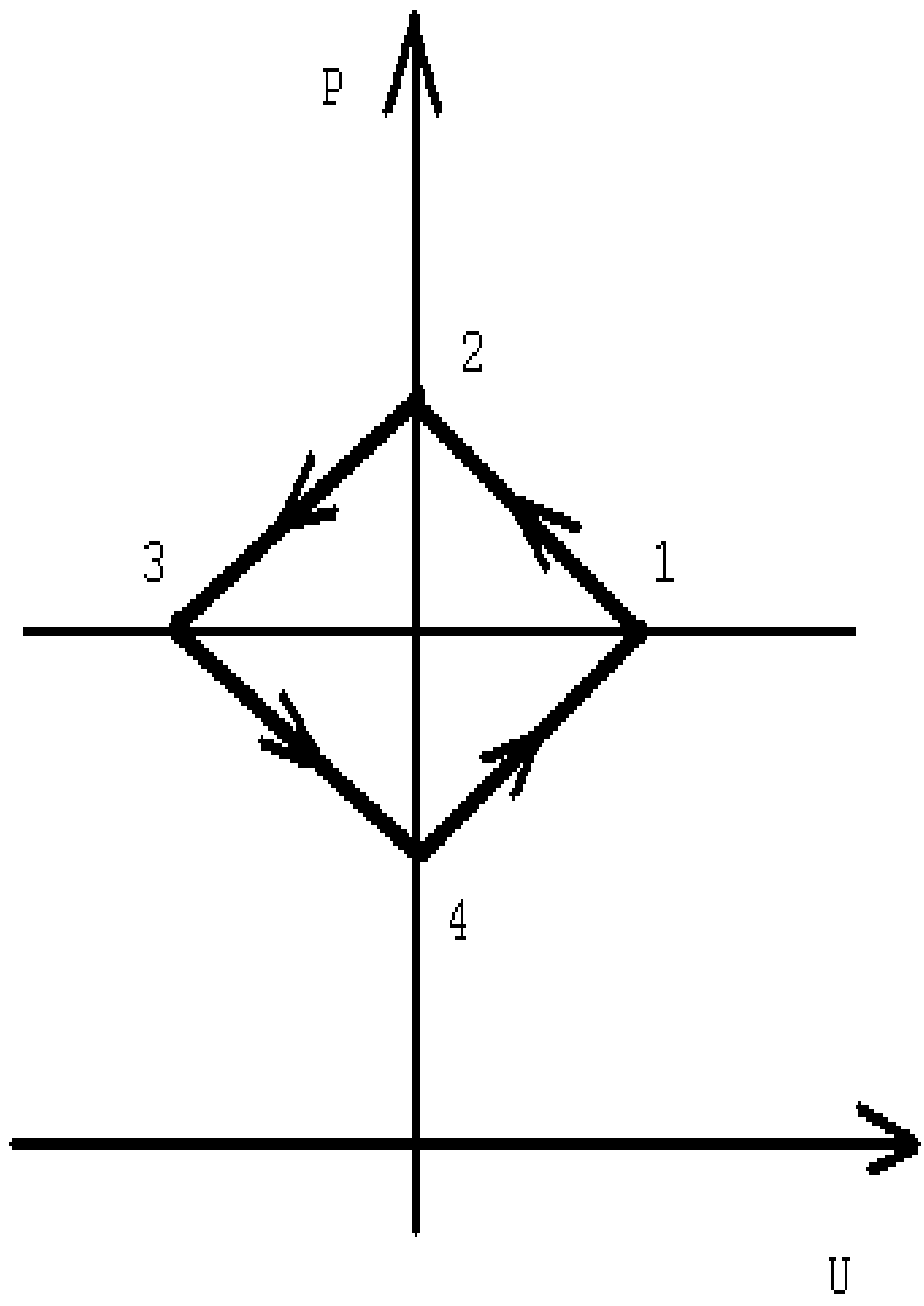
The ΔP vs ΔU equations allow us to connect unknown conditions at one end of a pipe at some point in time to known conditions at the other end back in time. They are known as the algebraic/graphical waterhammer equations.


F


f

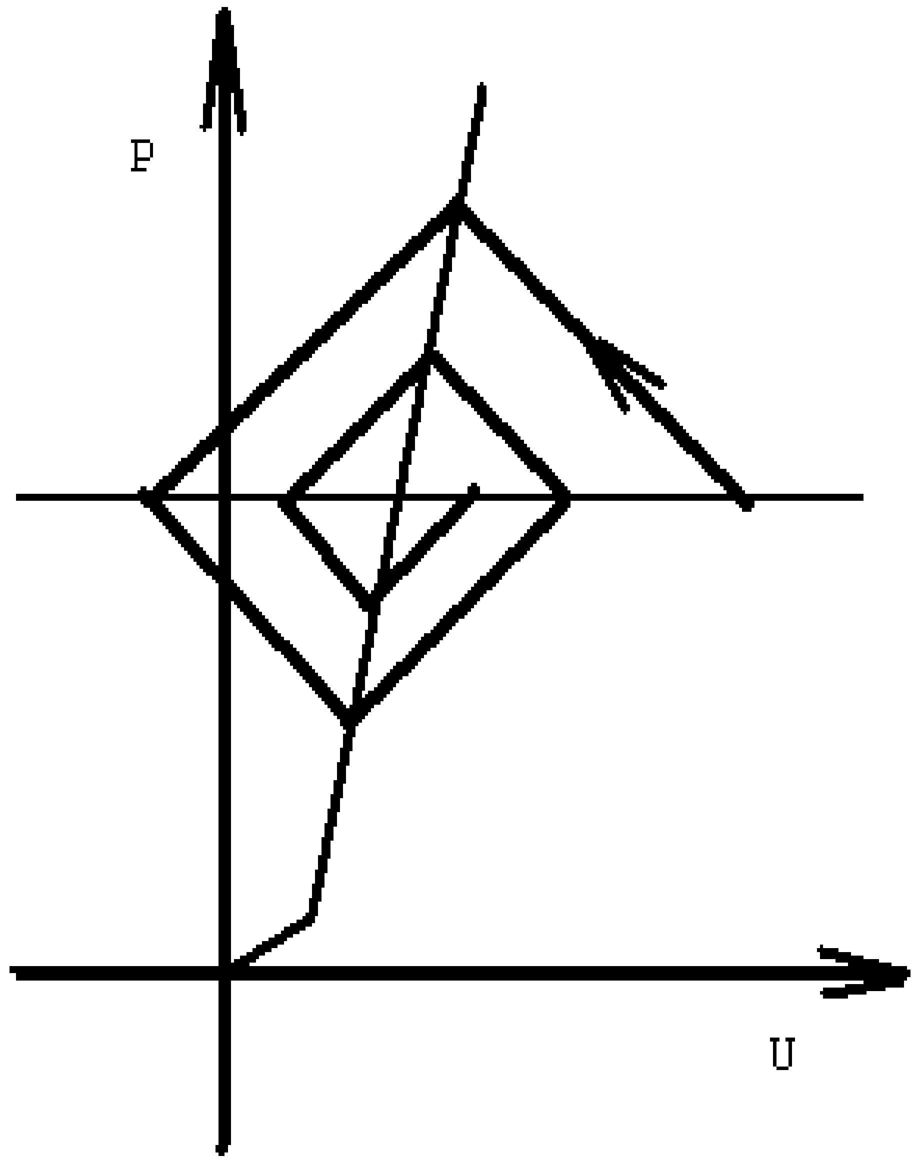
SUDDEN VALVE CLOSURE

Imagine a pipe with a reservoir at its upstream end and a valve at its downstream end. The valve is initially open. Then it is suddenly shut. From that point onward, the velocity at the valve is zero. We ignore losses. Because of this, the pressure at the reservoir is fixed at its initial level. We start at point 1 which is at the reservoir and move along an f wave to point 2 which is at the valve. A surge wave is created at the valve. We then move from the valve along an F wave to point 3 which is at the reservoir. A backflow wave is created at the reservoir. We then move from the reservoir along an f wave to point 4 which is at the valve. A suction wave is created at the valve. We then move from the valve along an F wave to point 1 which is at the reservoir. An inflow wave is created at the reservoir. From this point onward the cycle repeats. Friction gradually dissipates the waves and the velocity homes in on zero.

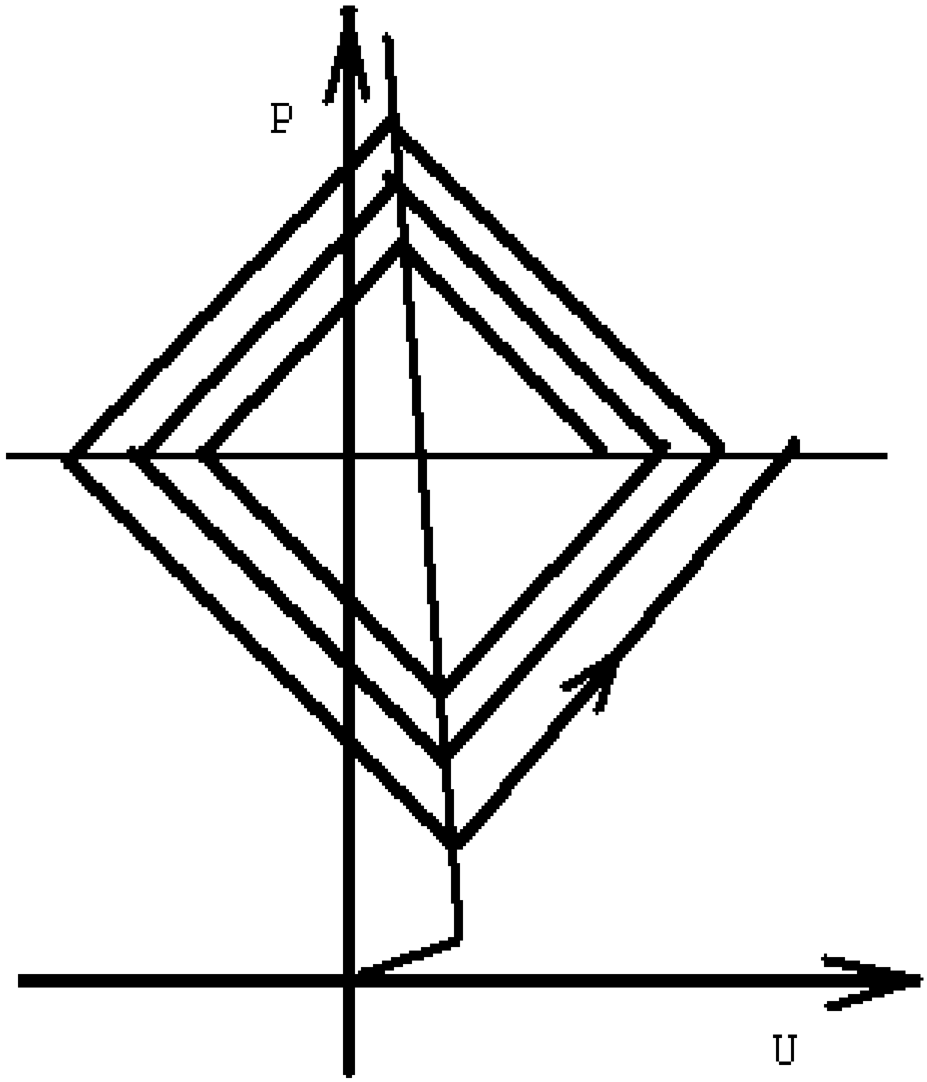

P
2
3
1
4
U

LEAKY VALVES

A stable leaky valve is basically one that has a P versus U characteristic which resembles that of a wide open valve. This has a parabolic shape with positive slope throughout. An unstable leaky valve has a characteristic that has a positive slope at low pressure but negative slope at high pressure. Basically, the valve tries to shut itself at high pressure. The flow rate just upstream of a valve is pipe flow speed times pipe area. The flow rate within the valve is valve flow speed times valve area. In a stable leaky valve, the areas are both constant. The valve flow speed increases with pipe pressure so the pipe flow speed also increases. In an unstable leaky valve, the flow speed within the valve also increases with pipe pressure but the valve area drops because of suction within the valve. The suction is generated by high speed flow through the small passageway within the valve. It pulls on flexible elements within the valve and attempts to shut it. Graphical waterhammer plots for stable and unstable leaky valves are given below. As can be seen, they both resemble the sudden valve closure plot, but the stable one is decaying while the unstable one is growing. In the unstable case, greater suction is needed each time a backflow wave comes up to the valve because the flow requirements of the valve keep getting bigger. In the stable case, less suction is needed because the flow requirements keep getting smaller.



STABLE LEAKY VALVE



UNSTABLE LEAKY VALVE

METHOD OF REACHES

Pipes in a pipe network often have different lengths. The method of reaches divides the pipes into segments that have the same transit time. The segments are known as reaches. The sketch on the next page shows a pipe divided into 4 reaches. Conditions at points i j k are known. Conditions at point J are unknown. Waterhammer analysis gives for point J:

$$\Delta P = - \rho a \ \Delta U$$

$$P_J = P_i - [\rho a][U_J - U_i]$$

$$\Delta P = + \rho a \ \Delta U$$

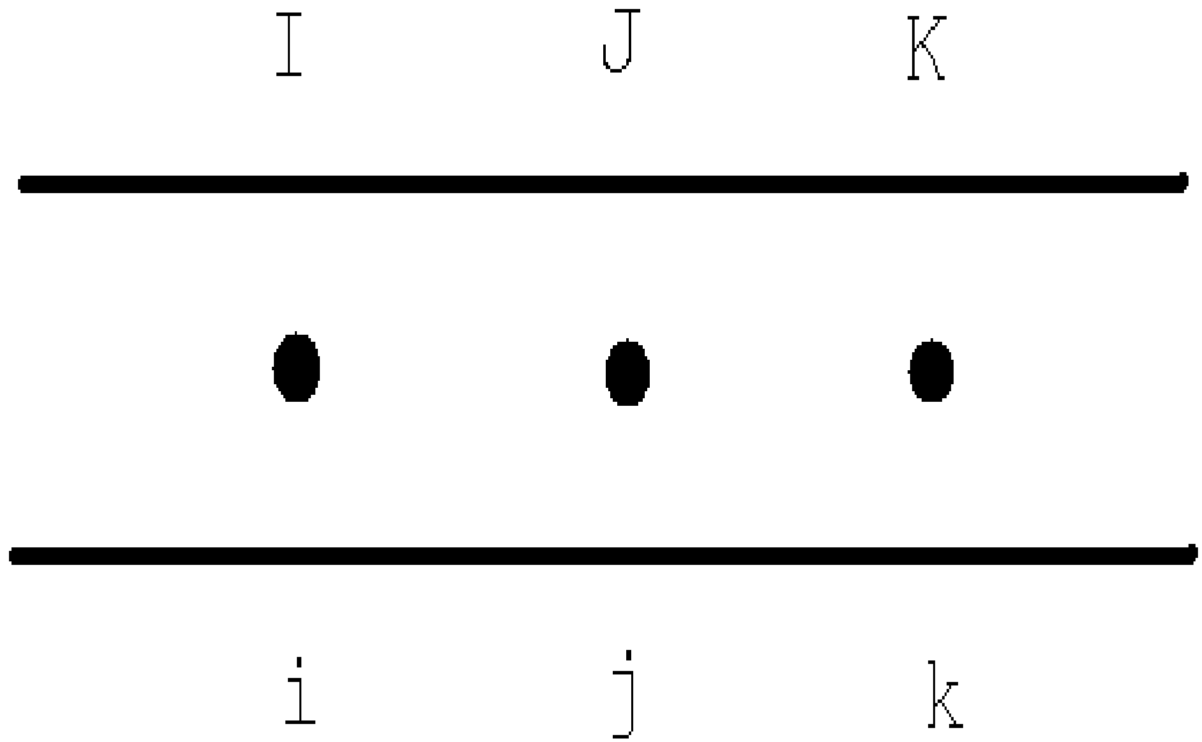
$$P_J = P_k + [\rho a][U_J - U_k]$$

Manipulation of these equations gives:

$$P_J = (P_k + P_i)/2 - [\rho a][U_k - U_i]/2$$

$$U_J = (U_k + U_i)/2 - [P_k - P_i]/[2\rho a]$$

This is the template for finding conditions at points inside the pipe. At the ends of a pipe, water hammer analysis would connect the end points to j points inside the pipe.


I J K
i j k

TREATMENT OF PIPE JUNCTIONS

Pipes in a pipe network are connected at junctions. The sketch on the next page shows a junction which connects 3 pipes. Lower case letters indicate known conditions. Upper case letters indicate unknown conditions. A junction is often small. This allows us to assume that the junction pressure is common to all pipes. It also allows us to assume that the net flow into or out of the junction is zero. Conservation of Mass considerations give:

$$\rho \, A_N \, U_N \; + \; \rho \, A_H \, U_H \; + \; \rho \, A_W \, U_W \; = \; 0$$

Waterhammer analysis gives:

$$P_N = P_m + [\rho a_N][U_N - U_m]$$

$$P_H = P_g + [\rho a_H][U_H - U_g]$$

$$P_W = P_v + [\rho a_W][U_W - U_v]$$

Manipulation gives

$$U_N = U_m + [P_N - P_m]/[\rho a_N]$$

$$U_H = U_g + [P_H - P_g]/[\rho a_H]$$

$$U_W = U_v + [P_W - P_v]/[\rho a_W]$$

In these equations $P_N = P_H = P_W = P_J$. Substitution into Conservation of Mass gives:

$$\rho\ A_N\ [\ U_m + [P_J - P_m]/[\rho a_N]\ ]$$
$$+\ \rho\ A_H\ [\ U_g + [P_J - P_g]/[\rho a_H]\ ]$$
$$+\ \rho\ A_W\ [\ U_v + [P_J - P_v]/[\rho a_W]\ ] = 0$$

Manipulation gives the junction pressure:

$$P_J = [X - Y] / Z$$

where

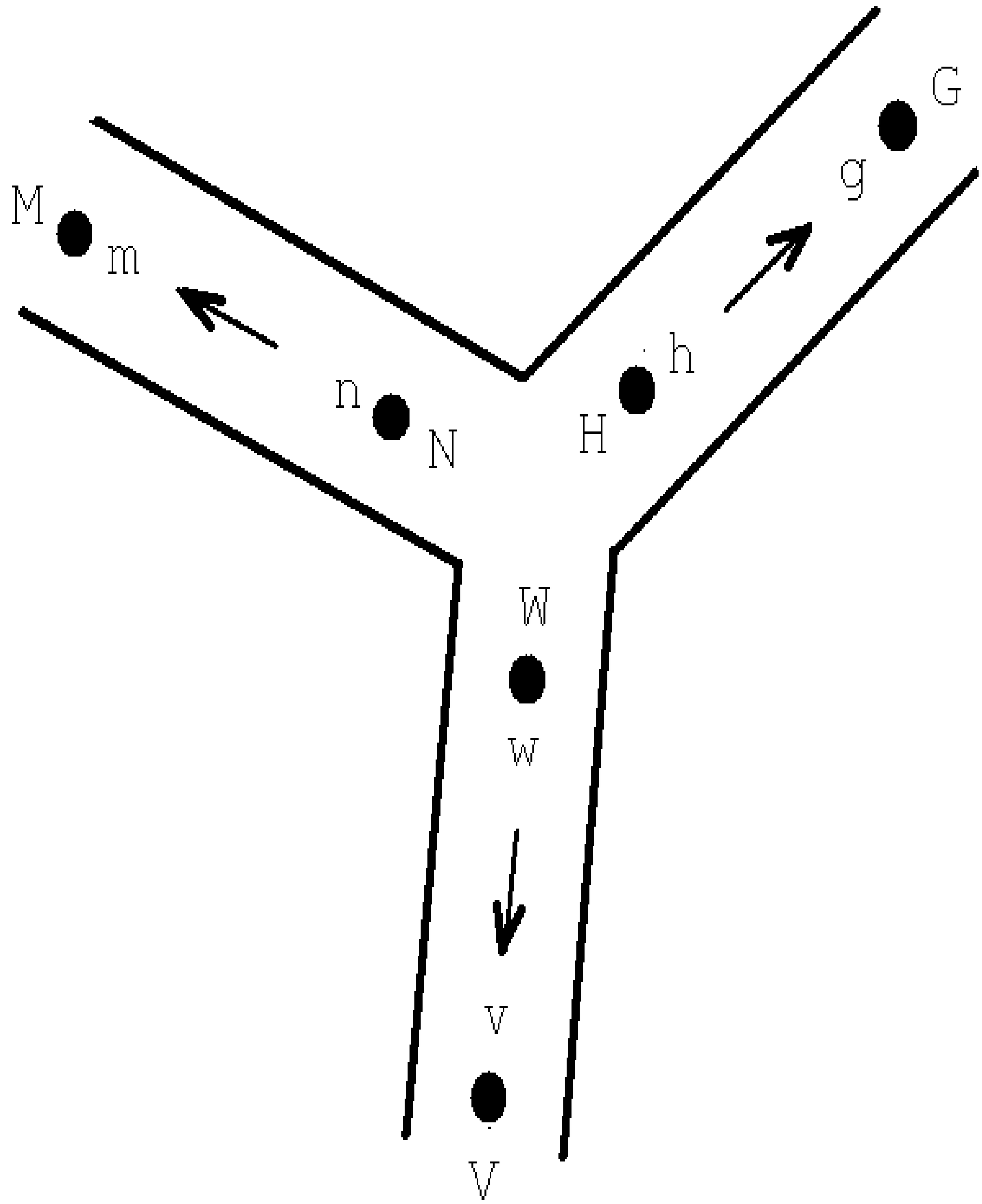
$$X = [\ A_N/a_N\ P_m + A_H/a_H\ P_g + A_W/a_W\ P_v\ ]$$

$$Y = \rho\ [\ A_N U_m + A_H U_g + A_W U_v\ ]$$

$$Z = [\ A_N/a_N + A_H/a_H + A_W/a_W\ ]$$

The velocities at the junction are:

$$U_N = U_m + [P_J - P_m]/[\rho a_N]$$
$$U_H = U_g + [P_J - P_g]/[\rho a_H]$$
$$U_W = U_v + [P_J - P_v]/[\rho a_W]$$


M
m
n
N
G
g
h
H
W
w
v
V

ACCUMULATOTS

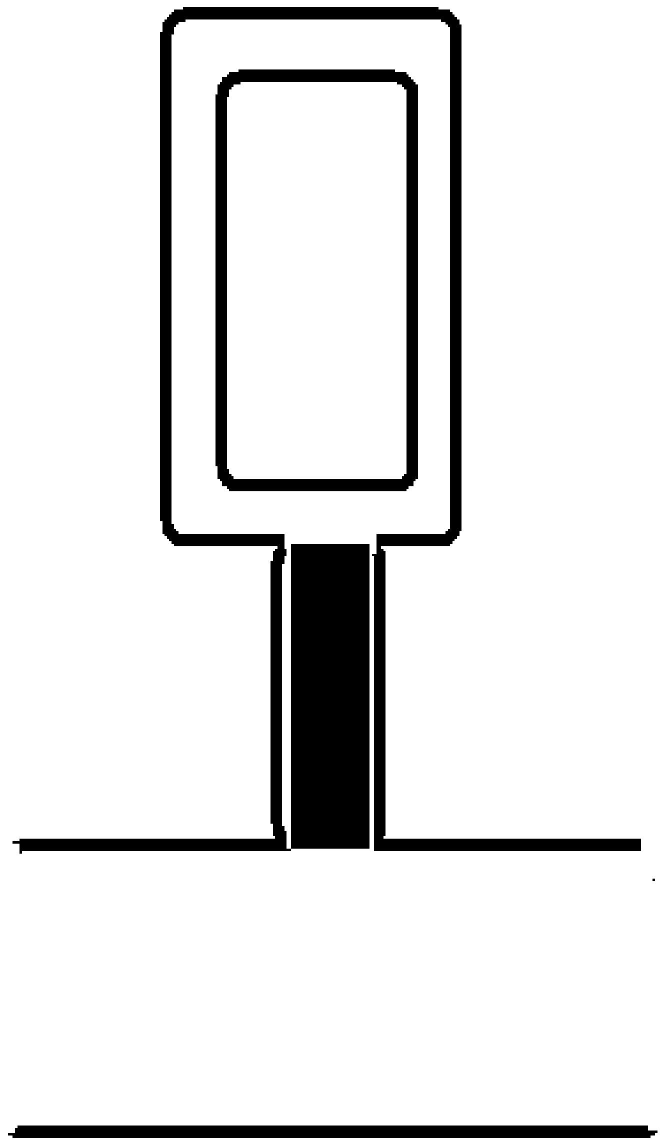
Accumulators are used to dampen transients in pipe networks. They generally consist of a neck or constriction containing liquid which is connected directly to the pipe network. A pocket of gas is at the other end of the neck. The gas is usually contained inside a flexible bladder.

There are two ways to model an accumulator. The first is the Helmholtz Resonator mass spring model where the slug of liquid in the neck bounces on the gas spring. This gives the natural frequency of the accumulator and one tries to match that to the natural period of the network. The second model is a transient model where the equation of motion of the slug of liquid in the neck and the equations for the gas pocket are solved step by step in time and this is coupled a water hammer analysis transient model.

The Helmholtz Resonator model starts with the equation of motion of a mass on a spring:

$$m \, d^2\Delta Z/dt^2 + k \, \Delta Z = f$$

where m is the mass of liquid in the neck and k is the spring due to gas compressibility.



The natural frequency and period of the accumulator are

$$\omega = \sqrt{[k/m]} \qquad T = 2\pi/\omega$$

The mass m of the slug of liquid in the neck is

$$m = \rho A L$$

where ρ is the density of the liquid in the neck, A is the area of the neck and L is the length of the neck.

Conservation of Mass for the gas pocket gives

$$\Delta [\sigma V] = V \Delta\sigma + \sigma \Delta V = 0$$

Thermodynamics gives

$$\Delta P/\Delta\sigma = a^2 \qquad a = \sqrt{[nRT]}$$

Geometry gives

$$\Delta V = - A \Delta Z$$

Substitution into mass gives

$$V \Delta P/a^2 - \sigma A \Delta Z = 0$$

$$\Delta P = [\sigma A a^2 / V] \Delta Z$$

The force on the slug of liquid is

$$\Delta F = \Delta P\ A = [\sigma\ A^2\ a^2\ /\ V]\ \Delta Z = k\ \Delta Z$$

This gives the spring constant k

$$k = [\sigma\ A^2\ a^2\ /\ V]$$

Substitution into the frequency equation gives

$$\omega = \sqrt{\ [\ [\sigma\ A^2\ a^2\ /\ V]\ /\ [\rho\ A\ L]\ ]}$$

$$= \sqrt{\ [\ [\sigma\ A\ a^2]\ /\ [\rho\ V\ L]\ ]}$$

For the transient model the equation governing the motion of the slug of liquid in the neck is:

$$m\ dU/dt = [\ P_J - P_G\ ]\ A - fL/D\ \rho\ U|U|/2\ A$$

where P_J is the junction pressure and P_G is the gas pressure. The volume of gas is governed by

$$dV/dt = -\ U\ A$$

The pressure of the gas is

$$P_G = N\ \sigma^n = N\ (M/V)^n$$

SUDDEN VALVE OPENING

A sketch of a valve is shown on the next page. The governing equation for the flow through it is:

$$P_N - P_X = K\ U|U|$$

For constant pipe properties

$$U = U_N = U_X \qquad P = P_N - P_X$$

$$P = K\ U|U|$$

Water hammer analysis gives

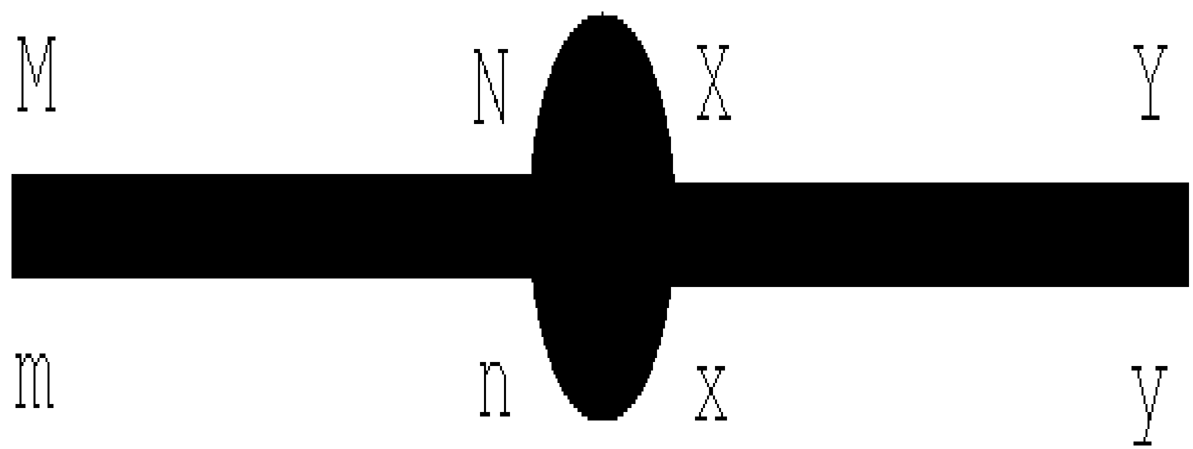
$$P_N - P_m = - \rho a\ (U_N - U_m)$$

$$P_X - P_y = + \rho a\ (U_X - U_y)$$

Substitution into the valve equation gives

$$[P_m - \rho a\ (U - U_m)] - [P_y + \rho a\ (U - U_y)] = K\ U|U|$$

This gives U at each time step. Back substitution gives the pressure upstream and downstream of the valve.


M
N
X
Y
m
n
x
y

METHOD OF CHARACTERISTICS

The method of characteristics is a way to determine the pressure and velocity variations in a pipe network when valves are adjusted or turbomachines undergo load changes. The equations governing flow in a typical pipe are:

$$\rho\ \partial U/\partial t + \rho U\ \partial U/\partial x + \partial P/\partial x - \rho g\ \mathrm{Sin}\alpha + f/D\ \rho U|U|/2 = 0$$

$$\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x = 0$$

where P is pressure, U is velocity, t is time, x is distance along the pipe, ρ is the fluid density, g is gravity, α is the pipe slope, f is the pipe friction factor, D is the pipe diameter and a is the wave speed. The wave speed is:

$$a^2 = \mathbf{K}/\rho \qquad \mathbf{K} = K\ /\ [1 + DK/Ee]$$

where K is the bulk modulus of the fluid, E is the Youngs Modulus of the pipe wall and e is its thickness.

The governing equations can be combined as follows:

$$\rho\ \partial U/\partial t\ +\ \rho\ U\ \partial U/\partial x\ +\ \partial P/\partial x\ +\ \rho\ C$$

$$+\ \lambda\ (\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x) = 0$$

where

$$C = f/D\ U|U|/2 - g\ \mathrm{Sin}\alpha$$

Manipulation gives

$$\rho\ (\partial U/\partial t + [U+\lambda a^2]\ \partial U/\partial x) + \lambda\ (\partial P/\partial t + [1/\lambda+U]\ \partial P/\partial x) + \rho C = 0$$

According to Calculus

$$dP/dt = \partial P/\partial t + dx/dt\ \partial P/\partial x$$

$$dU/dt = \partial U/\partial t + dx/dt\ \partial U/\partial x$$

Inspection of the last three equations suggests:

$$dx/dt = U + \lambda a^2 = 1/\lambda + U$$

In this case, the PDE becomes the ODE:

$$\rho\ dU/dt + \lambda\ dP/dt + \rho\ C = 0$$

The dx/dt equation gives

$$\lambda a^2 = 1/\lambda \quad \text{or} \quad \lambda^2 = 1/a^2 \quad \text{or} \quad \lambda = \pm\, 1/a$$

So there are 2 values of λ. They give

$$\rho\ dU/dt + 1/a\ dP/dt + \rho\ C = 0 \qquad dx/dt = U + a$$

$$\rho\ dU/dt - 1/a\ dP/dt + \rho\ C = 0 \qquad dx/dt = U - a$$

The dx/dt equations define directions in space and time along which the PDE becomes an ODE. Using finite differences, each ODE and dx/dt equation can be written as:

$$\rho\ \Delta U/\Delta t + 1/a\ \Delta P/\Delta t + \rho\ C = 0 \qquad \Delta x/\Delta t = U + a$$

$$\rho\ \Delta U/\Delta t - 1/a\ \Delta P/\Delta t + \rho\ C = 0 \qquad \Delta x/\Delta t = U - a$$

Manipulation gives

$$\rho a\ \Delta U + \Delta P + \Delta t\ \rho a\ C = 0$$

$$\rho a\ \Delta U - \Delta P + \Delta t\ \rho a\ C = 0$$

When the wave speed a is much greater than the flow speed U and when Δx is the length of the pipe L and Δt is the pipe transit time T, these equations are basically the water hammer equations but with friction added.

For pipes divided into reaches, one gets

$$U_P - U_L + (P_P-P_L)/[\rho a] + C_L(t_P-t_L) = 0 \quad x_P-x_L = (U_L+a)(t_P-t_L)$$

$$U_P - U_R - (P_P-P_R)/[\rho a] + C_R(t_P-t_R) = 0 \quad x_P-x_R = (U_R-a)(t_P-t_R)$$

Manipulation gives

$$U_P = 0.5\ (U_L + U_R + [P_L-P_R]/[\rho a] - \Delta t(C_L+C_R))$$

$$P_P = 0.5\ (P_L + P_R + [\rho a][U_L-U_R] - \Delta t[\rho a](C_L-C_R))$$

Linear interpolation gives U and P at points L and R in terms of known U and P at grid points A and B and C:

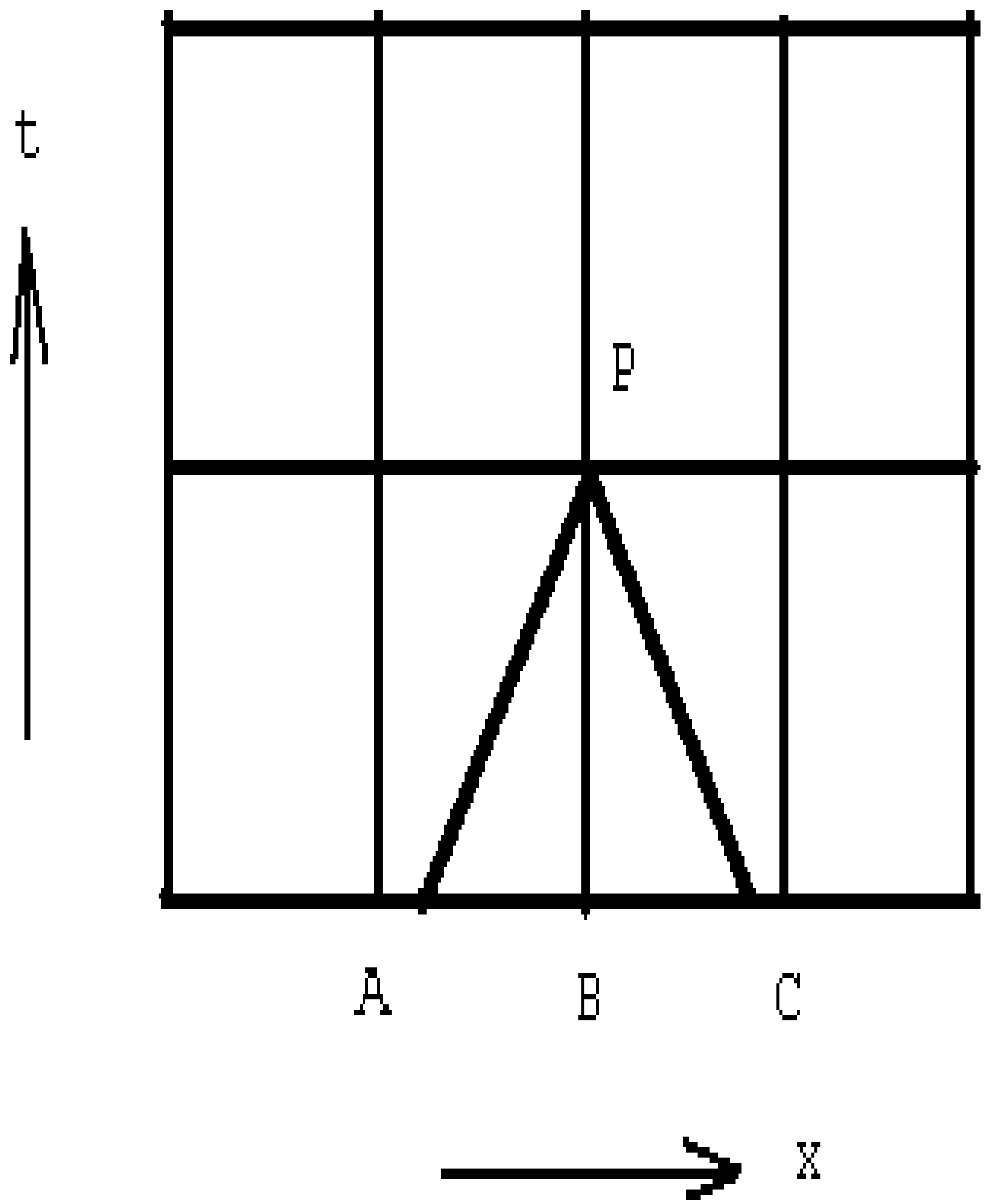
$$U_L = U_A + (x_L-x_A)/(x_B-x_A)\ (U_B-U_A)$$

$$U_R = U_C + (x_R-x_C)/(x_B-x_C)\ (U_B-U_C)$$

$$P_L = P_A + (x_L-x_A)/(x_B-x_A)\ (P_B-P_A)$$

$$P_R = P_C + (x_R-x_C)/(x_B-x_C)\ (P_B-P_C)$$

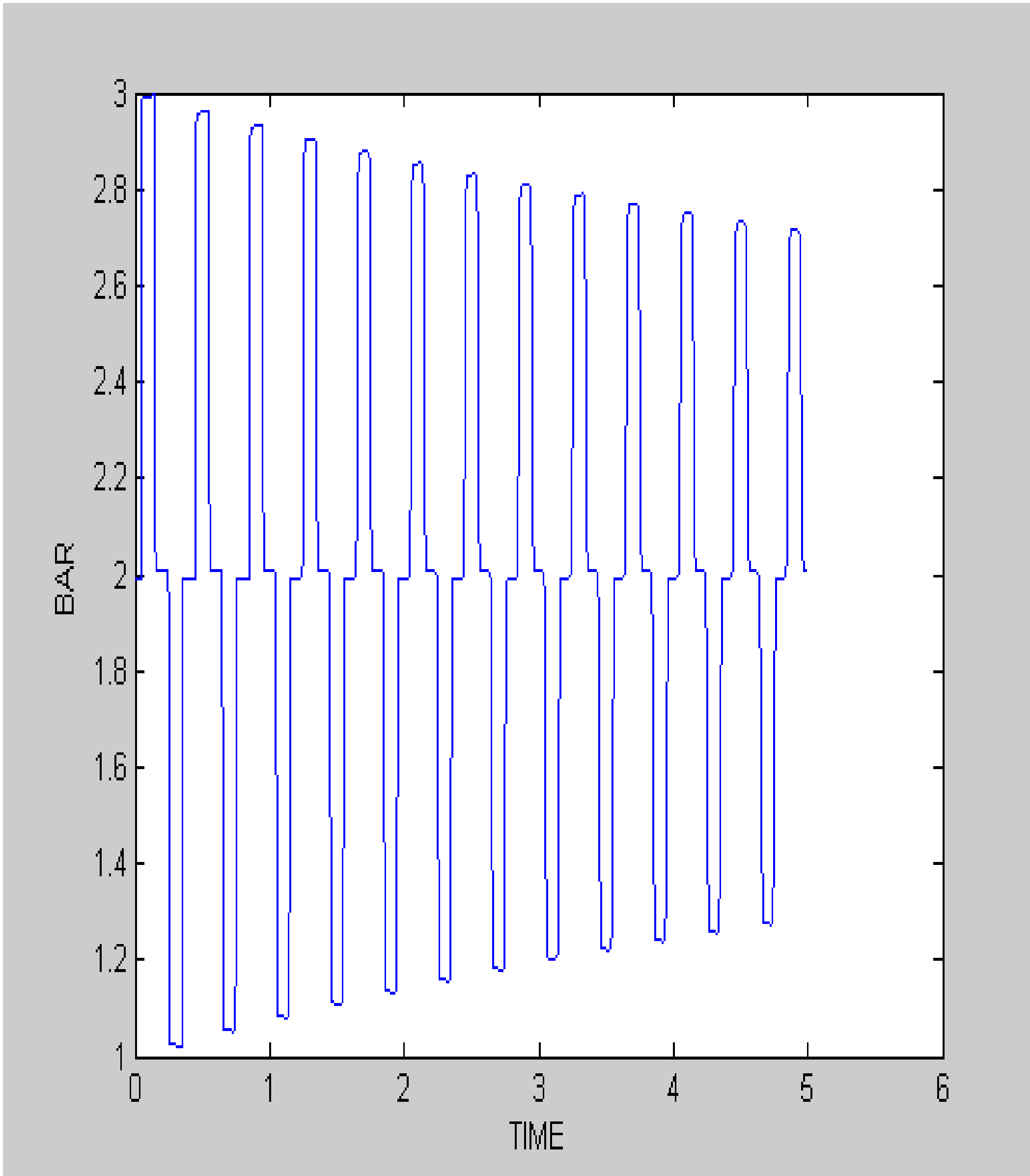
At each end of the pipe, a boundary condition relates the P_P and U_P there. A finite difference equation also relates the P_P and U_P there. So, one can solve for the P_P and U_P there.


t
P
A
B
C
x

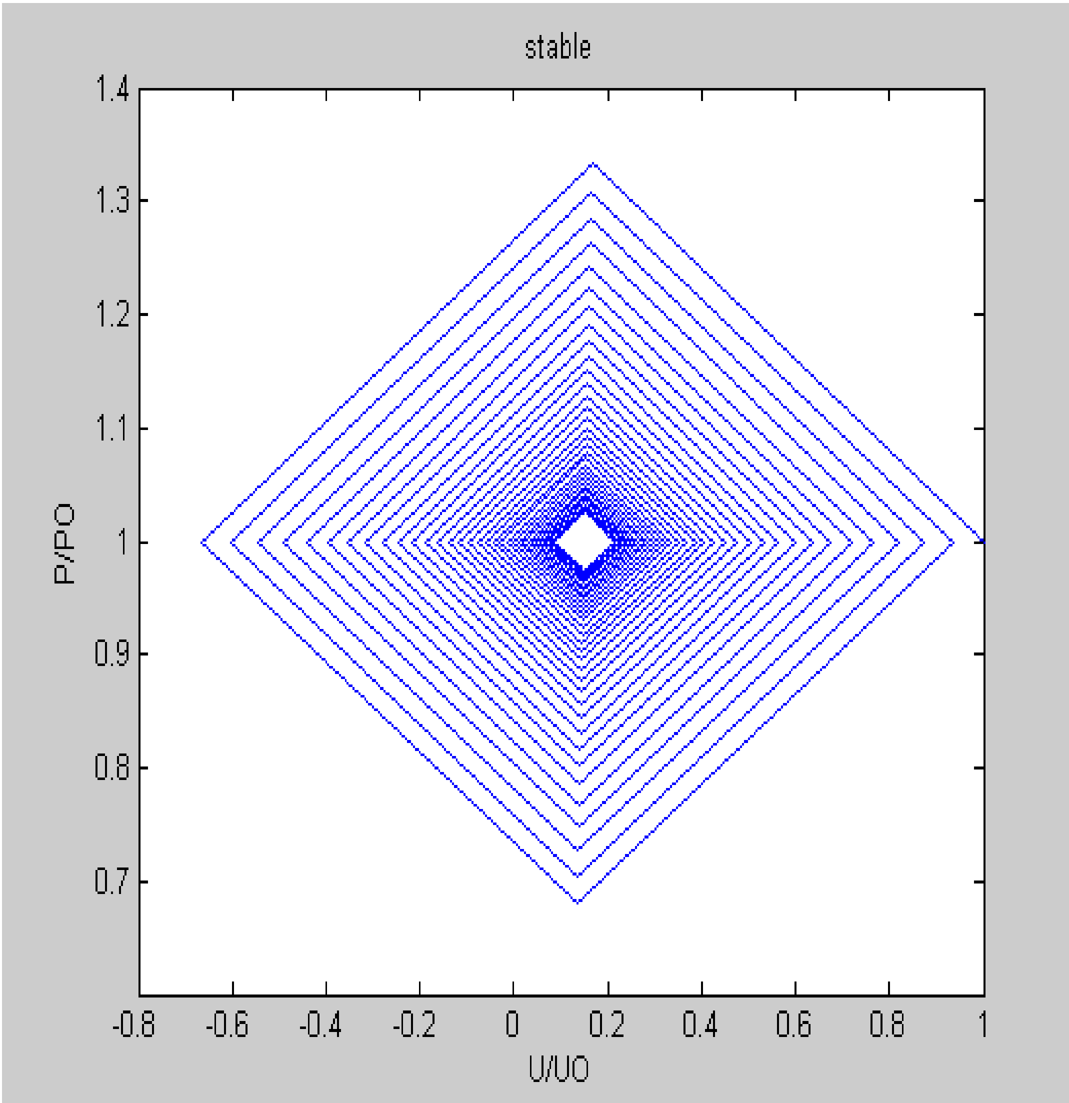
```
%     UNSTEADY FLOW IN A PIPE

%     METHOD OF CHARACTERISTICS

%     RESERVOIR / PIPE / VALVE

%     PRESSURE = POLD / PNEW
%     VELOCITY = UOLD / UNEW

%     HEAD = RESERVOIR HEAD
%     PIPE = HEAD PRESSURE

%     SLOPE = VALVE SLOPE

%     OD = PIPE DIAMETER
%     OL = PIPE LENGTH
%     CF = FRICTION FACTOR

%     SOUND = SOUND SPEED
%     GRAVITY = GRAVITY
%     DENSITY = DENSITY

%     NIT = NUMBER OF TIME STEPS
%     MIT = NUMBER OF PIPE NODES
%     DELT = STEP IN TIME

%     DATA
      DELT=0.001;
      CF=0.5;
      CMAX=+10.0;
      CMIN=0.0;
      OD=0.15;OL=100.0;
      SOUND=1000.0;
      GRAVITY=10.0;
      DENSITY=1000.0;
      SLOPE=-100000.0;
      HEAD=20.0;SPEED=0.1;
      NIT=5000;MIT=100;KIT=1;
      PIPE=HEAD*DENSITY*GRAVITY;
%
      ONE=PIPE;
      TWO=0.0;
      ZERO=0.0;
      BIT=MIT/2;
      GIT=MIT-1;
      DELX=OL/(MIT-1);
      FLD=CF*OL/OD;
      PMAX=CMAX*PIPE;
      PMIN=CMIN*PIPE;
      WAY=SPEED*SPEED/2.0;
```

```
      LOSS=FLD*WAY/GRAVITY;
      G=LOSS*DENSITY*GRAVITY;
      DELP=G/GIT;
      for IM=1:MIT
      POLD(IM)=ONE;
      UOLD(IM)=SPEED;
      X(IM)=TWO;
      ONE=ONE-DELP;
      TWO=TWO+DELX;
      end
      PV=POLD(MIT);
      UV=UOLD(MIT);

%     START LOOP ON TIME
      TIME=0.0;
      for IT=1:NIT
      TIME=TIME+DELT;
      T(IT)=TIME;
%     POINTS INSIDE PIPE
      for IM=2:MIT-1
      XA=X(IM-1);
      XB=X(IM);
      XC=X(IM+1);
      PA=POLD(IM-1);
      PB=POLD(IM);
      PC=POLD(IM+1);
      UA=UOLD(IM-1);
      UB=UOLD(IM);
      UC=UOLD(IM+1);
      XL=XB-(UB+SOUND)*DELT;
      XR=XB-(UB-SOUND)*DELT;
      UL=UA+(XL-XA)/(XB-XA)*(UB-UA);
      PL=PA+(XL-XA)/(XB-XA)*(PB-PA);
      UR=UC+(XR-XC)/(XB-XC)*(UB-UC);
      PR=PC+(XR-XC)/(XB-XC)*(PB-PC);
      UNEW(IM)=0.5*(UL+UR+(PL-PR)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*(UL*abs(UL)+UR*abs(UR))));
      PNEW(IM)=0.5*(PL+PR+(UL-UR)*DENSITY*SOUND-DENSITY ...
       *SOUND*CF/2.0/OD*DELT*(UL*abs(UL)-UR*abs(UR)));
      end
%     DOWNSTREAM END OF PIPE
      if(KIT==1) UNEW(MIT)=ZERO;end;
      if(KIT==2) UNEW(MIT)=UV ...
      +(POLD(MIT)-PV)/SLOPE;end;
      if(UNEW(MIT)<=ZERO) ...
      UNEW(MIT)=ZERO;end;
      XA=X(MIT-1);
      XB=X(MIT);
      PA=POLD(MIT-1);
      PB=POLD(MIT);
      UA=UOLD(MIT-1);
      UB=UOLD(MIT);
```

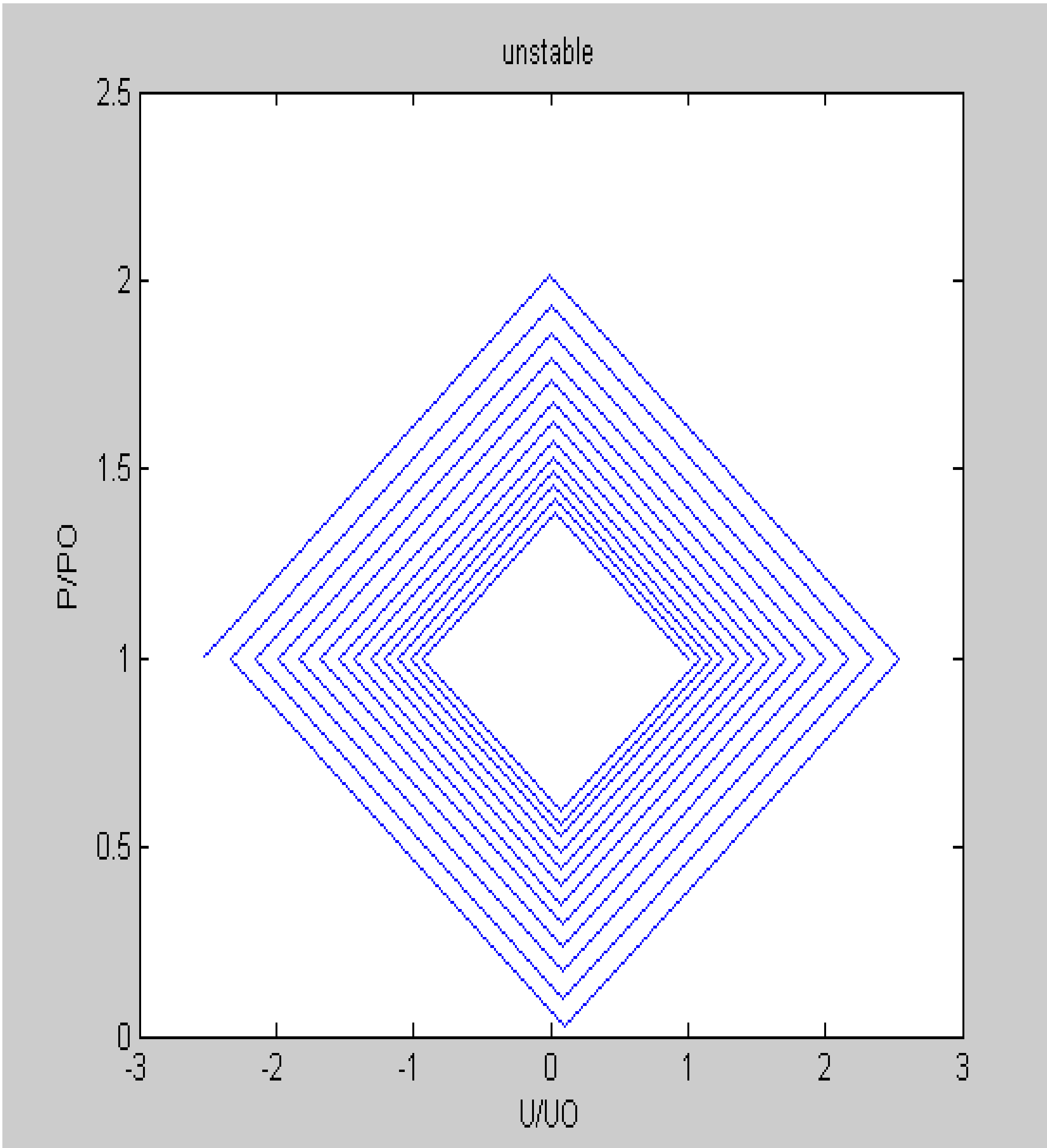
```
      XL=XB-(UB+SOUND)*DELT;
      UL=UA+(XL-XA)/(XB-XA)*(UB-UA);
      PL=PA+(XL-XA)/(XB-XA)*(PB-PA);
      PNEW(MIT)=PL-(UNEW(MIT)-UL)*DENSITY*SOUND ...
      -DELT*DENSITY*SOUND*(CF/2.0/OD*UL*abs(UL));
      if(PNEW(MIT)<=PMIN) PNEW(MIT)=PMIN;end;
      if(PNEW(MIT)>=PMAX) PNEW(MIT)=PMAX;end;
      if(PNEW(MIT)==PMAX | PNEW(MIT)==PMIN) ...
      UNEW(MIT)=UL-(PNEW(MIT)-PL)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*UL*abs(UL));end;
%     UPSTREAM END OF PIPE
      XB=X(1);
      XC=X(2);
      PB=POLD(1);
      PC=POLD(2);
      UB=UOLD(1);
      UC=UOLD(2);
      XR=XB-(UB-SOUND)*DELT;
      UR=UC+(XR-XC)/(XB-XC)*(UB-UC);
      PR=PC+(XR-XC)/(XB-XC)*(PB-PC);
      PNEW(1)=PIPE;
      UNEW(1)=UR+(PNEW(1)-PR)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*UR*abs(UR));
%     STORING P AND U
      for IM=1:MIT
      POLD(IM)=PNEW(IM);
      UOLD(IM)=UNEW(IM);
      if (IM==BIT) PIT(IT)=PNEW(IM); ...
      HIT(IT)=PIT(IT)/DENSITY/GRAVITY; ...
      BAR(IT)=HIT(IT)/10.0; ...
      UIT(IT)=UNEW(IM);end;
      end
%     END OF TIME LOOP
      end
%

      plot(T,UIT)
      plot(UIT,HIT)
      plot(UIT,BAR)
      plot(UIT,PIT)
      plot(T,PIT)
      plot(T,BAR)
      xlabel('TIME')
      ylabel('BAR')
```



SUDDEN VALVE CLOSURE



STABLE LEAKY VALVE



UNSTABLE LEAKY VALVE

REACHES WITH FRICTION

Pipes in a pipe network often have different lengths. The method of reaches divides the pipes into segments that have the same transit time. The segments are known as reaches. The sketch on the next page shows a pipe divided into 4 reaches. Conditions at points i j k are known. Conditions at point J are unknown. Waterhammer analysis gives for point J:

$$[\rho a]\ dU/dt + dP/dt + [\rho a]C = 0$$

$$P_J - P_i = -[\rho a][U_J - U_i] - \Delta t\ [\rho a]C_i$$

$$[\rho a]\ dU/dt - dP/dt + [\rho a]C = 0$$

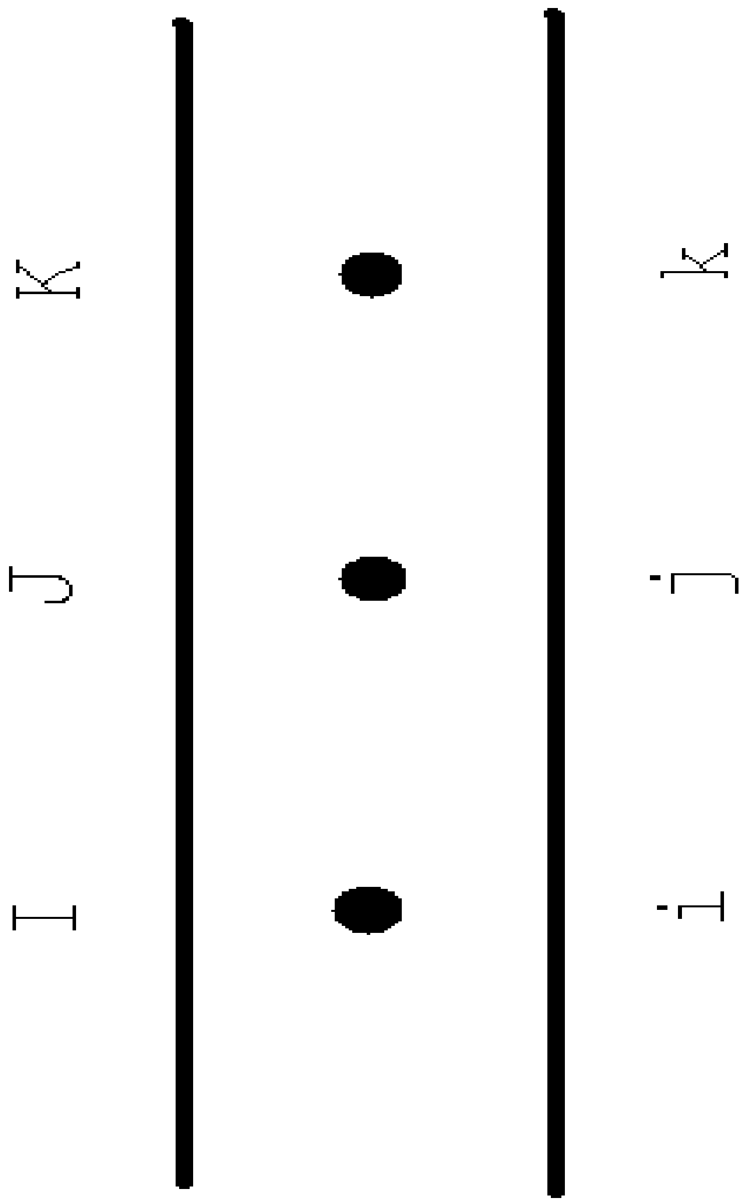
$$P_J - P_k = +[\rho a][U_J - U_k] + \Delta t\ [\rho a]C_k$$

Manipulation of these equations gives:

$$P_J = (P_k + P_i)/2 - [\rho a][U_k - U_i]/2 + \Delta t\ [\rho a][C_k - C_i]/2$$

$$U_J = (U_k + U_i)/2 - [P_k - P_i]/[2\rho a] - \Delta t\ [C_k + C_i]/2$$

This is the template for finding conditions at points inside the pipe. At the ends of a pipe, water hammer analysis would connect the end points to j points inside the pipe.



JUNCTIONS WITH FRICTION

Pipes in a pipe network are connected at junctions. The sketch on the next page shows a junction which connects 3 pipes. Lower case letters indicate known conditions. Upper case letters indicate unknown conditions. A junction is often small. This allows us to assume that the junction pressure is common to all pipes. It also allows us to assume that the net flow into or out of the junction is zero. Conservation of Mass considerations give:

$$+ \rho A_N U_N + \rho A_H U_H + \rho A_W U_W = 0$$

Waterhammer analysis gives:

$$P_N - P_m = + [\rho a_N][U_N - U_m] + \Delta t \ [\rho a]C_m$$

$$P_H - P_g = + [\rho a_H][U_H - U_g] + \Delta t \ [\rho a]C_g$$

$$P_W - P_v = + [\rho a_W][U_W - U_v] + \Delta t \ [\rho a]C_v$$

Manipulation gives

$$U_N = U_m + [P_N - P_m]/[\rho a_N] - \Delta t \ C_m$$

$$U_H = U_g + [P_H - P_g]/[\rho a_H] - \Delta t \ C_g$$

$$U_W = U_v + [P_W - P_v]/[\rho a_W] - \Delta t \ C_v$$

In these equations $P_N = P_H = P_W = P_J$. Substitution into Conservation of Mass gives:

$$+ \rho\ A_N\ [\ U_m + [P_J - P_m]/[\rho a_N] - \Delta t\ C_m]$$

$$+ \rho\ A_H\ [\ U_g + [P_J - P_g]/[\rho a_H] - \Delta t\ C_g]$$

$$+ \rho\ A_W\ [\ U_v + [P_J - P_v]/[\rho a_W] - \Delta t\ C_v] \quad = \quad 0$$

Manipulation gives the junction pressure:

$$P_J = [X - Y] / Z$$

$$X = [\ + A_N/a_N\ P_m + A_H/a_H\ P_g + A_W/a_W\ P_v\ ]$$

$$Y = \rho\ [\ + A_N[U_m - \Delta t C_m] + A_H[U_g - \Delta t C_g] + A_W[U_v - \Delta t C_v]\ ]$$

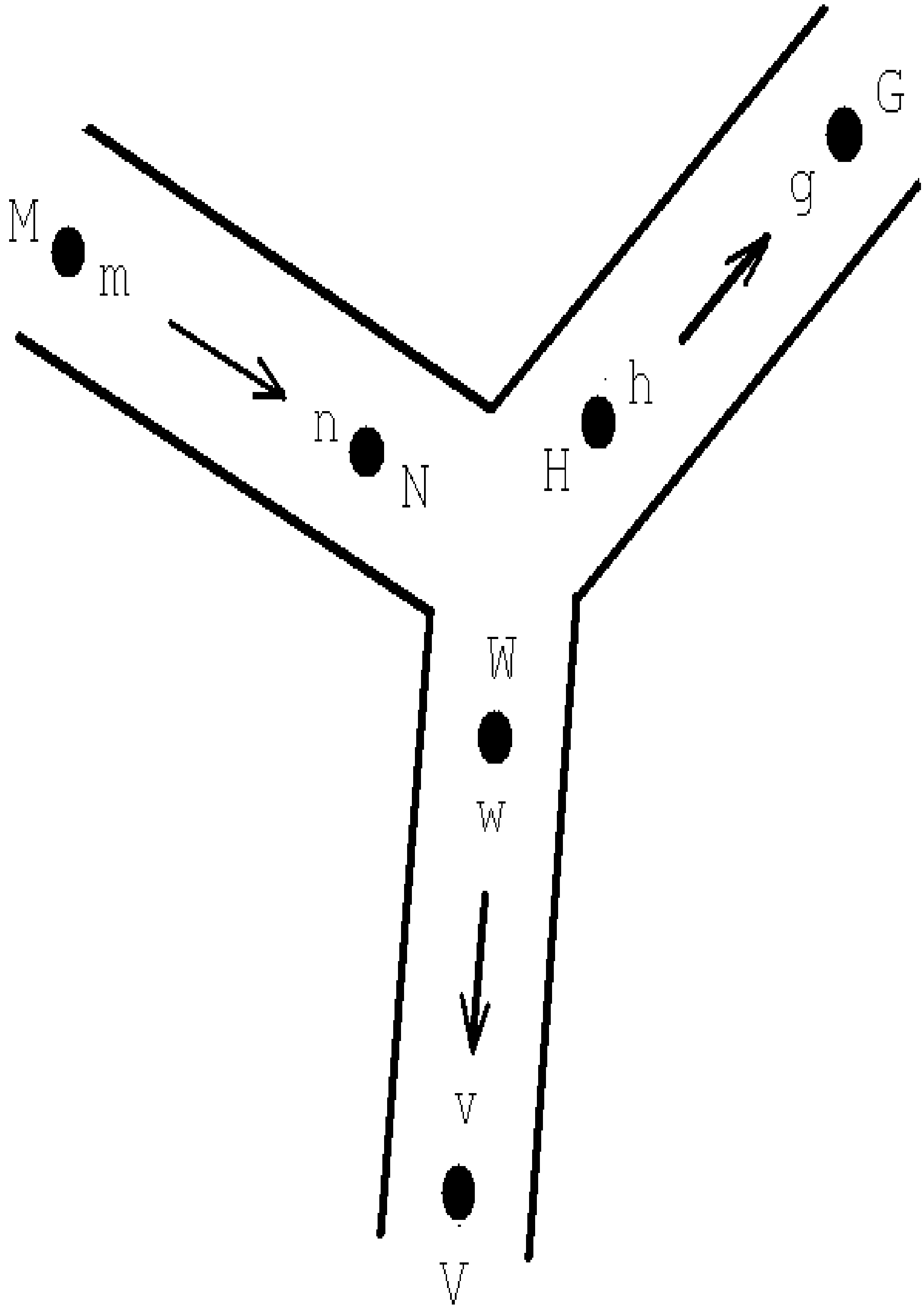
$$Z = [\ + A_N/a_N + A_H/a_H + A_W/a_W\ ]$$

The velocities at the junction are:

$$U_N = U_m + [P_J - P_m]/[\rho a_N] - \Delta t\ C_m$$

$$U_H = U_g + [P_J - P_g]/[\rho a_H] - \Delta t\ C_g$$

$$U_W = U_v + [P_J - P_v]/[\rho a_W] - \Delta t\ C_v$$



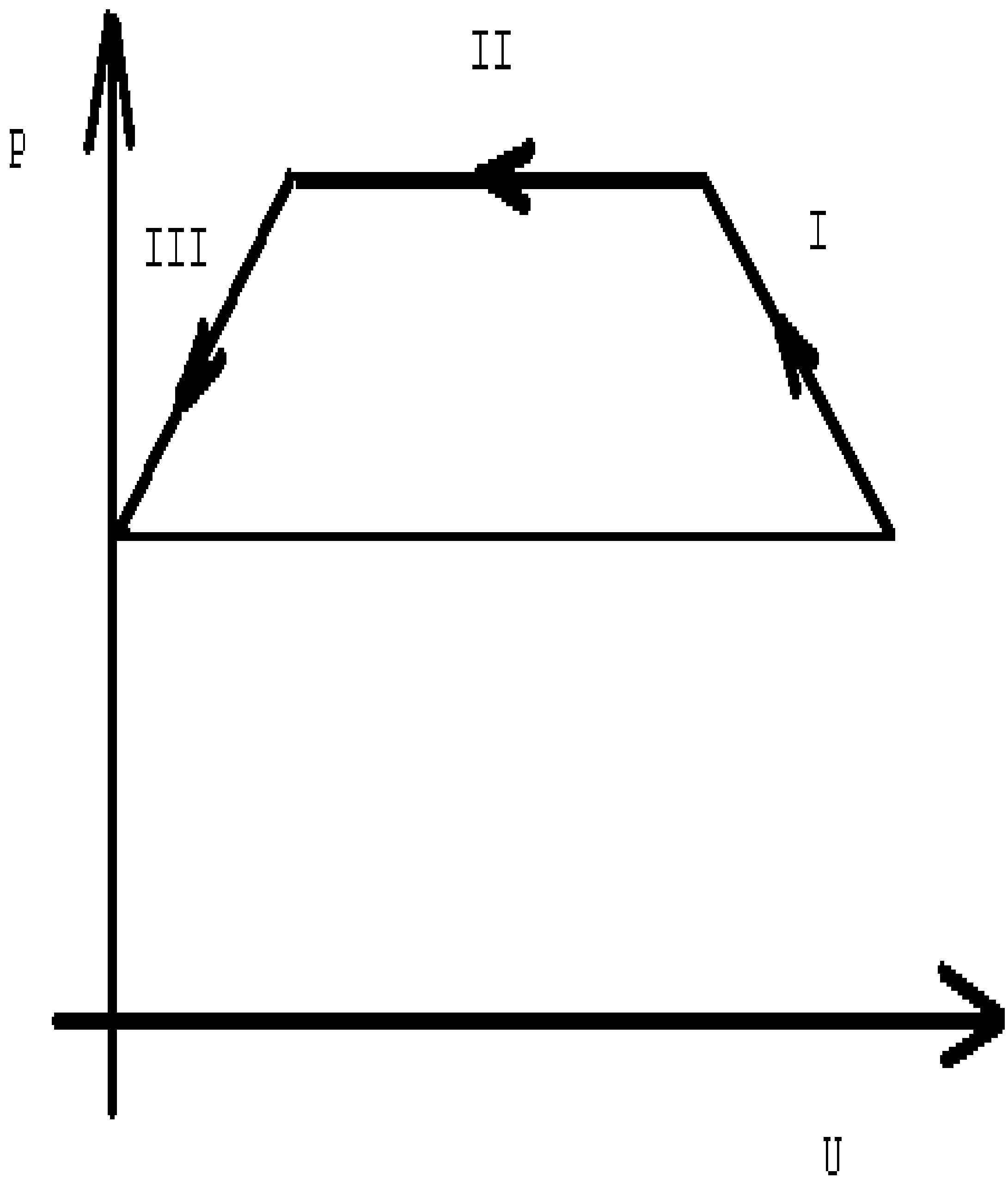
M
m
n
N
H
h
g
G
W
w
v
V

THREE PHASE VALVE STROKING

Three phase valve stroking is a process where a valve is opened or closed very fast in such a way that pressures are kept within preset limits and no waves are left at the end. It is described below for a complete closure case.

In phase I the valve is moved in such a way that the pressure at the valve rises linearly in time from P_{LOW} to P_{HIGH} in 2T pipe transit times. At the end of phase I the pressure variation along the pipe is linear and the velocity everywhere because of a combination of pressure surges and back flows has been reduced by $\Delta P/[\rho a]$ where ΔP is P_{HIGH} minus P_{LOW}. In phase II the valve is moved in such a way that the pressure variation along the pipe stays constant and the velocity drops by $2\Delta P/[\rho a]$ everywhere every 2T transit times. The pressure variation remains constant because pressure surges generated by valve motion are cancelled by suction waves at the valve caused by back flows. The constant pressure variation causes a constant deceleration of the fluid in the pipe. Phase III takes 2T pipe transit times to complete. During this time the velocity everywhere drops $\Delta P/[\rho a]$ and pressure falls linearly at the valve from P_{HIGH} to P_{LOW}. The valve is moved in such a way that suction waves at the valve caused by back flows are allowed to bring the pressure down again to P_{LOW}. Because phases I and III reduce the velocity by a total of $2\Delta P/[\rho a]$ phase II must take $(U-2\Delta P/[\rho a])/(2\Delta P/[\rho a])$ 2T seconds to complete. One can calculate what the valve area should be at each instant in time during stroking. A fast acting feedback control system can then be used to move the valve in the desired manner.

Phase I sets up conditions in the pipe for phase II. Similarly, phase II sets up conditions in the pipe for phase III. In phase II, the pressure surge rate is twice that of phases I and III. In a set period of time, one pressure surge maintains a backflow that would have otherwise been stopped by a suction wave. The other pressure surge balances a pressure release. There are no suction waves in phase II and all backflows are maintained. Every point in the pipe has a velocity reduction due to a surge wave and one due to a backflow. In phase III, the pressure surge rate is cut in half. This allows suction waves to form at the valve. These propagate up the pipe and eliminate backflows. Conditions in the pipe are controlled by these waves and by waves already there from phase II. During the first half of phase III, conditions in the pipe are still under the influence of phase II. Velocity falls faster at the reservoir than at the valve because of this. Half way through phase III, there is a linear pressure variation and a linear velocity variation along the pipe. During the second half of phase III, a wave travels down the pipe which brings the pressure back to P_{LOW} everywhere and the velocity to zero everywhere.


II
P
III
I
U

DISCRETE VALVE STROKING

The following sketches show a stroking maneuver that has been broken down into 16 discrete steps. The top sketch is for pressure and the bottom sketch is for velocity. The spacing of the steps in time is one quarter transit time.

The first four steps show small surge waves gradually propagating up the pipe causing velocity reductions as they go. The next four steps show where surge waves propagating up the pipe are superimposed on pressure releases and back flows propagating down the pipe. These eight steps are for phase I. The end result is a linear pressure variation along the pipe. The pressure at the valve is high and the velocity is the same everywhere. In phase II, the linear pressure variation is maintained and this causes a uniform deceleration of the fluid along the pipe. The last eight steps are for phase III. The first 4 steps there are influenced by phase II. After these 4 steps, the velocity is zero at the upstream end of the pipe and there is a new linear pressure variation along the pipe. During the last 4 steps, a wave propagates down the pipe making the velocity zero everywhere and bringing pressure back to low.

FLUID STRUCTURE INTERACTIONS

WATER WAVE INTERACTION

WITH STRUCTURES

PREAMBLE

Most water waves are generated by storms at sea. Many waves are present in a storm sea state: each has a different wavelength and period. Theory shows that the speed of propagation of a wave or its phase speed is a function of water depth. It travels faster in deeper water. Theory also shows that the speed of a wave is a function of its wavelength. Long wavelength waves travel faster than short wavelength waves. This explains why storm generated waves, which approach shore, are generally a single wavelength. Because waves travel at different speeds, they tend to separate or disperse. When waves approach shore, they are influenced by the seabed by a process known as refraction. This can focus or spread out wave energy onto a site. Close to shore water depth is not the same everywhere: so points on wave crests move at different speeds and crests become bent. This explains why crests which approach a shore line tend to line up with it: points in deep water travel faster than points in shallow water and overtake them. Wave energy travels at a speed known as the group speed. This is generally not the same as the phase speed. However for shallow water both speeds are the same and they depend only on the water depth. A large low pressure system moving over shallow water would generate an enormous wave if the system speed and the wave energy speed were the same. Basically wave energy gets trapped in the system frame when the system speed matches the wave energy speed. Tides are basically shallow water waves. Here the pull of the Moon mimics a low pressure system. Theory shows that if water depth was 22km everywhere on Earth the Moon pull would produce gigantic tides. They would probably drain the oceans and swamp the continents everyday. Fortunately the average water depth is only 3km.

There are two mechanisms that have been proposed for wave generation by winds. One is the classic Kelvin Helmholtz stability mechanism where water waves extract energy from the wind and grow. This mechanism explains the generation of small wavelength waves. Energy in small wavelength waves can leak into longer wavelength waves but not into very long wavelength swells. There must be another mechanism to explain them. This mechanism considers a storm to be made up of an infinite number of pressure waves each with a different speed and wavelength moving over the water surface. When the speed and wavelength of a pressure wave matches the speed and wavelength of a wave that can exist in the water, a resonance occurs which causes that water wave to grow. A pressure wave can be broken down into a series of infinitesimal pulses. Each pulse as it moves over the water generates a stern wave much like that directly behind a ship. Resonance occurs when all of the stern waves add up.

STRUCTURE SIZE

Water waves can interact with structures and cause them to move or experience loads. For wave structure interaction, an important parameter is $5D/\lambda$ where D is the characteristic dimension of the structure and λ is the wavelength. Structures are considered large if $5D/\lambda$ is much greater than unity: they are considered small if $5D/\lambda$ is much less than unity. Small structures are transparent to waves. Large structures scatter waves.

For large structures, wave energy can reflect from it or diffract around it. Panel Method CFD based on Potential Flow Theory can be used to study the scattering process. This is beyond the scope of this note.

When a wave passes a small structure, there can be two kinds of loads on the structure: wake load due to the formation of wakes back of the structure and inertia load due to pressures in the water caused by acceleration and deceleration of water particles in the wave. In deep water, water particles move in circular orbits. In finite depth water, the orbits are ellipses. Let the orbit dimension normal to the structure be d and let the characteristic dimension of the structure be D. When $5D \ll d$, a well defined wake forms behind the structure. When $5D \gg d$, such a wake does not form. When 5D is approximately equal to d, flows are extremely complex. Let T be the wave period and let T be the time it takes a water particle to move pass the structure. It turns out that 5T<<T corresponds to 5D<<d while 5T>>T corresponds to 5D>>d. When 5D<<d, wakes form because transit time is short relative to wave period. So, water is moving sufficiently long in one direction to pass the structure. When 5D>>d, wakes do not form because transit time is long relative to wave period. So, before water particles can pass the structure, they reverse direction.

WATER WAVES

The wave profile equation has the form:

$$\eta = \eta_O \mathrm{Sin}(kX)$$

where $X = x - C_P t$ where X is the horizontal coordinate of a wave fixed frame, x is the horizontal coordinate of an inertial frame, C_P is the wave phase speed, k is the wave number and $\omega = k C_P$ is the wave frequency. The wave number k is related to the wave length λ as follows: $k = 2\pi/\lambda$.

The water particle velocities are:

$$U = + H/2\ 2\pi/T\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX)$$
$$W = - H/2\ 2\pi/T\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

These can be used to get drag loads on small structures.

The water particle accelerations are:

$$dU/dt = - H/2\ (2\pi/T)^2\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

$$dW/dt = - H/2\ (2\pi/T)^2\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX)$$

These can be used to get inertia loads on small structures.

The water particle positions are:

$$x_p = x_o + H/2\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

$$Z_p = z_o + H/2\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX)$$

These give the water particle orbit size.

The wave pressure is:

$$\Delta P = \rho g\ \eta\ \mathrm{Cosh}[k(z+h)]/\mathrm{Cosh}[kh]$$

The dispersion relationships:

$$C_P = \sqrt{(g/k\ \mathrm{Tanh}[kh])}$$

$$\omega = \sqrt{(gk\ \mathrm{Tanh}[kh])}$$

These show that deep water waves travel faster than shallow water waves. They also show that long wave length waves travel faster than short wave length waves.

Wave energy travels at a speed known as the group speed. This is generally not the same as the phase speed of a wave. One can show that the group speed is given by:

$$C_G = d\omega/dk = C_P \left(1/2 + [kh]/\mathrm{Sinh}[2kh]\right)$$

The wave energy density is:

$$\mathbf{E} = 1/8\ \rho g\ H^2$$

One can show that wave energy flux is:

$$\mathbf{P} = C_G\ \mathbf{E}$$

Group speed is responsible for many important phenomena. Some of these were mentioned earlier.

Waves at sea after a storm are random. They are made up of an infinite number of frequencies. A spectrum shows how the energy in a wave field is spread out over a range of frequencies. A popular 2 parameter fit to a wave amplitude spectrum is the ITTC fit:

$$S_\eta = A/\omega^5\ e^{-B/\omega 4}$$

$$A=346H^2/T^4 \qquad B=691/T^4$$

where H is significant wave height and T is significant wave period. JONSWAP is a popular 3 parameter fit.

A Response Amplitude Operator or RAO can be used to connect a wave spectrum to a body motion or load response spectrum

$$S_R = RAO^2 \; S_\eta$$

An RAO is basically a Magnitude Ratio. For a specific wave period, it is the amplitude of body response divided by the wave amplitude. One can get RAOs from theoretical analysis. One can also get RAOs from experiments.

All sorts of statistical and probabilistic information can be obtained from spectra. For bodies, the analysis makes use of the following moments of the spectrum:

$$M_n = 1/2 \int_0^\infty S_R(\omega) \; \omega^n \; d\omega$$

One can show that the significant response height and period of a body motion or load are:

$$2 \; R_S = 4 \; \sqrt{M_0} \qquad T_S = 2\pi \; M_0/M_1$$

The probability of a response exceeding a certain level is:

$$P(R_o > R_\bullet) = e^{-X} \qquad X = R_\bullet R_\bullet / [2M_0]$$

WAVE INTERACTION WITH BODIES

REAL FLUID FORMULATION

PREAMBLE

At low speeds, fluid particles move along smooth paths: motion has a laminar or layered structure. At high speeds, particles have superimposed onto their basic streamwise observable motion a random walk or chaotic motion. Particles move as groups in small spinning bodies known as eddies. The flow pattern is said to be turbulent. A turbulent wake flow is one that contains some large eddies together with a lot of small ones. Such a flow could be found around the GBS on a stormy day. The large eddies generally stay roughly in one place. Fluid in them swirls around and around or recirculates in roughly closed orbits. The smaller eddies are associated with turbulence and are carried along by the local flow. The large eddies can usually be found inside wakes. Most of the smaller ones can be found near wake boundaries. They are generated in regions where velocity gradients are high like at the edges of wakes or in the boundary layers close to structures. They are dissipated in regions where gradients are low like in sheltered areas like corners. Turbulent wake flows are governed by the basic conservation laws. However, they are so complex that analytical solutions are impossible. One could develop computational fluid dynamics or CFD codes based on the conservation law equations. Unfortunately, the small eddies are so small that an extremely fine grid spacing and a very small time step would be needed to follow individual eddies in a flow. Small eddies are typically around 1mm in diameter. One would need a grid spacing smaller than 0.1mm to follow such eddies. CFD converts each governing equation into a set of algebraic equations or AEs: one AE for each PDE for each xyz grid point. Workable CFD is not possible because computers cannot handle the extremely large number of AEs generated. For example, a 100m x 100m x 100m volume of water near a structure like the GBS would need 10^6 x 10^6 x 10^6 or 10^{18} grid points if the grid spacing was 0.1mm. Also very many time steps would be needed to complete a simulation run. No computer currently exists that can handle so many grid points and so many time steps. The random motions of molecules in a gas diffuse momentum: they give gas its viscosity. Small eddies in a turbulent flow also diffuse momentum: they make fluid appear more viscous than it really is. This apparent increase in viscosity controls overall flow patterns and loads on structures. Models which account for this apparent increase in viscosity are known as eddy viscosity models. They can be obtained from the momentum equations by a complex time averaging process. The time averaging introduces the so called Reynolds Stresses into the momentum equations, and these are modelled using the eddy viscosity concept. Models have been developed which can estimate how eddy viscosity varies throughout a flow. Workable CFD is now possible because one can now use much larger grid spacing and time steps: it is no longer necessary to follow individual eddies around in a flow. When small eddies are accounted for in this way, they no longer show up in flow: they are suppressed by eddy viscosity. For the GBS case, a grid spacing around 1m would now be adequate. This means a 100m x 100m x 100m volume of water near the GBS would now need only $10^2 \times 10^2 \times 10^2$ or 10^6 grid points.

CONSERVATION LAWS FOR HYDRODYNAMICS FLOWS

Hydrodynamics flows are often turbulent. Conservation of momentum considerations for such flows give:

$$\rho \left(\partial U/\partial t + U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z \right) + A = - \partial P/\partial x + \left[\partial/\partial x \left(\mu\, \partial U/\partial x\right) + \partial/\partial y \left(\mu\, \partial U/\partial y\right) + \partial/\partial z \left(\mu\, \partial U/\partial z\right) \right]$$

$$\rho \left(\partial V/\partial t + U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z \right) + B = - \partial P/\partial y + \left[\partial/\partial x \left(\mu\, \partial V/\partial x\right) + \partial/\partial y \left(\mu\, \partial V/\partial y\right) + \partial/\partial z \left(\mu\, \partial V/\partial z\right) \right]$$

$$\rho \, (\partial W/\partial t + U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z) + C = - \partial P/\partial z - \rho g$$

$$+ \, [\partial/\partial x \, (\mu \, \partial W/\partial x) + \partial/\partial y \, (\mu \, \partial W/\partial y) + \partial/\partial z \, (\mu \, \partial W/\partial z)]$$

where U V W are respectively the velocity components in the x y z directions, P is pressure, ρ is the density of water and μ is its effective viscosity. The time averaging process introduces source like terms A B C into the momentum equations. Each is a complex function of velocity and viscosity gradients as indicated below:

$$A = \partial\mu/\partial y \, \partial V/\partial x - \partial\mu/\partial x \, \partial V/\partial y + \partial\mu/\partial z \, \partial W/\partial x - \partial\mu/\partial x \, \partial W/\partial z$$

$$B = \partial\mu/\partial x \, \partial U/\partial y - \partial\mu/\partial y \, \partial U/\partial x + \partial\mu/\partial z \, \partial W/\partial y - \partial\mu/\partial y \, \partial W/\partial z$$

$$C = \partial\mu/\partial y \, \partial V/\partial z - \partial\mu/\partial z \, \partial V/\partial y + \partial\mu/\partial x \, \partial U/\partial z - \partial\mu/\partial z \, \partial U/\partial x$$

Conservation of mass considerations give:

$$\partial P/\partial t + \rho \, c^2 \, (\partial U/\partial x + \partial V/\partial y + \partial W/\partial z) = 0$$

where c is the speed of sound in water. Although water is basically incompressible, CFD takes it to be compressible. Mass is used to adjust pressure at points in the grid when the divergence of the velocity vector is not zero.

A special function F known as the volume of fluid or VOF function is used to locate the water surface. For water, F is taken to be unity: for air, it is taken to be zero. Regions with F between unity and zero must contain the water surface. Material volume considerations give:

$$\partial F/\partial t + U\partial F/\partial x + V\partial F/\partial y + W\partial F/\partial z = 0$$

TURBULENCE MODEL

Engineers are usually not interested in the details of the eddy motion. Instead they need models which account for the diffusive character of turbulence. One such model is the k-ε model, where k is the local intensity of turbulence and ε is its local dissipation rate. Its governing equations are:

$$\partial k/\partial t + U\partial k/\partial x + V\partial k/\partial y + W\partial k/\partial z = T_P - T_D$$
$$+ \left[\partial/\partial x\ (\mu/a\ \partial k/\partial x) + \partial/\partial y\ (\mu/a\ \partial k/\partial y) + \partial/\partial z\ (\mu/a\ \partial k/\partial z) \right]$$

$$\partial \varepsilon/\partial t + U\partial \varepsilon/\partial x + V\partial \varepsilon/\partial y + W\partial \varepsilon/\partial z = D_P - D_D$$
$$+ \left[\partial/\partial x\ (\mu/b\ \partial \varepsilon/\partial x) + \partial/\partial y\ (\mu/b\ \partial \varepsilon/\partial y) + \partial/\partial z\ (\mu/b\ \partial \varepsilon/\partial z) \right]$$

where

$$T_P = G\ \mu_t\ /\ \rho \qquad D_P = T_P\ C_1\ \varepsilon\ /\ k$$

$$T_D = C_D\ \varepsilon \qquad D_D = C_2\ \varepsilon^2 / k$$

$$\mu_t = C_3\ k^2 / \varepsilon \qquad \mu = \mu_t + \mu_l$$

where

$$G = 2\ [\ (\partial U/\partial x)^2 + (\partial V/\partial y)^2 + (\partial W/\partial z)^2\ ]$$
$$+ [\ \partial U/\partial y + \partial V/\partial x\ ]^2 + [\ \partial U/\partial z + \partial W/\partial x\ ]^2$$
$$+ [\ \partial W/\partial y + \partial V/\partial z\ ]^2$$

where C_D=1.0 C_1=1.44 C_2=1.92 C_3=0.9 a=1.0 b=1.3 are constants based on data from geometrically simple experiments, μ_l is the laminar viscosity, μ_t is extra viscosity due to eddy motion and G is a production function. The k-ε equations account for the convection, diffusion, production and dissipation of turbulence. Special wall functions are used to simplify consideration of the sharp normal gradients in velocity and turbulence near walls.

COMPUTATIONAL FLUID DYNAMICS

For CFD, the flow field is discretized by a Cartesian or xyz system of grid lines. Small volumes or cells surround points where grid lines cross. Flow is not allowed in cells occupied by fixed bodies. Ways to handle moving bodies are still under development. Flow can enter or leave the region of interest through its boundaries. For hydrodynamics problems, an oscillating pressure over a patch of the water surface could be used to generate waves. An oscillating flow at a vertical wall could also be used for this. For CFD, each governing equation is put into the form:

$$\partial M/\partial t = N$$

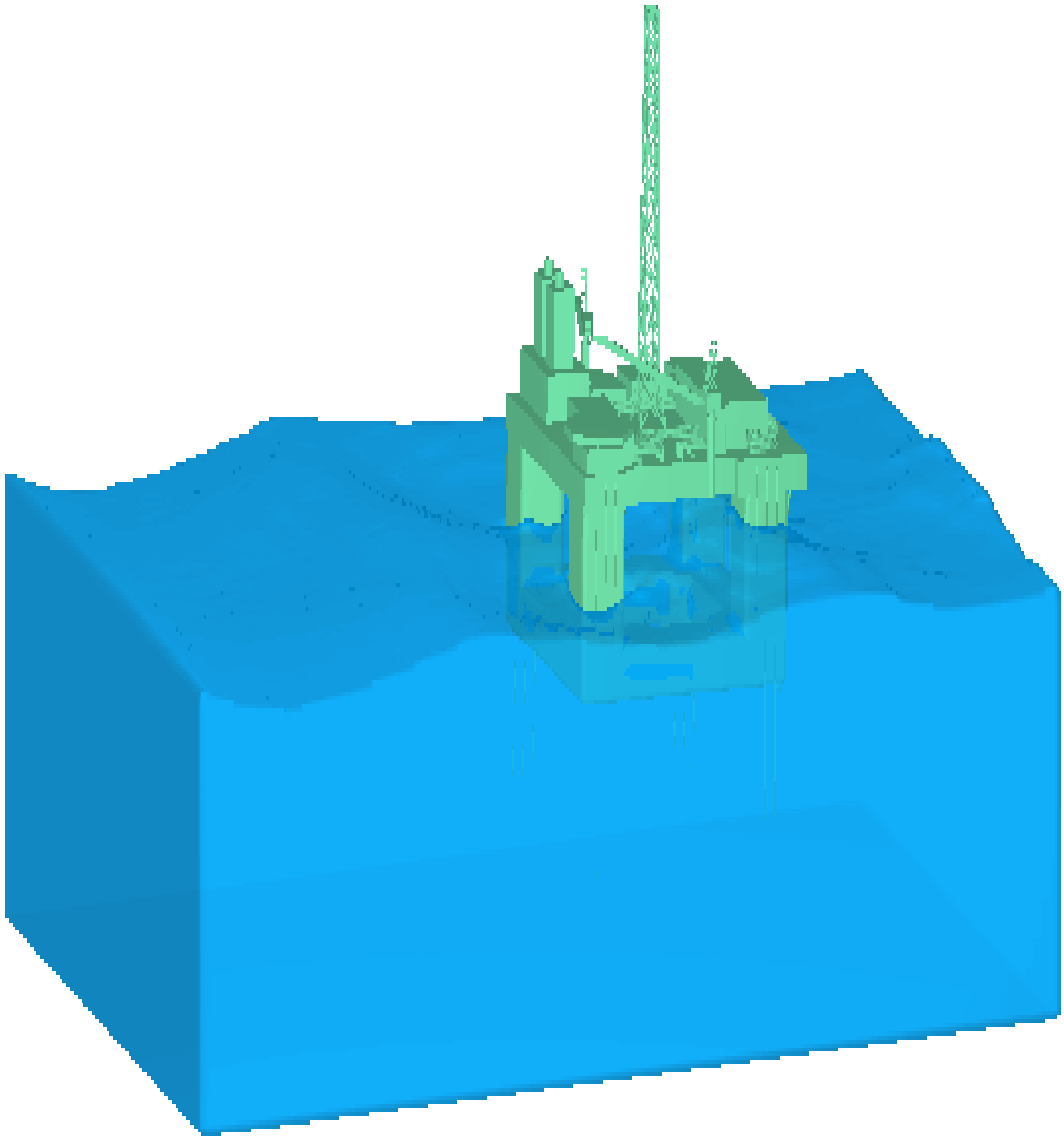
At points within the CFD grid, each governing equation is integrated numerically across a time step to get:

$$M(t+\Delta t) = M(t) + \Delta t \; N(t)$$

where the various derivatives in N are discretized using finite difference approximations. The discretization gives algebraic equations for the scalars P F k ε at points where grid lines cross and equations for the velocity components at staggered positions between the grid points. Central differences are used to discretize the viscous terms in the momentum and turbulence equations. To ensure numerical stability, a combination of central and upwind differences is used for the convective terms. Collocation or lumping is used for the T and D terms. To march the unknowns forward in time, the momentum equations are used to update U V W, the mass equation is used to update P and correct U V W, the VOF equation is used to update F and the location of the water surface and the turbulence equations are used to update k ε.

APPLICATIONS OF FLOW-3D CODE

FLOW-3D is a CFD software package for hydrodynamics and other flows <www.flow3d.com>. It can handle all sorts of complex phenomena such as wave breaking and phase changes such as vaporization and solidification. No other CFD package can handle these phenomena. A unique feature of FLOW-3D known as the General Moving Object or GMO can simulate the complex motions of floating bodies in steep waves. The motions of the bodies can be prescribed or they can be coupled to the motion of the fluid. It allows for extremely complicated motions and flows. One can think of a GMO as a bubble in a flow where the pressure on the inside surface of the bubble is adjusted in such a way that its boundary matches the shape of a body. FLOW 3D uses a complex interpolation scheme to fit the body into the Cartesian grid. The sketch on the next page shows a FLOW-3D simulation of an oil rig in waves.



SPECTRAL ANALYSIS OF SENSOR SIGNALS

FOURIER SERIES

A Fourier Series breaks down a periodic signal with a known period into its harmonics. The general equation for a Fourier Series representation of a signal is

$$f(t) = \Sigma \; [\; A_k \, \mathrm{Sin}[k\omega t] + B_k \, \mathrm{Cos}[k\omega t] \;]$$

Manipulation shows that

$$A_k = 2/T \int_{-T/2}^{+T/2} f(t)\, \mathrm{Sin}[k\omega t]\, dt$$

$$B_k = 2/T \int_{-T/2}^{+T/2} f(t)\, \mathrm{Cos}[k\omega t]\, dt$$

For discrete data these take the form

$$A_k = 2/T \sum_{n=1}^{N} f[n\Delta t]\, \mathrm{Sin}[k\omega[n\Delta t]]\, \Delta t$$

$$B_k = 2/T \sum_{n=1}^{N} f[n\Delta t]\, \mathrm{Cos}[k\omega[n\Delta t]]\, \Delta t$$

$$F_k = B_k + A_k\, j$$

FOURIER TRANSFORM

The general equation for a Fourier Transform is

$$A(\omega) = \int_{-\infty}^{+\infty} f(t)\ \mathrm{Sin}[\omega t]\ dt$$

$$B(\omega) = \int_{-\infty}^{+\infty} f(t)\ \mathrm{Cos}[\omega t]\ dt$$

While the Fourier Series deals with the harmonics of a periodic signal with a known period, the Fourier Transform deals with a signal with an infinite number of periods.

For discrete data the Fourier Transform becomes

$$A(\omega) = \sum_{n=1}^{N} f[n\Delta t]\ \mathrm{Sin}[\omega[n\Delta t]]\ \Delta t$$

$$B(\omega) = \sum_{n=1}^{N} f[n\Delta t]\ \mathrm{Cos}[\omega[n\Delta t]]\ \Delta t$$

$$F(\omega) = B(\omega) + A(\omega)j$$

This acts on data streams that are not infinitely long. Windows are used to compress data at the extremes to avoid errors due to the finite length of the stream.

HYDRODYNAMICS

SCALING LAWS

All sorts of probabilistic and statistical information can be obtained from the response spectrum of a structure in random waves. The desired spectrum needs to be known before the structure is actually built. Getting it would be part of the design process. There are two ways to get the response spectrum. One way is to measure response profile data and use that to generate the spectrum. One can get this response profile data by putting a small scale model of the structure in random seas in a physical wave tank. Obviously, we need to know how the model data scales to prototype size. One can also put a model or prototype in random seas in a numerical wave tank. The other way to get a response spectrum is to measure the response amplitude operator of the structure. It connects the wave spectrum to the response spectrum:

$$S_R = RAO^2 \, S_\eta$$

A wave spectrum for a particular location can be obtained from historical data or it can be obtained from wave profile data measured at the location. For motion studies, the RAO is usually a ratio of amplitudes. In this case, the RAO is already a dimensionless number. One would expect the peak RAO to be the same at model and prototype scales.

Model frequencies are usually higher than prototype frequencies. Wave theory connects the frequencies. The dispersion relationship for deep water waves is

$$\omega^2 = gk$$

$$\omega = 2\pi/T \qquad k = 2\pi/\lambda$$

Manipulation gives

$$[2\pi/T]^2 = g[2\pi/\lambda]$$

$$[2\pi/T_M]^2 = g[2\pi/\lambda_M] \qquad [2\pi/T_P]^2 = g[2\pi/\lambda_P]$$

Division of model by prototype gives

$$[T_P/T_M]^2 = \lambda_P/\lambda_M$$

Geometric scaling requires that

$$\lambda_P/\lambda_M = D_P/D_M$$

This gives

$$[T_P/T_M]^2 = D_P/D_M$$

$$[\omega_P/\omega_M]^2 = D_M/D_P$$

This implies that for a 100:1 geometry ratio the period ratio is 10:1 while the frequency ratio is 0.1:1. Note that a response spectrum by definition is

$$S_R = [R_O]^2/\Delta\omega$$

This implies that for a 100:1 geometry ratio the spectrum ratio would be 100000:1. For finite depth water

$$\omega^2 = gk\ \mathrm{Tanh}[kh]$$

This gives the same scaling laws as those for deep water.

The resistance to forward motion of ships is often studied at model scale. Most of the resistance is due to wave generation by the ship. One usually plots the resistance force coefficient C_D versus the Froude Number F_R:

$$C_D = R / [A \rho U^2/2] \qquad F_R = U / \sqrt{[g D]}$$

For a ship moving at a steady speed, the phase speed of generated waves directly behind it matches the ship speed. The dispersion relationship gives for phase speed:

$$C_W = \sqrt{[g/k]} = \sqrt{[g\lambda/[2\pi]]}$$

Manipulation gives

$$C_P/\sqrt{[g\lambda_P]} = C_M/\sqrt{[g\lambda_M]}$$

$$U_P/\sqrt{[gD_P]} = U_M/\sqrt{[gD_M]}$$

So the Froude Number definition follows from the dispersion relationship. It is basically a speed coefficient.

```
%
% HYDRODYNAMICS LAB
%
clear all
DRAFT=0.55; DEPTH=1.5;
MASS=27.7; POT=11.9;
SPRING=359.0; HEAVE=0.1;
OMEGA=0.00001; STEP=0.01;
HEIGHT=0.1; PERIOD=2.0;
MOMENT0=0.0; MOMENT1=0.0;
GRAVITY=9.81; PI=3.14159;
for count=1:1000
  A=346.0*HEIGHT^2/PERIOD^4;
  B=691/PERIOD^4;
  WAVE=A/OMEGA^5*exp(-B/OMEGA^4);
  HZ(count)=OMEGA/(2.0*PI); SZ(count)=WAVE;
  WAVENUMBER=OMEGA^2/GRAVITY;
  CORRECT=tanh(WAVENUMBER*DEPTH);
  WAVENUMBER=WAVENUMBER/CORRECT;
  G=exp(-WAVENUMBER*DRAFT);
  ONE=cosh(WAVENUMBER*(-DRAFT+DEPTH));
  TWO=cosh(WAVENUMBER*DEPTH); G=ONE/TWO;
  P=(SPRING-MASS*OMEGA^2); Q=POT*OMEGA;
  M=SPRING*G*P/(P^2+Q^2);
  N=-SPRING*G*Q/(P^2+Q^2);
  M=G*(SPRING*P+Q*Q)/(P^2+Q^2);
  N=G*(P*Q-SPRING*Q)/(P^2+Q^2);
  RAO=sqrt(M^2+N^2); RESPONSE=RAO*RAO*WAVE;
  MR(count)=RAO; SR(count)=RESPONSE;
  MOMENT0=MOMENT0+RESPONSE*STEP/2.0;
  MOMENT1=MOMENT1+RESPONSE*OMEGA*STEP/2.0;
  OMEGA=OMEGA+STEP;
end
SIGR=4.0*sqrt(MOMENT0)/2.0
SIGT=2.0*PI*MOMENT0/MOMENT1
PROB=exp(-HEAVE^2/(2.0*MOMENT0))
plot(HZ,SZ*500,HZ,SR*500,HZ,MR)
```

```
%
% SIGNALS LAB DATA
%
% FAST FOURIER TRANSFORM
%
clear all
name='Book1.txt';
S=load(name);
out=254/2;
depth=2.0;
gravity=9.81;
delt=0.00004; cycle=1/delt;
nit=1000000; mit=2*nit;
bit=nit+1; f(1)=0.0;
w(1)=0.0; wn(1)=0.0;
% period=2.0;
% omega=2*pi/period;
CFW=1.0;CFP=1.0;
SW=S(1,2); SP=S(1,3);
for it=1:nit
 iot=it+1;
 time=it*delt;
 t(it)=time;
 wave(it)=(S(it,2)-SW)*CFW;
 heave(it)=(S(it,3)-SP)*CFP;
 % shake(it)=sin(omega*time);
 f(iot)=cycle/2*iot/nit;
end
width=cycle/mit;
wave=wave-mean(wave);
zw=fft(wave,mit)/nit;
pw=2*abs(zw(1:bit));
qw=2*pw(1:bit).*conj(2*pw(1:bit))/width;
```

```
figure(6)
plot(f(1:out),qw(1:out))
xlabel('hz')
ylabel('data')
title('spectrum')
figure(5)
plot(f(1:out),pw(1:out))
xlabel('hz')
ylabel('data')
title('FFT')
figure(4)
plot(t,wave)
xlabel('time')
ylabel('data')
title('test')
heave=heave-mean(heave);
zp=fft(heave,mit)/nit;
pp=2*abs(zp(1:bit));
qp=2*zp(1:bit).*conj(2*zp(1:bit))/width;
figure(3)
plot(f(1:out),qp(1:out))
xlabel('hz')
ylabel('data')
title('spectrum')
figure(2)
plot(f(1:out),pp(1:out))
xlabel('hz')
ylabel('data')
title('FFT')
figure(1)
plot(t,heave)
xlabel('time')
ylabel('data')
title('test')
```

```
%
% SIGNALS LAB DATA
%
% STANDARD FOURIER TRANSFORM
%
%
clear all
name='Book1.txt';
S=load(name);
gravity=9.81;
depth=2.0;
nit=1000000; mit=300;
sat=1000; nit=nit/sat;
delt=0.00004*sat;
span=delt*nit;
SW=S(1,2); SP=S(1,3);
CFW=1.0;CFP=1.0;
% period=2.0;
% omega=2*pi/period;
for nat=1:nit
time=nat*delt;
t(nat)=time;
window=sin(pi*time/span);window=1.0;
wave(nat)=(S(nat*sat,2)-SW)*window*CFW;
heave(nat)=(S(nat*sat,3)-SP)*window*CFP;
% shake(nat)=sin(omega*time)*window;
end
```

```
wave=wave-mean(wave);
abc=0.0;
xyz=0.0;
omega=0.0;
for mat=1:mit
omega=omega+pi/100;
hz(mat)=omega/(2*pi);
time=0.0;
for nat=1:nit
time=time+delt;
one=sin(omega*time)*delt;
two=cos(omega*time)*delt;
abc=abc+wave(nat)*one;
xyz=xyz+wave(nat)*two;
end
sum=(abc^2+xyz^2)^0.5;
uvw=2*sum/span;
level(mat)=uvw;
energy(mat)=uvw*uvw ...
    /(pi/100);
abc=0.0; xyz=0.0;
end
figure(6)
plot(hz,energy)
xlabel('hz')
ylabel('data')
title('spectrum')
figure(5)
plot(hz,level)
xlabel('hz')
ylabel('data')
title('SFT')
figure(4)
plot(t,wave)
xlabel('time')
ylabel('data')
title('test')
```

```
heave=heave-mean(heave);
abc=0.0;
xyz=0.0;
omega=0.0;
for mat=1:mit
omega=omega+pi/100;
hz(mat)=omega/(2*pi);
time=0.0;
for nat=1:nit
time=time+delt;
one=sin(omega*time)*delt;
two=cos(omega*time)*delt;
abc=abc+heave(nat)*one;
xyz=xyz+heave(nat)*two;
end
sum=(abc^2+xyz^2)^0.5;
uvw=2*sum/span;
level(mat)=uvw;
energy(mat)=uvw*uvw ...
    /(pi/100);
abc=0.0; xyz=0.0;
end
figure(3)
plot(hz,energy)
xlabel('hz')
ylabel('data')
title('spectrum')
figure(2)
plot(hz,level)
xlabel('hz')
ylabel('data')
title('SFT')
figure(1)
plot(t,heave)
xlabel('time')
ylabel('data')
title('test')
```