FLUID STRUCTURE INTERACTIONS

UNSTEADY FLOW

IN PIPE NETWORKS

PREAMBLE

Unsteady flow in pipe networks can be caused by a number of factors. A turbomachine with blades can send pressure waves down a pipe. If the period of these waves matches a natural period of the pipe wave speed resonance develops. A piston pump can send similar waves down a pipe. Waves on the surface of a water reservoir can also excite resonance of inlet pipes. One way to avoid resonance is to change the wave speed of the pipes in the network. For liquids, one can do this by adding a gas such as air. This can be bled into the network at critical locations or it can be held in a flexible tube which runs inside the pipes. One could also use a flexible pipe to change the wave speed. Sudden valve or turbomachine changes can send waves up and down pipes. These can cause the pipes to explode or implode. In some cases interaction between pipes and devices is such that oscillations develop automatically. Examples include oscillations set up by leaky valves and those set up by slow turbomachine controllers. To lessen the severity of transients in a hydraulic network, one can use gas accumulators. Hydro plants use surge pipes. Another way to lessen the severity of transients is use of relief valves. These are spring loaded valves which open when the pressure reaches a preset level. This can be high or low. For high pressure liquids, they create a pathway back to a sump. For low pressure liquids, they allow a gas such as air to enter the pipe. Bypass valves and check valves can be used to isolate turbomachines when they fail.

There are three procedures that can be used to study unsteady flow in pipe networks. The most complex of these is the Method of Characteristics. This finds directions in space and time along which the partial differential equations of mass and momentum reduce to an ordinary differential equation in time. Computational Fluid Dynamics codes have been developed based on this method that can handle extremely complex pipe networks. A second procedure is known as Graphical Waterhammer. It is a graphical form of a procedure known as Algebraic Waterhammer. It makes extensive use of PU plots. A third procedure is known as the Impedance Method. This makes use of Laplace Transforms. It employs something called the Impedance Transfer Function. It resembles closely a method used to study Electrical Transmission Lines.

These notes start with a physical description of how pressure waves propagate along a pipe. This is followed by a derivation of the basic wave equations. Then, wave speeds for waves in flexible tubes and mixtures are given. Next, an outline of Algebraic/Graphical Waterhammer is given. Finally, the Method of Characteristics is presented.

WAVE PROPAGATION IN PIPES

Consider flow in a rigid pipe with a valve at its downstream end and a reservoir at its upstream end. Assume that there are no friction losses. This implies that the pressure and flow speed are the same everywhere along the pipe.

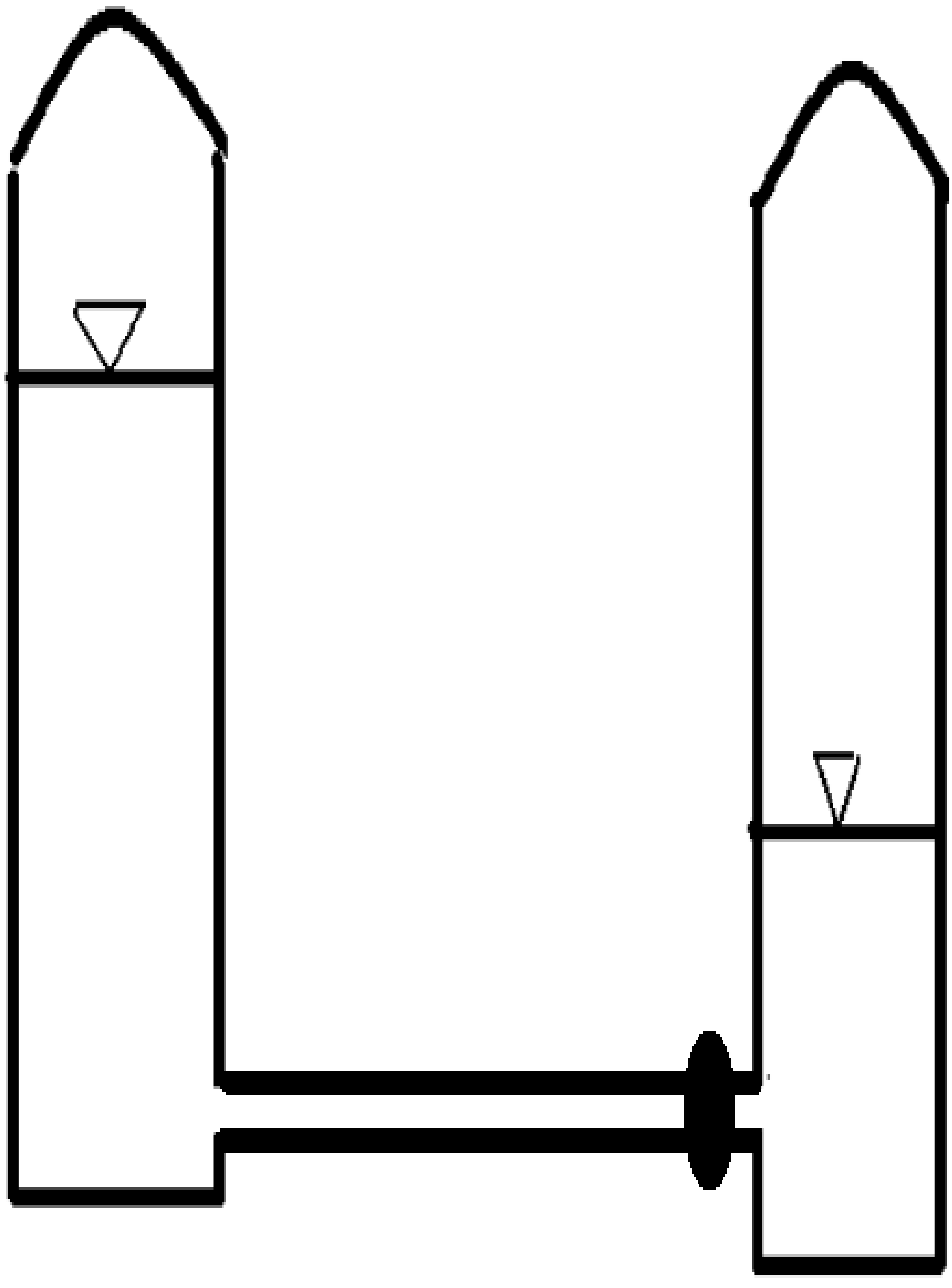
Imagine now that the valve is suddenly closed. This causes a high pressure or surge wave to propagate up the pipe. As it does so, it brings the fluid to rest. The fluid immediately next to the valve is stopped first. The valve is like a wall. Fluid enters an infinitesimal layer next to this wall and pressurizes it and stops. This layer becomes like a wall for an infinitesimal layer just upstream. Fluid then enters that layer and pressurizes it and stops. As the surge wave propagates up the pipe, it causes an infinite number of these pressurizations. When it reaches the reservoir, all of the inflow has been stopped, and pressure is high everywhere along the pipe. The pipe resembles a compressed spring.

When the surge wave reaches the reservoir, it creates a pressure imbalance. The layer of fluid just inside the pipe has high pressure fluid downstream of it and reservoir pressure upstream. Fluid exits the layer on its upstream side and depressurizes it. The pressure drops back to the reservoir level. A backflow wave is created. The speed of the backflow is exactly the same as the speed of the original inflow. The pressure that was generated by taking the original inflow away is exactly what is available to generate the backflow. The backflow wave propagates down the pipe restoring pressure everywhere to its original level.

When the backflow wave reaches the valve, it creates a flow imbalance. This causes a low pressure or suction wave to propagate up the pipe. As it does so, it brings the fluid to rest. Again, the valve is like a wall. Because of backflow, fluid exits an infinitesimal layer next to this wall and depressurizes it and stops. The pressure drops below the reservoir level by exactly the amount it was above the reservoir level in the surge wave.

When the suction wave reaches the reservoir, all of the backflow has been stopped, and pressure is low everywhere along the pipe. The pipe resembles a stretched spring. At the reservoir, the suction wave creates a pressure imbalance. An inflow wave is created. The speed of the inflow is exactly the same as the speed of the backflow. The inflow wave travels down the pipe restoring pressure to its original level. Conditions in the pipe become what they were just before the valve was closed.

During one cycle of vibration, there are 4 transits of the pipe by pressure waves. This means that the natural period of the pipe is 4 times the length of the pipe divided by the wave speed. Without friction, the vibration cycle repeats over and over. With friction, it gradually dies away.



BASIC WAVE EQUATIONS

Consider a wave travelling up a rigid pipe. In a reference frame moving with the wave, mass considerations give

$$\rho\ A\ (U+a) = (\rho+\Delta\rho)\ A\ (U+\Delta U+a)$$

where ρ is density, A is pipe area, U is flow velocity and a is wave speed. When a >> U, this reduces to

$$0 = \rho\ \Delta U + a\ \Delta\rho$$

Momentum considerations give

$$[(\rho+\Delta\rho)A(U+\Delta U+a)\ (U+\Delta U+a) - \rho A(U+a)\ (U+a)] = [P - [P+\Delta P]]\ A$$

$$\rho A(U+a)\ [(U+\Delta U+a) - (U+a)] = - \Delta P\ A$$

where P is pressure. When a >> U, this reduces to

$$\rho\ a\ \Delta U = - \Delta P$$

Manipulations give

$$a = \sqrt{[\Delta P/\Delta \rho]}$$

For a gas such as air moving down a pipe, one can assume ideal gas behavior for which:

$$P/\rho = R\ T$$

R is the ideal gas constant and T is the absolute temperature of the gas. For a wave propagating through a gas, one can assume processes are isentropic: in other words, adiabatic and frictionless. The wave moves so fast through the gas that there is no time for heat transfer or friction. The isentropic equation of state is:

$$P = K\ \rho^{k}$$

where K is another constant and k is the ratio of specific heats. Differentiation of this equation gives

$$\Delta P/\Delta \rho = K\ k\ \rho^{k-1} = K\ k\ \rho^{k} / \rho$$
$$= k/\rho\ \ K\ \rho^{k} = k\ P/\rho$$

The ideal gas law into this gives

$$\Delta P/\Delta \rho = k\ R\ T$$

So wave speed for a gas becomes

$$a = \sqrt{[k\ R\ T]}$$

For a liquid, fluid mechanics shows that

$$\Delta P = - K\ \Delta V/V$$

where K is the bulk modulus of the liquid. It is a measure of its compressibility. For a bit of fluid mass

$$\Delta M = \Delta\ [\rho\ V] = V\ \Delta\rho + \rho\ \Delta V = 0$$

This implies that

$$\Delta P = K\ \Delta\rho/\rho \qquad \Delta P/\Delta\rho = K/\rho$$

So wave speed for a liquid becomes

$$a = \sqrt{[K/\rho]}$$

The bulk modulus of a gas follows from

$$a = \sqrt{[k\ R\ T]} = \sqrt{[K/\rho]}$$

$$K/\rho = k\ R\ T \qquad K = k\ \rho\ R\ T$$

$$K = k\ P$$

WAVES IN FLEXIBLE TUBES

Conservation of Mass for a flexible tube is

$$\rho\ A\ (U+a) = (\rho+\Delta\rho)\ (A+\Delta A)\ (U+\Delta U+a)$$

Manipulation of this equation gives when U<<a

$$\rho A\ \Delta U\ +\ (U+a)A\ \Delta\rho\ +\ \rho(U+a)\ \Delta A = 0$$

$$\Delta U/a\ +\ \Delta\rho/\rho\ +\ \Delta A/A = 0$$

Conservation of Momentum for a flexible tube is

$$[\rho A(U+a)]\ [(U+\Delta U+a)-(U+a)] =$$

$$PA + [P+\Delta P]\ \Delta A - [P+\Delta P][A+\Delta A]$$

Manipulation of this equation gives when U<<a

$$\rho A(U+a)\ \Delta U\ +\ A\ \Delta P\ =\ 0$$

$$\rho a\ \Delta U\ +\ \Delta P\ =\ 0$$

More manipulation gives

$$\Delta U = -\ \Delta P/[\rho a] \qquad\qquad \Delta U/a = -\Delta P/[\rho a^2]$$

Experiments show that

$$\Delta P = K\ \Delta\rho/\rho \qquad\qquad \Delta\rho/\rho = \Delta P/K$$

For a thin wall tube, the hoop stress follows from

$$[2e]\ \sigma = \Delta P\ D \qquad\qquad \sigma = \Delta P\ D/[2e]$$

where e is the wall thickness and D is the tube diameter. The hoop strain is

$$\varepsilon = [\pi\Delta D]/[\pi D] = \Delta D/D$$

Substitution into the stress strain connection gives

$$\sigma = E\ \varepsilon \qquad\qquad \Delta P\ D/[2e] = E\ \Delta D/D$$

where E is the Elastic Modulus of the wall material.

Geometry gives

$$A = \pi\ D^2/4 \qquad\qquad \Delta A = \pi\ 2D/4\ \Delta D$$

$$\Delta A/A = 2\ \Delta D/D = \Delta P\ D/[Ee]$$

With this Conservation of Mass becomes

$$-\ \Delta P/[\rho a^2]\ +\ \Delta P/K\ +\ \Delta P\ D/[Ee] = 0$$

Manipulation of Conservation of Mass gives

$$a = \sqrt{\ [\mathbf{K}/\rho]}$$

$$\mathbf{K} = K\ /\ [\ 1 + [DK]/[Ee]\ ]$$

WAVES IN MIXTURES

For a mixture the wave speed is:

$$a_M = \sqrt{[K_M/\rho_M]}$$

For a two component mixture the density follows from:

$$M_M = M_A + M_B \qquad \rho_M V_M = \rho_A V_A + \rho_B V_B$$

$$\rho_M = [\ \rho_A V_A + \rho_B V_B\ ]\ /\ V_M$$

Experiments show that

$$\Delta P = -\ K_M\ [\Delta V_M/V_M]$$

Manipulation gives the bulk modulus

$$K_M = -\ \Delta P\ /\ [\Delta V_M/V_M]$$

$$V_M = V_A + V_B \qquad \Delta V_M = \Delta V_A + \Delta V_B$$

For each component in the mixture:

$$\Delta P = -\ K_A\ [\Delta V_A/V_A] \qquad \Delta V_A = -\ [V_A/K_A]\ \Delta P$$

$$\Delta P = -\ K_B\ [\Delta V_B/V_B] \qquad \Delta V_B = -\ [V_B/K_B]\ \Delta P$$

The mixture bulk modulus becomes:

$$K_M = [\ V_A + V_B\ ]\ /\ [\ V_A/K_A + V_B/K_B\ ]$$

The mixture analysis is also valid for mixtures of small solid particles and a fluid, such as a dusty gas.

ALGEBRAIC/GRAPHICAL WATERHAMMER

Waterhammer analysis allows one to connect unknown pressure and flow velocity at one end of a pipe to known pressure and velocity at the other end of the pipe one transit time back in time. The derivation of the waterhammer equations starts with the conservation of momentum and mass equations for unsteady flow in a pipe. These are:

$$\rho\ \partial U/\partial t + \rho U\ \partial U/\partial x + \partial P/\partial x - \rho g\ \mathrm{Sin}\alpha + f/D\ \rho U|U|/2 = 0$$

$$\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x = 0$$

where P is pressure and U is velocity. For the case where gravity and friction are insignificant and the mean flow speed is approximately zero, these reduce to:

$$\rho\ \partial U/\partial t + \partial P/\partial x = 0$$

$$\partial P/\partial t + \rho a^2\ \partial U/\partial x = 0$$

Manipulation gives the wave equations:

$$\partial^2 P/\partial t^2 = a^2\ \partial^2 P/\partial x^2$$

$$\partial^2 U/\partial t^2 = a^2\ \partial^2 U/\partial x^2$$

The general solution consists of two waves: one wave which travels up the pipe known as the F wave and the other which travels down the pipe known as the f wave.

In terms of these waves, pressure and velocity are:

$$P - P_o = f(N) + F(M)$$

$$U - U_o = [f(N) - F(M)] / [\rho a]$$

where N and M are wave fixed frames given by:

$$N = x - a\,t \qquad M = x + a\,t$$

For a given point N on the f wave, the N equation shows that x must increase as time increases, which means the wave must be moving down the pipe. For a given point M on the F wave, the M equation shows that x must decrease as time increases, which means the wave must be moving up the pipe. Substitution of the general solution into mass or momentum or the wave equations shows that they are valid solutions.

Multiplying U by ρa and subtracting it from P gives:

$$[P-P_o] - \rho a[U-U_o] = 2F(M)$$

Let the F wave travel from the downstream end of the pipe to the upstream end. For a point on the wave, the value of F would be the same. This implies

$$\Delta P = + \rho a \ \Delta U$$

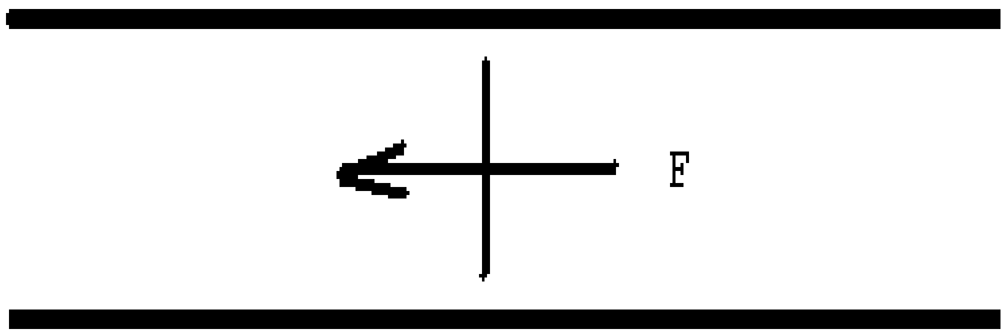
Multiplying U by ρa and adding it to P gives:

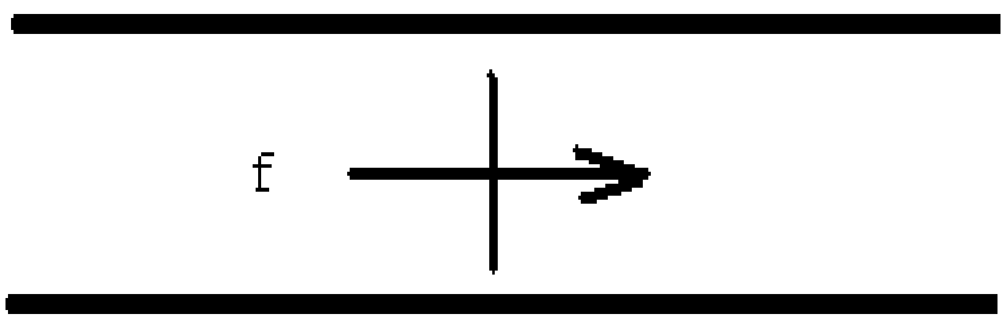
$$[P-P_o] + \rho a[U-U_o] = 2f(N)$$

Let the f wave travel from the upstream end of the pipe to the downstream end. For a point on the wave, the value of f would be the same. This implies

$$\Delta P = - \rho a \ \Delta U$$

The ΔP vs ΔU equations allow us to connect unknown conditions at one end of a pipe at some point in time to known conditions at the other end back in time. They are known as the algebraic/graphical waterhammer equations.


F


f

DERIVATION OF WAVE EQUATIONS

Conservation of Momentum and Mass are:

$$\rho\ \partial U/\partial t + \partial P/\partial x = 0$$

$$\partial P/\partial t + \rho a^2\ \partial U/\partial x = 0$$

Differentiation of Momentum with respect to t gives

$$\rho\ \partial^2 U/\partial t^2 + \partial(\partial P/\partial x)/\partial t = 0$$

Differentiation of Mass with respect to x gives

$$\partial(\partial P/\partial t)/\partial x + \rho a^2\ \partial^2 U/\partial x^2 = 0$$

Subtraction of Mass from Momentum gives

$$\rho\ \partial^2 U/\partial t^2 = \rho a^2\ \partial^2 U/\partial x^2$$

$$\partial^2 U/\partial t^2 = a^2\ \partial^2 U/\partial x^2$$

Similar manipulations of Momentum and Mass give

$$\rho a^2\ \partial(\partial U/\partial t)/\partial x + a^2\ \partial^2 P/\partial x^2 = 0$$

$$\partial^2 P/\partial t^2 + \rho a^2\ \partial(\partial U/\partial x)/\partial t = 0$$

$$\partial^2 P/\partial t^2 = a^2\ \partial^2 P/\partial x^2$$

GENERAL SOLUTIONS FOR WAVE EQUATIONS

The general solutions are

$$P - P_o = f(N) + F(M)$$

$$U - U_o = [f(N) - F(M)] / [\rho a]$$

where N and M are wave fixed coordinates

$$N = x - a\,t \qquad M = x + a\,t$$

Substitution into Momentum gives

$$\rho\ \partial U/\partial t + \partial P/\partial x = 0$$

$$(\ \rho\ \partial f/\partial N\ \partial N/\partial t\ -\ \rho\ \partial F/\partial M\ \partial M/\partial t\ )\ /\ [\rho a]$$
$$+\ \partial f/\partial N\ \partial N/\partial x\ +\ \partial F/\partial M\ \partial M/\partial x$$
$$= 0$$

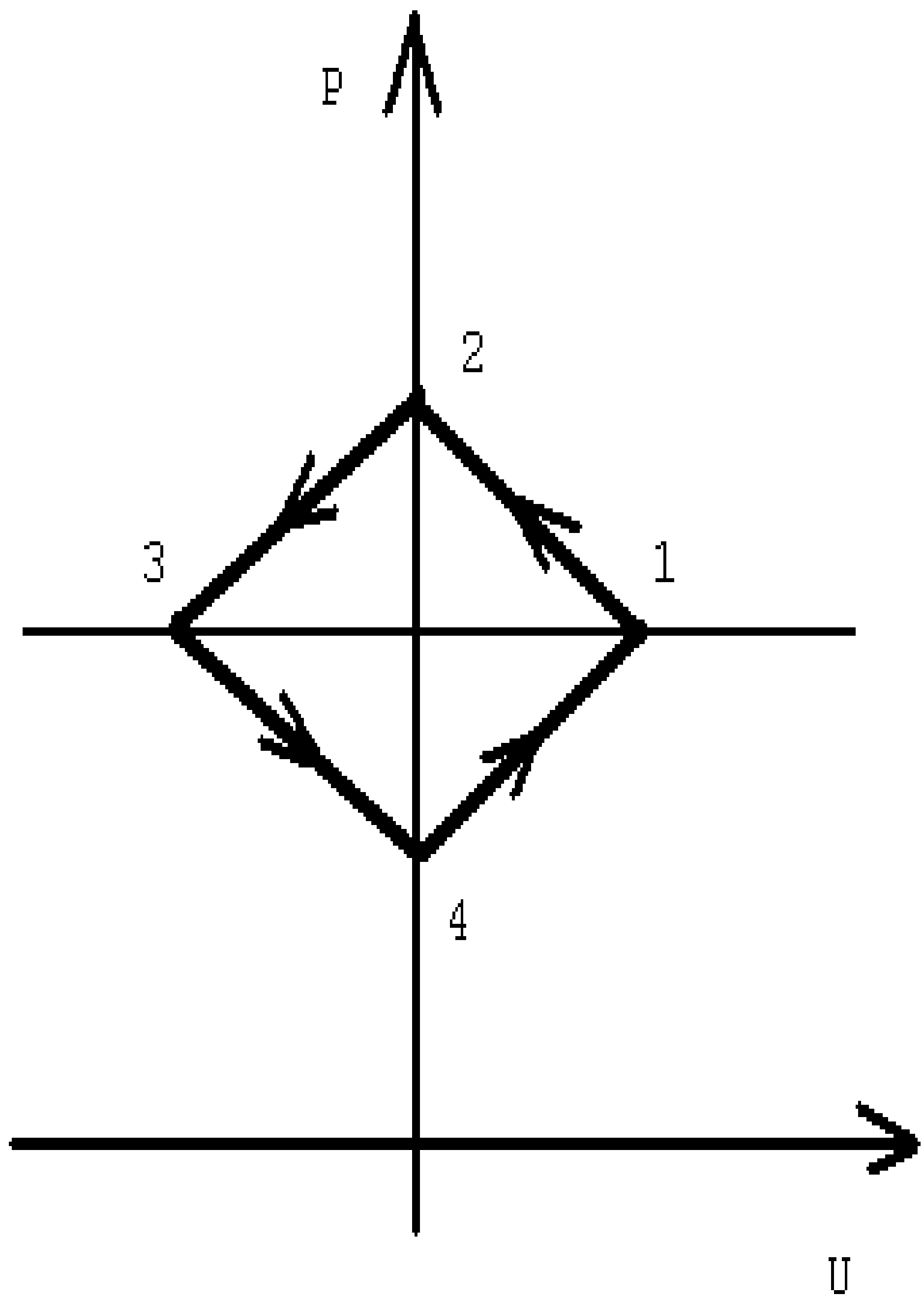
$$(\ \rho\ \partial f/\partial N\ [-a]\ -\ \rho\ \partial F/\partial M\ [+a]\ )\ /\ [\rho a]$$
$$+\ \partial f/\partial N\ [1]\ +\ \partial F/\partial M\ [1]$$
$$= 0$$

$$-\ \partial f/\partial N\ -\ \partial F/\partial M\ +\ \partial f/\partial N\ +\ \partial F/\partial M\ =\ 0$$
$$0 = 0$$

So the general solutions satisfy Momentum. One can also show that they satisfy Mass and the Wave equations.

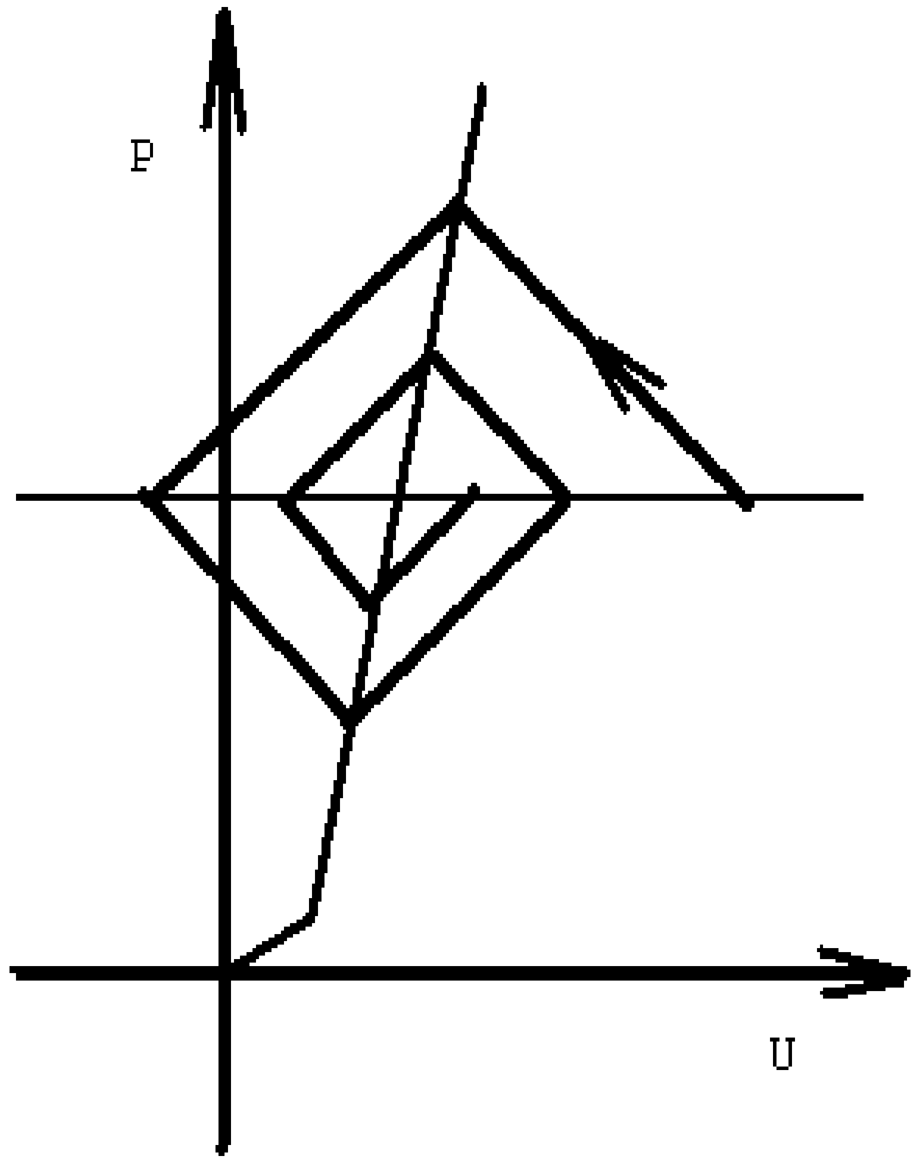
SUDDEN VALVE CLOSURE

Imagine a pipe with a reservoir at its upstream end and a valve at its downstream end. The valve is initially open. Then it is suddenly shut. From that point onward, the velocity at the valve is zero. We ignore losses. Because of this, the pressure at the reservoir is fixed at its initial level. We start at point 1 which is at the reservoir and move along an f wave to point 2 which is at the valve. A surge wave is created at the valve. We then move from the valve along an F wave to point 3 which is at the reservoir. A backflow wave is created at the reservoir. We then move from the reservoir along an f wave to point 4 which is at the valve. A suction wave is created at the valve. We then move from the valve along an F wave to point 1 which is at the reservoir. An inflow wave is created at the reservoir. From this point onward the cycle repeats. Friction gradually dissipates the waves and the velocity homes in on zero.

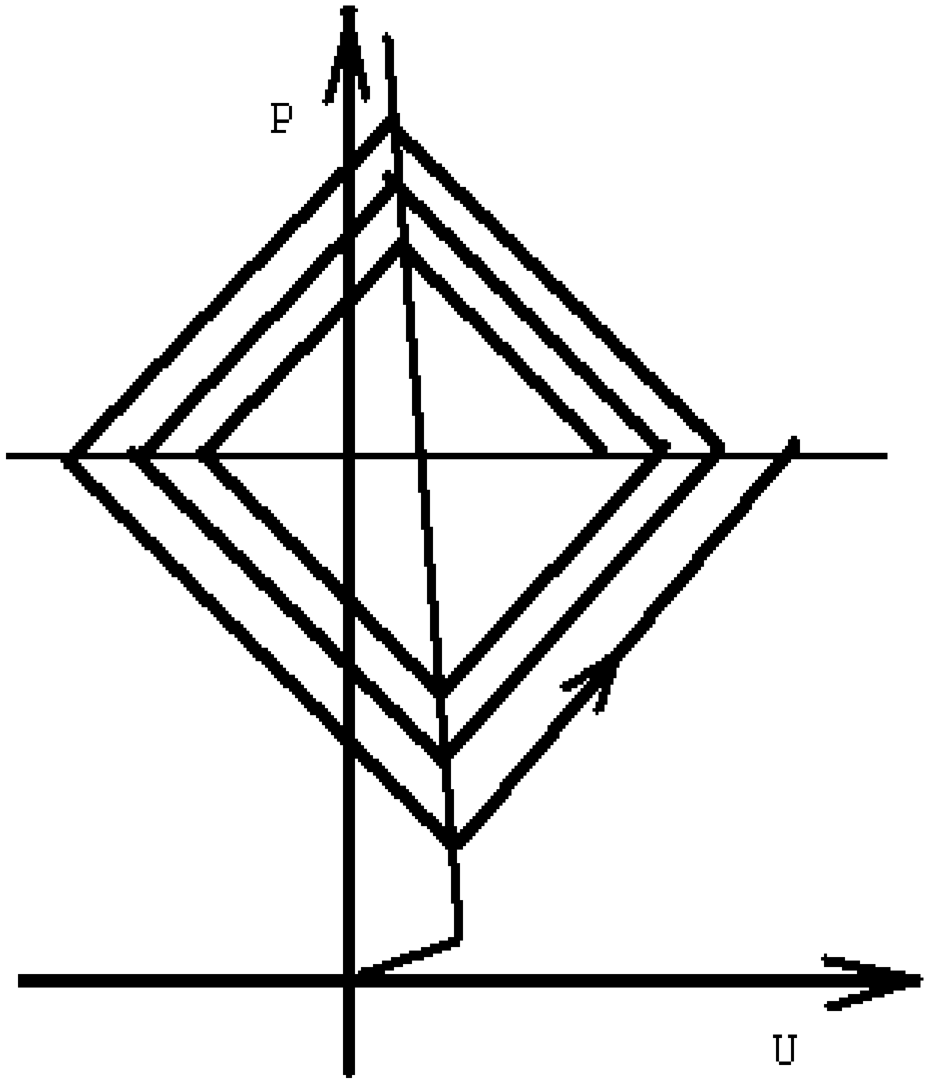

P
2
3
1
4
U

LEAKY VALVES

A stable leaky valve is basically one that has a P versus U characteristic which resembles that of a wide open valve. This has a parabolic shape with positive slope throughout. An unstable leaky valve has a characteristic that has a positive slope at low pressure but negative slope at high pressure. Basically, the valve tries to shut itself at high pressure. The flow rate just upstream of a valve is pipe flow speed times pipe area. The flow rate within the valve is valve flow speed times valve area. In a stable leaky valve, the areas are both constant. The valve flow speed increases with pipe pressure so the pipe flow speed also increases. In an unstable leaky valve, the flow speed within the valve also increases with pipe pressure but the valve area drops because of suction within the valve. The suction is generated by high speed flow through the small passageway within the valve. It pulls on flexible elements within the valve and attempts to shut it. Graphical waterhammer plots for stable and unstable leaky valves are given below. As can be seen, they both resemble the sudden valve closure plot, but the stable one is decaying while the unstable one is growing. In the unstable case, greater suction is needed each time a backflow wave comes up to the valve because the flow requirements of the valve keep getting bigger. In the stable case, less suction is needed because the flow requirements keep getting smaller.



STABLE LEAKY VALVE



UNSTABLE LEAKY VALVE

METHOD OF REACHES

Pipes in a pipe network often have different lengths. The method of reaches divides the pipes into segments that have the same transit time. The segments are known as reaches. The sketch on the next page shows a pipe divided into 4 reaches. Conditions at points i j k are known. Conditions at point J are unknown. Waterhammer analysis gives for point J:

$$\Delta P = - \rho a \ \Delta U$$

$$P_J = P_i - [\rho a][U_J - U_i]$$

$$\Delta P = + \rho a \ \Delta U$$

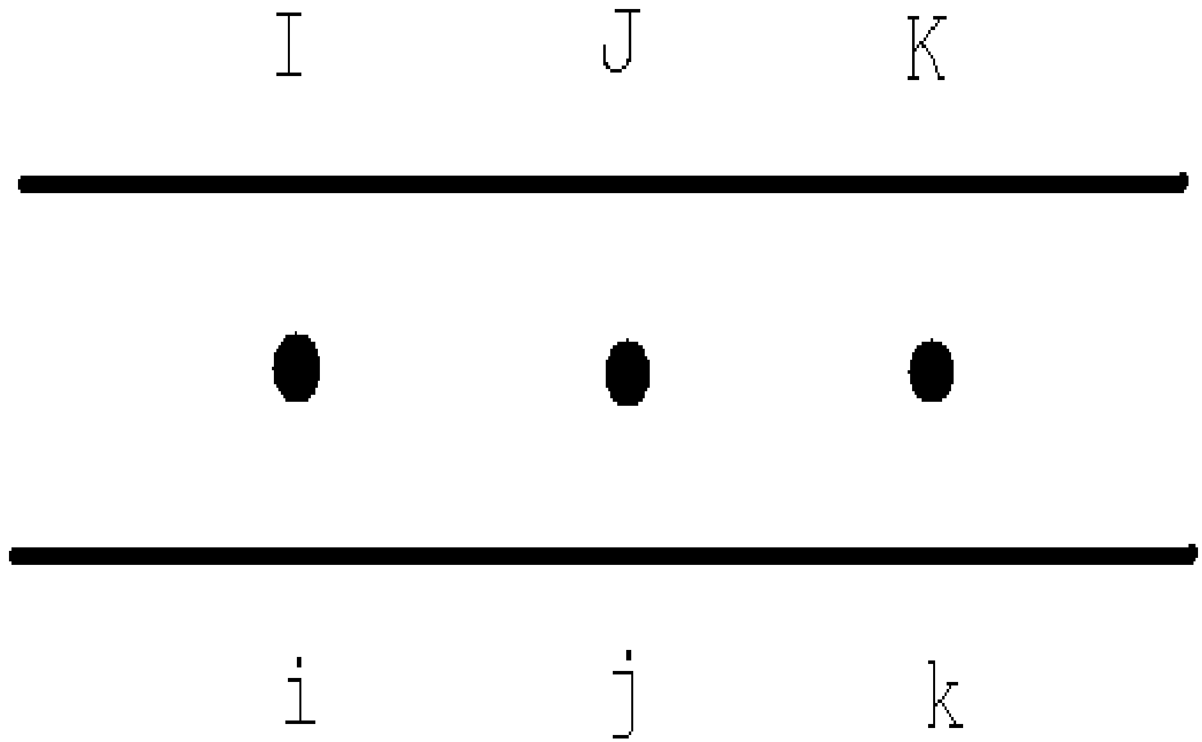
$$P_J = P_k + [\rho a][U_J - U_k]$$

Manipulation of these equations gives:

$$P_J = (P_k + P_i)/2 - [\rho a][U_k - U_i]/2$$

$$U_J = (U_k + U_i)/2 - [P_k - P_i]/[2\rho a]$$

This is the template for finding conditions at points inside the pipe. At the ends of a pipe, water hammer analysis would connect the end points to j points inside the pipe.


I
J
K
i
j
k

TREATMENT OF PIPE JUNCTIONS

Pipes in a pipe network are connected at junctions. The sketch on the next page shows a junction which connects 3 pipes. Lower case letters indicate known conditions. Upper case letters indicate unknown conditions. A junction is often small. This allows us to assume that the junction pressure is common to all pipes. It also allows us to assume that the net flow into or out of the junction is zero. Conservation of Mass considerations give:

$$\rho\ A_N\ U_N \ + \ \rho\ A_H\ U_H \ + \ \rho\ A_W\ U_W \ = \ 0$$

Waterhammer analysis gives:

$$P_N - P_m = + [\rho a_N]\ [U_N-U_m]$$

$$P_H - P_g = + [\rho a_H]\ [U_H-U_g]$$

$$P_W - P_v = + [\rho a_W]\ [U_W-U_v]$$

Manipulation gives

$$U_N = U_m + [P_N-P_m]/[\rho a_N]$$

$$U_H = U_g + [P_H-P_g]/[\rho a_H]$$

$$U_W = U_v + [P_W-P_v]/[\rho a_W]$$

In these equations $P_N = P_H = P_W = P_J$. Substitution into Conservation of Mass gives:

$$\rho A_N [U_m + [P_J - P_m]/[\rho a_N]] + \rho A_H [U_g + [P_J - P_g]/[\rho a_H]] + \rho A_W [U_v + [P_J - P_v]/[\rho a_W]] = 0$$

Manipulation gives the junction pressure:

$$P_J = [X - Y] / Z$$

where

$$X = [A_N/a_N \, P_m + A_H/a_H \, P_g + A_W/a_W \, P_v]$$

$$Y = \rho [A_N U_m + A_H U_g + A_W U_v]$$

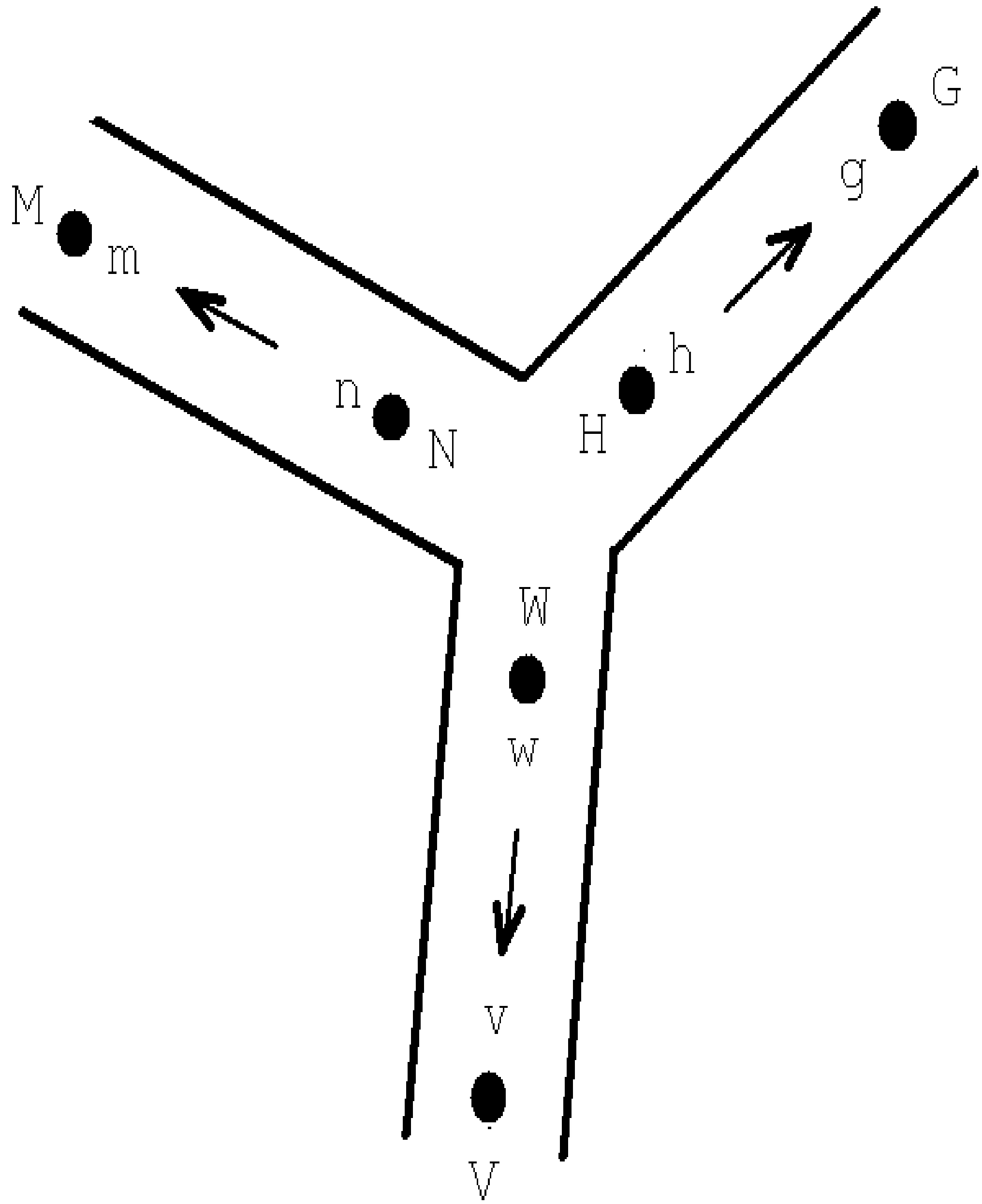
$$Z = [A_N/a_N + A_H/a_H + A_W/a_W]$$

The velocities at the junction are:

$$U_N = U_m + [P_J - P_m]/[\rho a_N]$$

$$U_H = U_g + [P_J - P_g]/[\rho a_H]$$

$$U_W = U_v + [P_J - P_v]/[\rho a_W]$$



M
m
G
g
h
n
H
N
W
w
v
V

ACCUMULATOTS

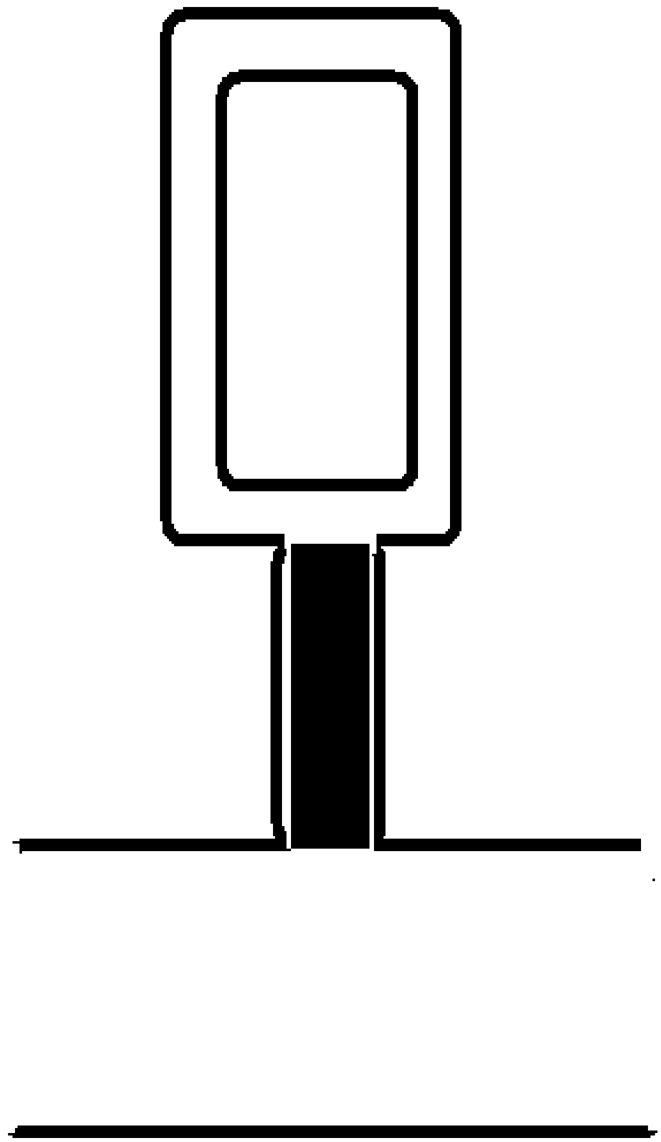
Accumulators are used to dampen transients in pipe networks. They generally consist of a neck or constriction containing liquid which is connected directly to the pipe network. A pocket of gas is at the other end of the neck. The gas is usually contained inside a flexible bladder.

There are two ways to model an accumulator. The first is the Helmholtz Resonator mass spring model where the slug of liquid in the neck bounces on the gas spring. This gives the natural frequency of the accumulator and one tries to match that to the natural period of the network. The second model is a transient model where the equation of motion of the slug of liquid in the neck and the equations for the gas pocket are solved step by step in time and this is coupled a water hammer analysis transient model.

The Helmholtz Resonator model starts with the equation of motion of a mass on a spring:

$$m \, d^2\Delta Z/dt^2 + k \, \Delta Z = f$$

where m is the mass of liquid in the neck and k is the spring due to gas compressibility.



The natural frequency and period of the accumulator are

$$\omega = \sqrt{[k/m]} \qquad T = 2\pi/\omega$$

The mass m of the slug of liquid in the neck is

$$m = \rho A L$$

where ρ is the density of the liquid in the neck, A is the area of the neck and L is the length of the neck.

Conservation of Mass for the gas pocket gives

$$\Delta [\sigma V] = V \Delta\sigma + \sigma \Delta V = 0$$

Thermodynamics gives

$$\Delta P/\Delta\sigma = a^2 \qquad a = \sqrt{[nRT]}$$

Geometry gives

$$\Delta V = - A \Delta Z$$

Substitution into mass gives

$$V \Delta P/a^2 - \sigma A \Delta Z = 0$$

$$\Delta P = [\sigma A a^2 / V] \Delta Z$$

The force on the slug of liquid is

$$\Delta F = \Delta P\ A = [\sigma\ A^2\ a^2\ /\ V]\ \Delta Z = k\ \Delta Z$$

This gives the spring constant k

$$k = [\sigma\ A^2\ a^2\ /\ V]$$

Substitution into the frequency equation gives

$$\omega = \sqrt{\ [\ [\sigma\ A^2\ a^2\ /\ V]\ /\ [\rho\ A\ L]\ ]}$$

$$= \sqrt{\ [\ [\sigma\ A\ a^2]\ /\ [\rho\ V\ L]\ ]}$$

For the transient model the equation governing the motion of the slug of liquid in the neck is:

$$m\ dU/dt = [\ P_J - P_G\ ]\ A - fL/D\ \rho\ U|U|/2\ A$$

where P_J is the junction pressure and P_G is the gas pressure. The volume of gas is governed by

$$dV/dt = -\ U\ A$$

The pressure of the gas is

$$P_G = N\ \sigma^n = N\ (M/V)^n$$

TREATMENT OF VALVES

A sketch of a valve is shown on the next page. The governing equation for the flow through it is:

$$P_N - P_X = K\ U|U|$$

For constant pipe properties

$$U = U_N = U_X \qquad P = P_N - P_X$$

$$P = K\ U|U|$$

Water hammer analysis gives

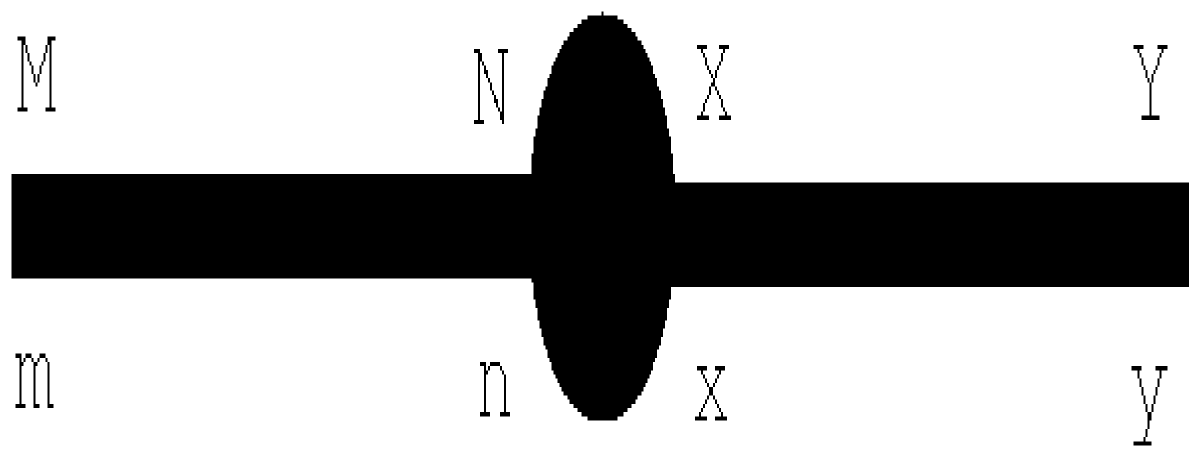
$$P_N - P_m = -\ \rho a\ (U_N - U_m)$$

$$P_X - P_y = +\ \rho a\ (U_X - U_y)$$

Substitution into the valve equation gives

$$[P_m - \rho a\ (U - U_m)] - [P_y + \rho a\ (U - U_y)] = K\ U|U|$$

This gives U at each time step. Back substitution gives the pressure upstream and downstream of the valve.


M
N
X
Y
m
n
x
y

METHOD OF CHARACTERISTICS

The method of characteristics is a way to determine the pressure and velocity variations in a pipe network when valves are adjusted or turbomachines undergo load changes. The equations governing flow in a typical pipe are:

$$\rho\ \partial U/\partial t + \rho U\ \partial U/\partial x + \partial P/\partial x - \rho g\ \mathrm{Sin}\alpha + f/D\ \rho U|U|/2 = 0$$
$$\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x = 0$$

where P is pressure, U is velocity, t is time, x is distance along the pipe, ρ is the fluid density, g is gravity, α is the pipe slope, f is the pipe friction factor, D is the pipe diameter and a is the wave speed. The wave speed is:

$$a^2 = \mathbf{K}/\rho \qquad \mathbf{K} = K\ /\ [1 + DK/Ee]$$

where K is the bulk modulus of the fluid, E is the Youngs Modulus of the pipe wall and e is its thickness.

The governing equations can be combined as follows:

$$\rho\ \partial U/\partial t\ +\ \rho\ U\ \partial U/\partial x\ +\ \partial P/\partial x\ +\ \rho\ C$$
$$+\ \lambda\ (\partial P/\partial t + U\ \partial P/\partial x + \rho a^2\ \partial U/\partial x)\ =\ 0$$

where

$$C = f/D\ U|U|/2 - g\ \sin\alpha$$

Manipulation gives

$$\rho\ (\partial U/\partial t + [U+\lambda a^2]\ \partial U/\partial x) + \lambda\ (\partial P/\partial t + [1/\lambda+U]\ \partial P/\partial x) + \rho C = 0$$

According to Calculus

$$dP/dt = \partial P/\partial t + dx/dt\ \partial P/\partial x$$

$$dU/dt = \partial U/\partial t + dx/dt\ \partial U/\partial x$$

Inspection of the last three equations suggests:

$$dx/dt = U + \lambda a^2 = 1/\lambda + U$$

In this case, the PDE becomes the ODE:

$$\rho\ dU/dt + \lambda\ dP/dt + \rho\ C = 0$$

The dx/dt equation gives

$$\lambda a^2 = 1/\lambda \quad \text{or} \quad \lambda^2 = 1/a^2 \quad \text{or} \quad \lambda = \pm 1/a$$

So there are 2 values of λ. They give

$$\rho\, dU/dt \;+\; 1/a\, dP/dt \;+\; \rho\, C \;=\; 0 \qquad dx/dt = U + a$$
$$\rho\, dU/dt \;-\; 1/a\, dP/dt \;+\; \rho\, C \;=\; 0 \qquad dx/dt = U - a$$

The dx/dt equations define directions in space and time along which the PDE becomes an ODE. Using finite differences, each ODE and dx/dt equation can be written as:

$$\rho\, \Delta U/\Delta t \;+\; 1/a\, \Delta P/\Delta t \;+\; \rho\, C \;=\; 0 \qquad \Delta x/\Delta t = U + a$$
$$\rho\, \Delta U/\Delta t \;-\; 1/a\, \Delta P/\Delta t \;+\; \rho\, C \;=\; 0 \qquad \Delta x/\Delta t = U - a$$

Manipulation gives

$$\rho a\, \Delta U \;+\; \Delta P \;+\; \Delta t\, \rho a\, C \;=\; 0$$
$$\rho a\, \Delta U \;-\; \Delta P \;+\; \Delta t\, \rho a\, C \;=\; 0$$

When the wave speed a is much greater than the flow speed U and when Δx is the length of the pipe L and Δt is the pipe transit time T, these equations are basically the water hammer equations but with friction added.

For pipes divided into reaches, one gets

$$U_P - U_L + (P_P-P_L)/[\rho a] + C_L(t_P-t_L) = 0 \quad x_P-x_L = (U_L+a)(t_P-t_L)$$

$$U_P - U_R - (P_P-P_R)/[\rho a] + C_R(t_P-t_R) = 0 \quad x_P-x_R = (U_R-a)(t_P-t_R)$$

Manipulation gives

$$U_P = (U_L+U_R)/2 + (P_L-P_R)/[2\rho a] - \Delta t(C_L+C_R)/2$$

$$P_P = (P_L+P_R)/2 + [\rho a](U_L-U_R)/2 - \Delta t[\rho a](C_L-C_R)/2$$

Linear interpolation gives U and P at points L and R in terms of known U and P at grid points A and B and C:

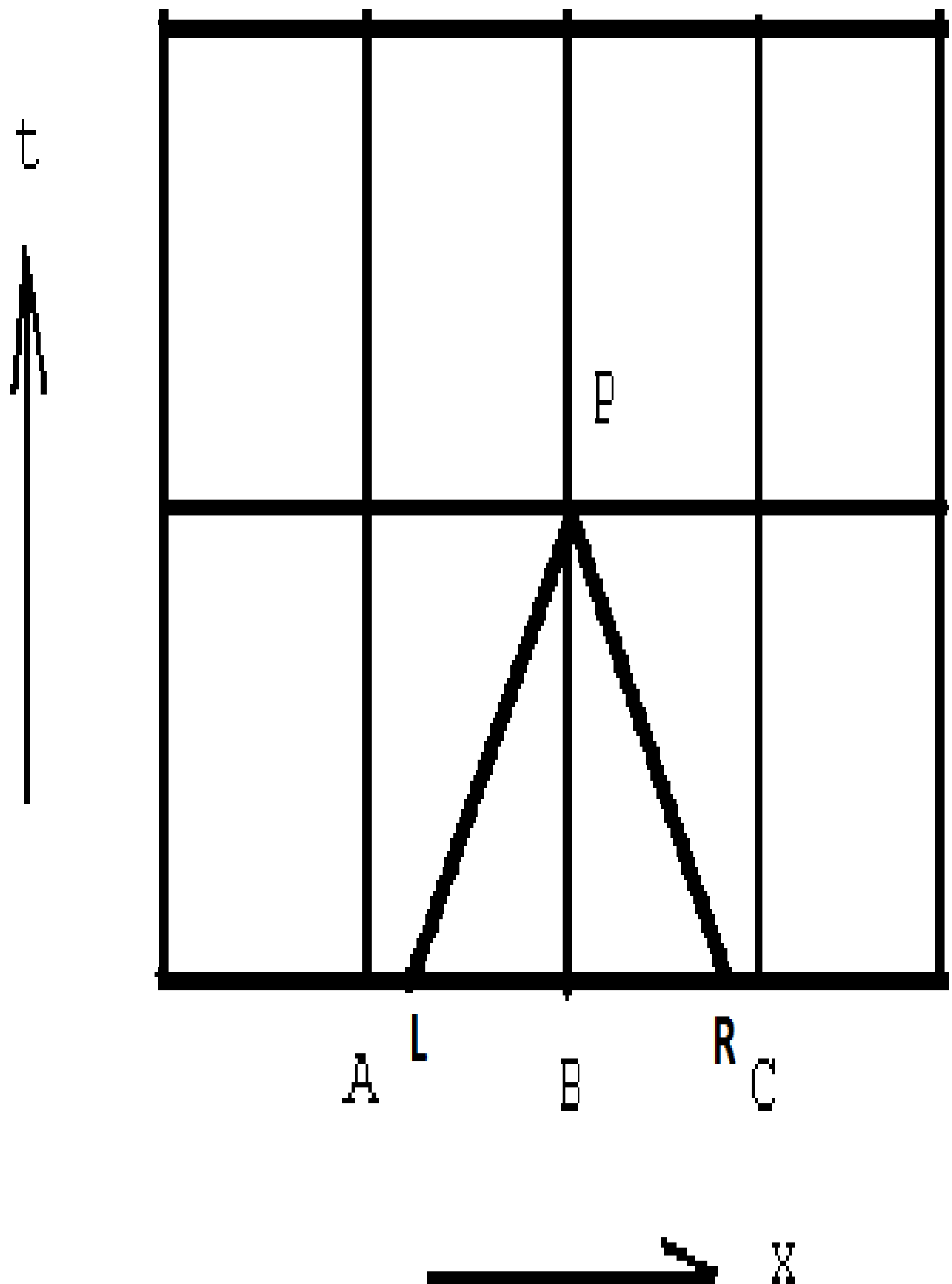
$$U_L = U_A + (x_L-x_A)/(x_B-x_A)\ (U_B-U_A)$$

$$U_R = U_C + (x_R-x_C)/(x_B-x_C)\ (U_B-U_C)$$

$$P_L = P_A + (x_L-x_A)/(x_B-x_A)\ (P_B-P_A)$$

$$P_R = P_C + (x_R-x_C)/(x_B-x_C)\ (P_B-P_C)$$

At each end of the pipe, a boundary condition relates the P_P and U_P there. A finite difference equation also relates the P_P and U_P there. So, one can solve for the P_P and U_P there.

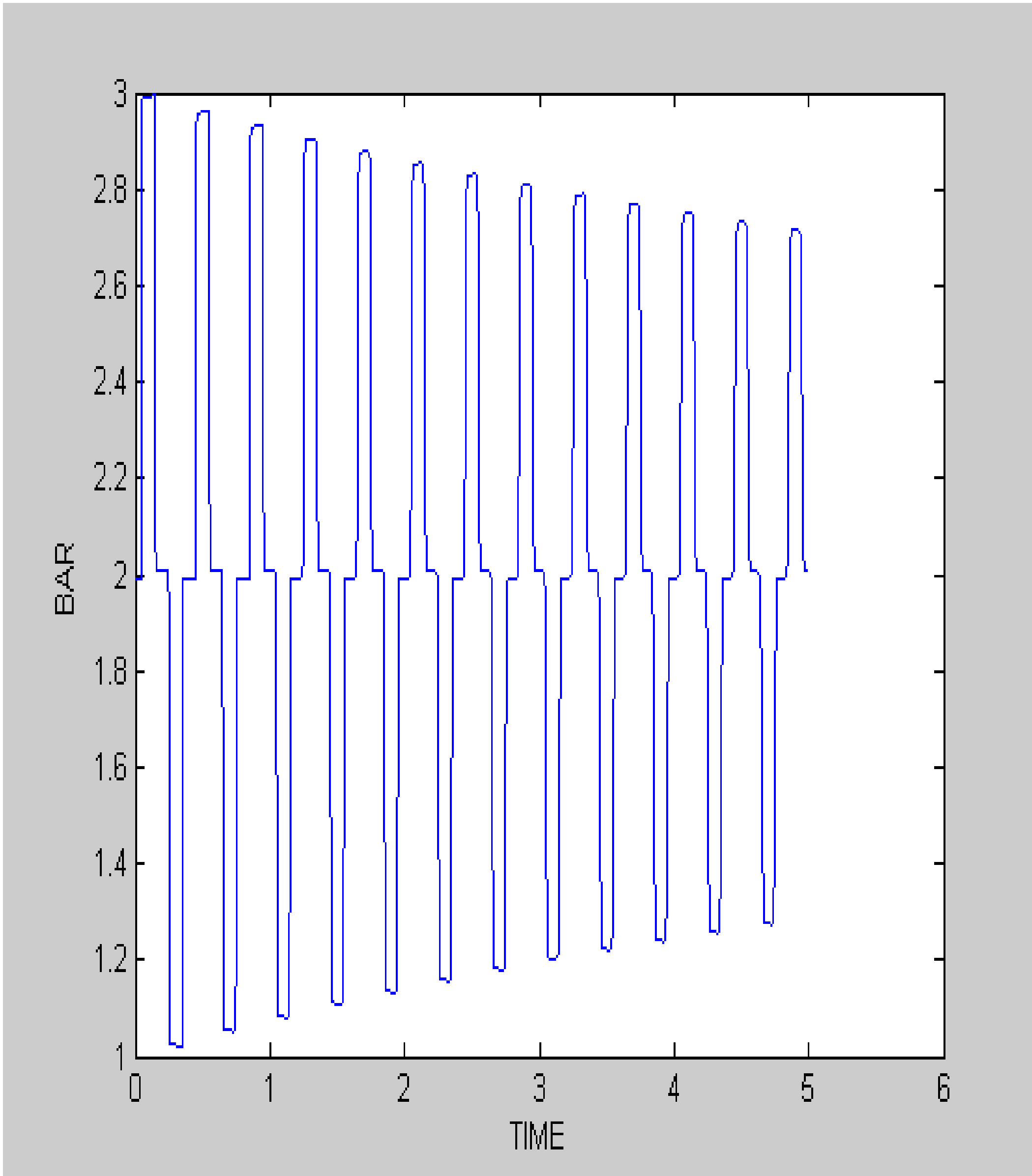


t
P
A
L
B
R
C
x

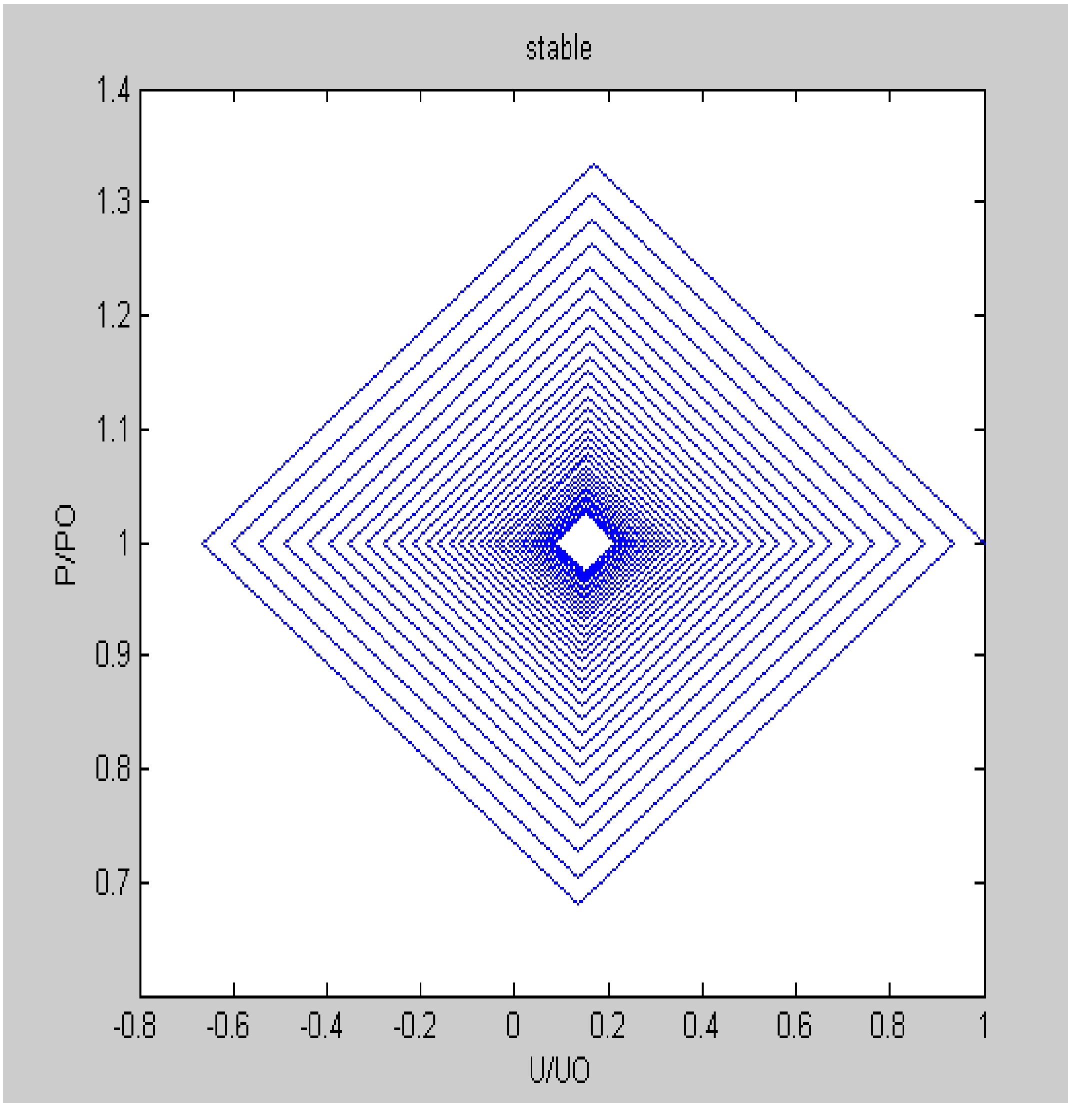
```
%     UNSTEADY FLOW IN A PIPE

%     METHOD OF CHARACTERISTICS

%     RESERVOIR / PIPE / VALVE

%     PRESSURE = POLD / PNEW
%     VELOCITY = UOLD / UNEW

%     HEAD = RESERVOIR HEAD
%     PIPE = HEAD PRESSURE

%     SLOPE = VALVE SLOPE

%     OD = PIPE DIAMETER
%     OL = PIPE LENGTH
%     CF = FRICTION FACTOR

%     SOUND = SOUND SPEED
%     GRAVITY = GRAVITY
%     DENSITY = DENSITY

%     NIT = NUMBER OF TIME STEPS
%     MIT = NUMBER OF PIPE NODES
%     DELT = STEP IN TIME

%     DATA
      DELT=0.001;
      CF=0.5;
      CMAX=+10.0;
      CMIN=0.0;
      OD=0.15;OL=100.0;
      SOUND=1000.0;
      GRAVITY=10.0;
      DENSITY=1000.0;
      SLOPE=-100000.0;
      HEAD=20.0;SPEED=0.1;
      NIT=5000;MIT=100;KIT=1;
      PIPE=HEAD*DENSITY*GRAVITY;
%
      ONE=PIPE;
      TWO=0.0;
      ZERO=0.0;
      BIT=MIT/2;
      GIT=MIT-1;
      DELX=OL/(MIT-1);
      FLD=CF*OL/OD;
      PMAX=CMAX*PIPE;
      PMIN=CMIN*PIPE;
      WAY=SPEED*SPEED/2.0;
```

```
      LOSS=FLD*WAY/GRAVITY;
      G=LOSS*DENSITY*GRAVITY;
      DELP=G/GIT;
      for IM=1:MIT
      POLD(IM)=ONE;
      UOLD(IM)=SPEED;
      X(IM)=TWO;
      ONE=ONE-DELP;
      TWO=TWO+DELX;
      end
      PV=POLD(MIT);
      UV=UOLD(MIT);

%     START LOOP ON TIME
      TIME=0.0;
      for IT=1:NIT
      TIME=TIME+DELT;
      T(IT)=TIME;
%     POINTS INSIDE PIPE
      for IM=2:MIT-1
      XA=X(IM-1);
      XB=X(IM);
      XC=X(IM+1);
      PA=POLD(IM-1);
      PB=POLD(IM);
      PC=POLD(IM+1);
      UA=UOLD(IM-1);
      UB=UOLD(IM);
      UC=UOLD(IM+1);
      XL=XB-(UB+SOUND)*DELT;
      XR=XB-(UB-SOUND)*DELT;
      UL=UA+(XL-XA)/(XB-XA)*(UB-UA);
      PL=PA+(XL-XA)/(XB-XA)*(PB-PA);
      UR=UC+(XR-XC)/(XB-XC)*(UB-UC);
      PR=PC+(XR-XC)/(XB-XC)*(PB-PC);
      UNEW(IM)=0.5*(UL+UR+(PL-PR)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*(UL*abs(UL)+UR*abs(UR))));
      PNEW(IM)=0.5*(PL+PR+(UL-UR)*DENSITY*SOUND-DENSITY ...
       *SOUND*CF/2.0/OD*DELT*(UL*abs(UL)-UR*abs(UR)));
      end
%     DOWNSTREAM END OF PIPE
      if(KIT==1) UNEW(MIT)=ZERO;end;
      if(KIT==2) UNEW(MIT)=UV ...
      +(POLD(MIT)-PV)/SLOPE;end;
      if(UNEW(MIT)<=ZERO) ...
      UNEW(MIT)=ZERO;end;
      XA=X(MIT-1);
      XB=X(MIT);
      PA=POLD(MIT-1);
      PB=POLD(MIT);
      UA=UOLD(MIT-1);
      UB=UOLD(MIT);
```

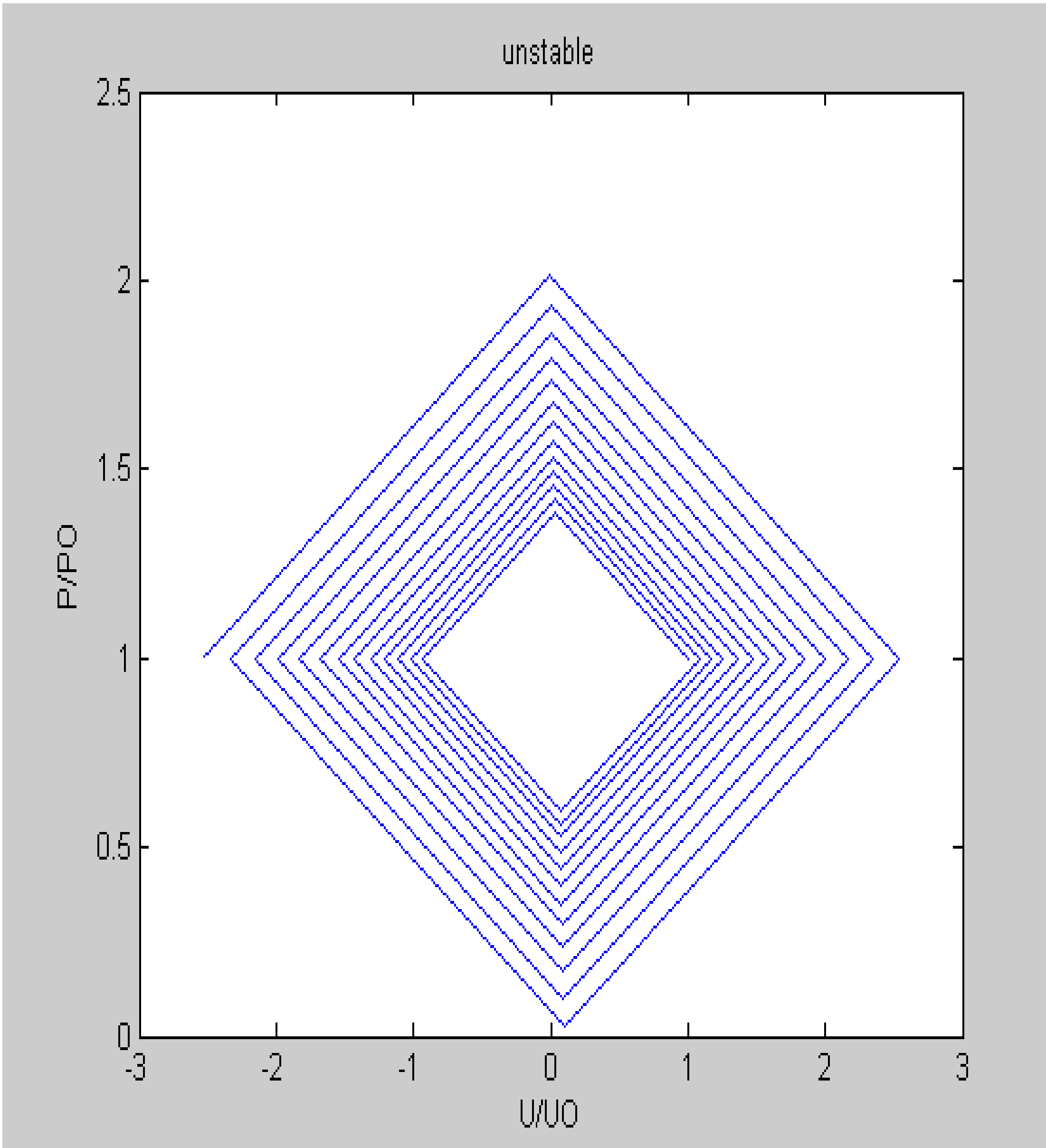
```
      XL=XB-(UB+SOUND)*DELT;
      UL=UA+(XL-XA)/(XB-XA)*(UB-UA);
      PL=PA+(XL-XA)/(XB-XA)*(PB-PA);
      PNEW(MIT)=PL-(UNEW(MIT)-UL)*DENSITY*SOUND ...
      -DELT*DENSITY*SOUND*(CF/2.0/OD*UL*abs(UL));
      if(PNEW(MIT)<=PMIN) PNEW(MIT)=PMIN;end;
      if(PNEW(MIT)>=PMAX) PNEW(MIT)=PMAX;end;
      if(PNEW(MIT)==PMAX | PNEW(MIT)==PMIN) ...
      UNEW(MIT)=UL-(PNEW(MIT)-PL)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*UL*abs(UL));end;
%     UPSTREAM END OF PIPE
      XB=X(1);
      XC=X(2);
      PB=POLD(1);
      PC=POLD(2);
      UB=UOLD(1);
      UC=UOLD(2);
      XR=XB-(UB-SOUND)*DELT;
      UR=UC+(XR-XC)/(XB-XC)*(UB-UC);
      PR=PC+(XR-XC)/(XB-XC)*(PB-PC);
      PNEW(1)=PIPE;
      UNEW(1)=UR+(PNEW(1)-PR)/DENSITY/SOUND ...
      -DELT*(CF/2.0/OD*UR*abs(UR));
%     STORING P AND U
      for IM=1:MIT
      POLD(IM)=PNEW(IM);
      UOLD(IM)=UNEW(IM);
      if (IM==BIT) PIT(IT)=PNEW(IM); ...
      HIT(IT)=PIT(IT)/DENSITY/GRAVITY; ...
      BAR(IT)=HIT(IT)/10.0; ...
      UIT(IT)=UNEW(IM);end;
      end
%     END OF TIME LOOP
      end
%

      plot(T,UIT)
      plot(UIT,HIT)
      plot(UIT,BAR)
      plot(UIT,PIT)
      plot(T,PIT)
      plot(T,BAR)
      xlabel('TIME')
      ylabel('BAR')
```



SUDDEN VALVE CLOSURE



STABLE LEAKY VALVE



UNSTABLE LEAKY VALVE

REACHES WITH FRICTION

Pipes in a pipe network often have different lengths. The method of reaches divides the pipes into segments that have the same transit time. The segments are known as reaches. The sketch on the next page shows a pipe divided into 4 reaches. Conditions at points i j k are known. Conditions at point J are unknown. Waterhammer analysis gives for point J:

$$[\rho a]\ dU/dt + dP/dt + [\rho a]C = 0$$

$$P_J - P_i = -\ [\rho a][U_J - U_i] - \Delta t\ [\rho a]C_i$$

$$[\rho a]\ dU/dt - dP/dt + [\rho a]C = 0$$

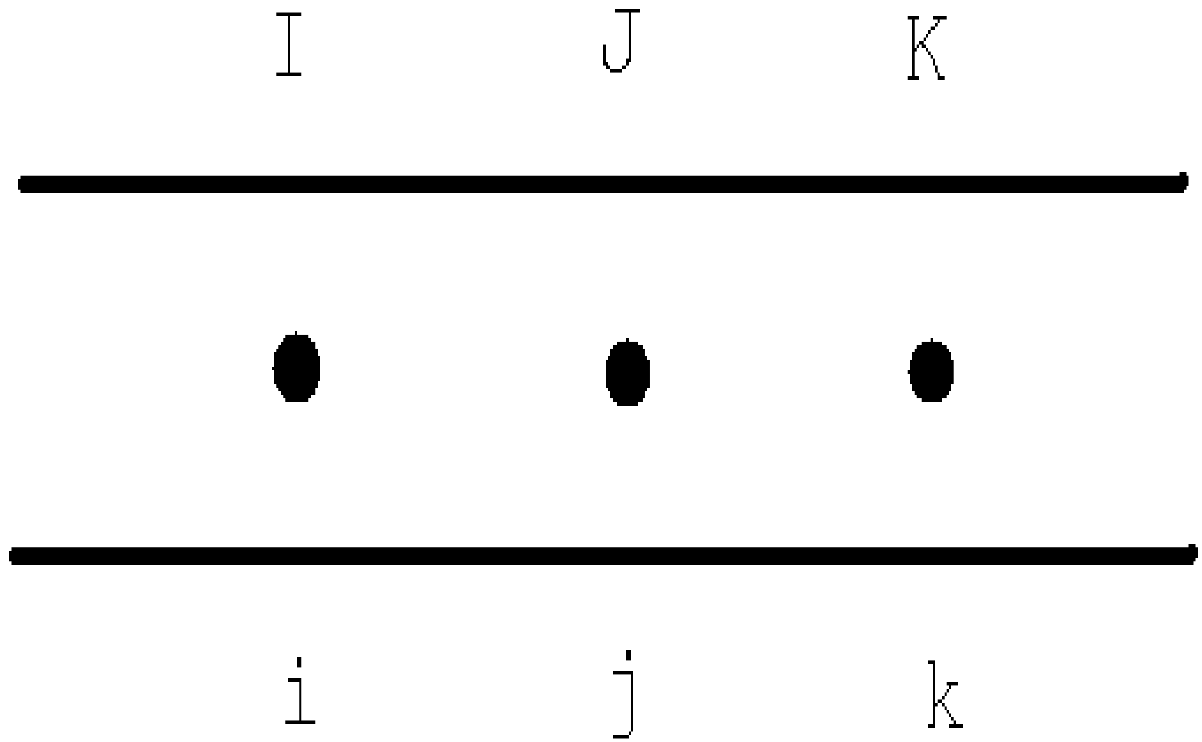
$$P_J - P_k = +\ [\rho a][U_J - U_k] + \Delta t\ [\rho a]C_k$$

Manipulation of these equations gives:

$$P_J = (P_k + P_i)/2 - [\rho a][U_k - U_i]/2 + \Delta t\ [\rho a][C_k - C_i]/2$$

$$U_J = (U_k + U_i)/2 - [P_k - P_i]/[2\rho a] - \Delta t\ [C_k + C_i]/2$$

This is the template for finding conditions at points inside the pipe. At the ends of a pipe, water hammer analysis would connect the end points to j points inside the pipe.


I
J
K
i
j
k

JUNCTIONS WITH FRICTION

Pipes in a pipe network are connected at junctions. The sketch on the next page shows a junction which connects 3 pipes. Lower case letters indicate known conditions. Upper case letters indicate unknown conditions. A junction is often small. This allows us to assume that the junction pressure is common to all pipes. It also allows us to assume that the net flow into or out of the junction is zero. Conservation of Mass considerations give:

$$+ \rho A_N U_N + \rho A_H U_H + \rho A_W U_W = 0$$

Waterhammer analysis gives:

$$P_N - P_m = + [\rho a_N][U_N - U_m] + \Delta t \ [\rho a]C_m$$

$$P_H - P_g = + [\rho a_H][U_H - U_g] + \Delta t \ [\rho a]C_g$$

$$P_W - P_v = + [\rho a_W][U_W - U_v] + \Delta t \ [\rho a]C_v$$

Manipulation gives

$$U_N = U_m + [P_N - P_m]/[\rho a_N] - \Delta t \ C_m$$

$$U_H = U_g + [P_H - P_g]/[\rho a_H] - \Delta t \ C_g$$

$$U_W = U_v + [P_W - P_v]/[\rho a_W] - \Delta t \ C_v$$

In these equations $P_N = P_H = P_W = P_J$. Substitution into Conservation of Mass gives:

$$+ \rho A_N [U_m + [P_J-P_m]/[\rho a_N] - \Delta t C_m]$$

$$+ \rho A_H [U_g + [P_J-P_g]/[\rho a_H] - \Delta t C_g]$$

$$+ \rho A_W [U_v + [P_J-P_v]/[\rho a_W] - \Delta t C_v] = 0$$

Manipulation gives the junction pressure:

$$P_J = [X - Y] / Z$$

$$X = [+ A_N/a_N P_m + A_H/a_H P_g + A_W/a_W P_v]$$

$$Y = \rho [+ A_N[U_m-\Delta t C_m] + A_H[U_g-\Delta t C_g] + A_W[U_v-\Delta t C_v]]$$

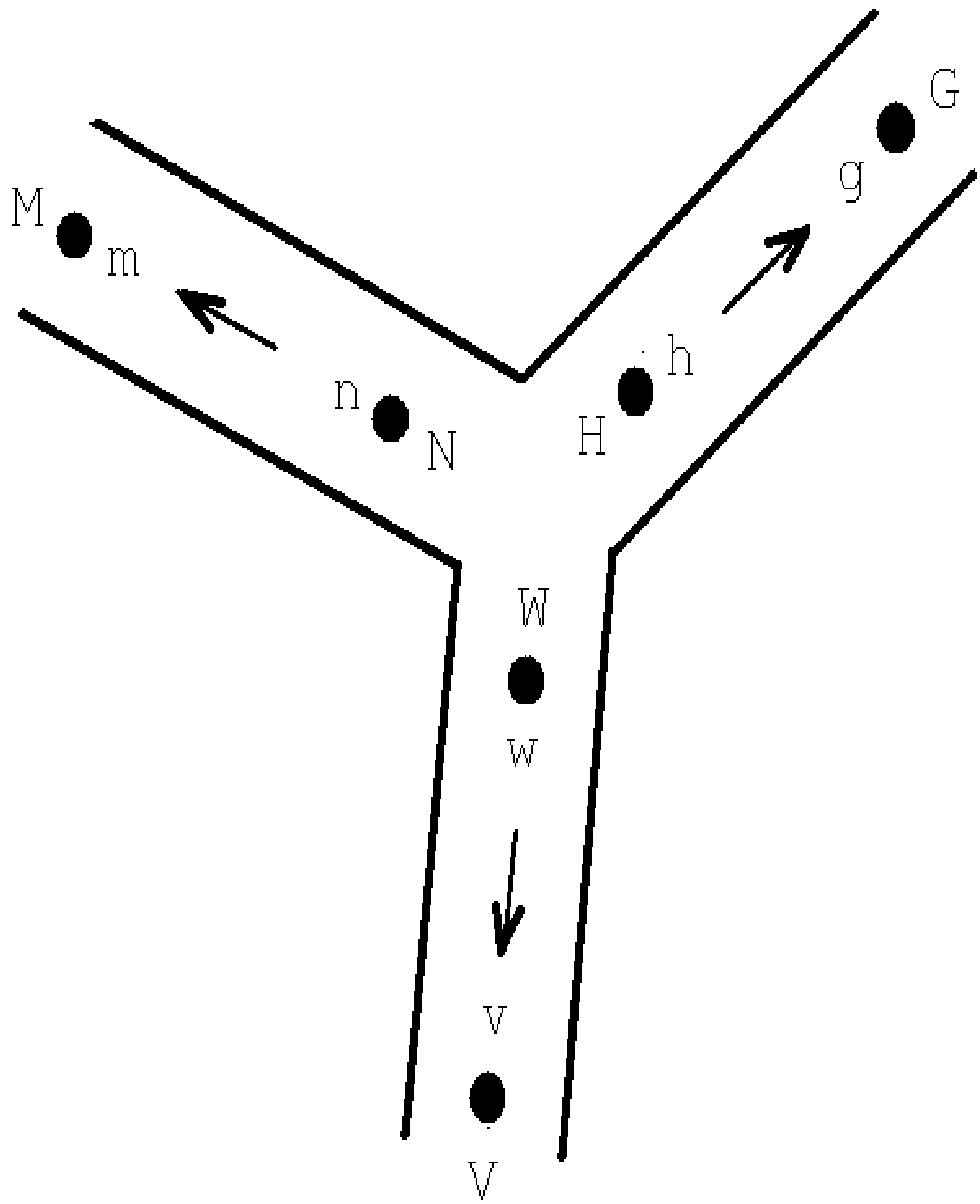
$$Z = [+ A_N/a_N + A_H/a_H + A_W/a_W]$$

The velocities at the junction are:

$$U_N = U_m + [P_J-P_m]/[\rho a_N] - \Delta t C_m$$

$$U_H = U_g + [P_J-P_g]/[\rho a_H] - \Delta t C_g$$

$$U_W = U_v + [P_J-P_v]/[\rho a_W] - \Delta t C_v$$

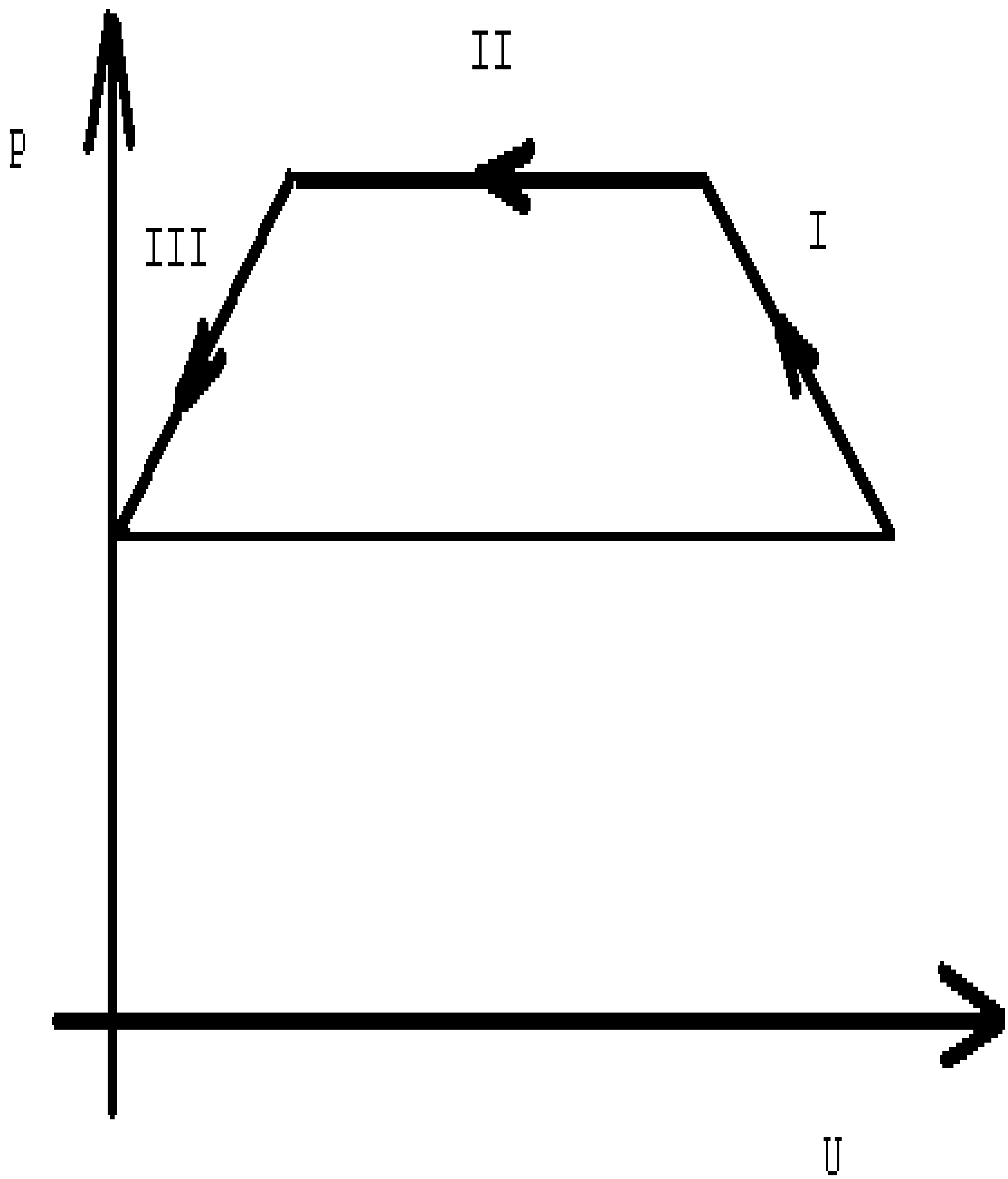

M
m
G
g
n
N
H
h
W
w
v
V

THREE PHASE VALVE STROKING

Three phase valve stroking is a process where a valve is opened or closed very fast in such a way that pressures are kept within preset limits and no waves are left at the end. It is described below for a complete closure case.

In phase I the valve is moved in such a way that the pressure at the valve rises linearly in time from P_{LOW} to P_{HIGH} in 2T pipe transit times. At the end of phase I the pressure variation along the pipe is linear and the velocity everywhere because of a combination of pressure surges and back flows has been reduced by $\Delta P/[\rho a]$ where ΔP is P_{HIGH} minus P_{LOW}. In phase II the valve is moved in such a way that the pressure variation along the pipe stays constant and the velocity drops by $2\Delta P/[\rho a]$ everywhere every 2T transit times. The pressure variation remains constant because pressure surges generated by valve motion are cancelled by suction waves at the valve caused by back flows. The constant pressure variation causes a constant deceleration of the fluid in the pipe. Phase III takes 2T pipe transit times to complete. During this time the velocity everywhere drops $\Delta P/[\rho a]$ and pressure falls linearly at the valve from P_{HIGH} to P_{LOW}. The valve is moved in such a way that suction waves at the valve caused by back flows are allowed to bring the pressure down again to P_{LOW}. Because phases I and III reduce the velocity by a total of $2\Delta P/[\rho a]$ phase II must take $(U-2\Delta P/[\rho a])/(2\Delta P/[\rho a])$ 2T seconds to complete. One can calculate what the valve area should be at each instant in time during stroking. A fast acting feedback control system can then be used to move the valve in the desired manner.

Phase I sets up conditions in the pipe for phase II. Similarly, phase II sets up conditions in the pipe for phase III. In phase II, the pressure surge rate is twice that of phases I and III. In a set period of time, one pressure surge maintains a backflow that would have otherwise been stopped by a suction wave. The other pressure surge balances a pressure release. There are no suction waves in phase II and all backflows are maintained. Every point in the pipe has a velocity reduction due to a surge wave and one due to a backflow. In phase III, the pressure surge rate is cut in half. This allows suction waves to form at the valve. These propagate up the pipe and eliminate backflows. Conditions in the pipe are controlled by these waves and by waves already there from phase II. During the first half of phase III, conditions in the pipe are still under the influence of phase II. Velocity falls faster at the reservoir than at the valve because of this. Half way through phase III, there is a linear pressure variation and a linear velocity variation along the pipe. During the second half of phase III, a wave travels down the pipe which brings the pressure back to P_{LOW} everywhere and the velocity to zero everywhere.


II
P
III
I
U