FLUID STRUCTURE INTERACTIONS

FLOW INDUCED VIBRATIONS

OF STRUCTURES

PREAMBLE

There are two types of vibrations: resonance and instability. Resonance occurs when a structure is excited at a natural frequency. When damping is low, the structure is able to absorb energy each oscillation cycle and dangerous amplitudes can build up. There are two types of instability: static and dynamic. Static instability occurs when a negative fluid stiffness overcomes a positive structural stiffness. Usually, because of nonlinearity, this instability is oscillatory: oscillations are often referred to as relaxation oscillations. Examples are wing stall flutter and gate valve vibration. Dynamic instability occurs when a negative fluid damping overcomes a positive structural damping. Examples include galloping of slender structures and tube bundle vibrations. In many cases, a system oscillates at a structural natural frequency. In these cases, frequency is a parameter in a semi empirical critical speed equation. Natural frequencies depend on the inertia of the structure and its stiffness. Usually the damping of the structure is ignored. It usually has only a small influence on periods. If the structure has a heavy fluid surrounding it, some of the fluid mass must be considered part of the structure. The structure appears more massive than it really is. For a simple discrete mass stiffness system, there is only one natural period. For distributed mass/stiffness systems, like wires and beams, there are an infinite number of natural periods. For each period, there is a mode shape. This shows the level of vibration at points along the structure. Structural frequencies can be obtained analytically for discrete mass/spring systems and for uniform wires and beams. For complex structures, they can be obtained using approximate procedures like the Galerkin Method of Weighted Residuals. In some cases, the fluid structure interaction is so complex that vibration frequencies depend on both the structure and the fluid. Examples include flutter of wings and panels and pipe whip due to internal flow.

These notes start with a description of some flow induced vibrations of slender structures. Next vibration of lifting bodies like wings and propellors is considered. Then vibration of panels exposed to flow is discussed. Finally, vibration in pipe networks is considered.

FLOW INDUCED VIBRATION OF SLENDER STRUCTURES

VORTEX INDUCED RESONANCE

Vortices shed from most slender structures in an asymmetric pattern. The shedding causes a lateral vibration of the structure. When the vortex shedding frequency is close to a natural frequency of the structure, the structure undergoes resonance. Once the structure begins to oscillate, it causes a phenomenon known as lock in. The vortices shed at the natural frequency. In other words, the structure motion controls the vortex shedding. It also increases the correlation length along the span. This means that vortex shedding along the span occurs at the same time. This gives rise to greater lateral loads. So, once shedding starts, it quickly amplifies motion.

VORTEX INDUCED INSTABILITY

Beyond a certain critical flow speed, a shear layer that has separated from a structure can reattach and create a very strong attached vortex. This occurs only for certain shapes. When such a shape is moving laterally in a flow, the attached vortex pulls it even more laterally! The phenomenon is known as galloping. The structure moves until its stiffness stops it. The vortex disappears and the structure starts moving back the other way. As it does so, the vortex appears on the other side of the structure which pulls it the other way. Another type of galloping is known as wake galloping. This is an oval shaped orbit motion of a cylindrical structure in the outer wake of another structure which is just upstream.

WAKE BREATHING OF A CYLINDER IN A FLOW

There are two modes of wake breathing. In the first mode, the Reynolds Number is near the point where the boundary layer becomes turbulent and the wake becomes smaller. When the cylinder moves upstream into such a flow, its drag drops, whereas when it moves downstream away from such a flow, its drag rises. This promotes a streamwise vibration of the cylinder. In the second mode of wake breathing, when the cylinder moves into a wake, added mass phenomena cause the wake to grow, whereas when the cylinder moves away from the wake, it causes it to shrink. This promotes a streamwise vibration of the cylinder.

FLOW INDUCED VIBRATIONS OF TUBE BUNDLES

There are three mechanisms that can cause tube bundles in a flow to vibrate. One is known as the displacement mechanism. As tubes move relative to each other, some passageways narrow while others widen. Fluid speeds up in narrowed passageways and slows down in widened passageways. Bernoulli shows that in the narrowed passageways pressure decreases while in the widened passageways it increases. Common sense would suggest that if tube stiffness and damping are low, at some point as flow increases, tubes must flutter or vibrate. The displacement mechanism has one serious drawback. It predicts that a single flexible tube in an otherwise rigid bundle cannot flutter but it can undergo a nonlinear oscillation called divergence. It is known from experiments that a single flexible tube in an otherwise rigid bundle can flutter. Another mechanism known as the velocity mechanism does predict flutter in the single flexible tube case. This mechanism is based on the idea that, when a tube is moving, the fluid force on it due its motion lags behind the motion because the upstream flow which influences the force needs time to redistribute. This time lag introduces a negative damping which can overcome the positive damping due to structural and viscous phenomena. The time lag is roughly the tube spacing divided by the flow speed within the bundle. Details of this model are beyond the scope of this note. The third mechanism for tube vibration involves vortex shedding and turbulence within the bundle.

CRITICAL SPEED EQUATIONS

For a slender structure, the Strouhal Number S is the transit time T divided by the vortex shedding period τ: $S=T/\tau$. The transit time T is D/U. Solving for flow speed U gives: $U = D/[S\tau]$. During resonance, $\tau=\mathbf{T}$ where $\mathbf{T}$ is the structural period. So the critical flow speed is:

$$U = D/[S\ \mathbf{T}]$$

For the lateral vibration of a slender structure known as galloping, the critical flow speed U is

$$U = U_\circ\ M/M_\circ\ \zeta\ \mathbf{a} \qquad U_\circ = D/\mathbf{T} \qquad M_\circ = \rho D^2$$

The factor ζ accounts for damping: it is typically in the range 0.01 to 0.1. The parameter **a** accounts for the shape of the structure. For a square cross section structure **a** is 8 while for a circular cross section structure **a** is ∞.

For tube bundle vibration, the critical flow speed is

$$U = \beta/\mathbf{T}\ \sqrt{[M\delta/\rho]} \qquad U = \beta U_\circ\ \sqrt{[\delta M/M_\circ]}$$

The factor δ accounts for damping, and the parameter β accounts for the bundle geometry. Typically δ is in the range 0.05 to 0.25 while β is in the range 2.5 to 6.0.

VIBRATION MODES OF SIMPLE WIRES AND BEAMS

For a wire under tension free to undergo lateral motion, the governing equation is:

$$\partial/\partial x\ (T\partial Y/\partial x) = M\ \partial^2 Y/\partial t^2$$

where Y is the lateral deflection, T is the tension in the wire, M is its mass per unit length, x is position along the wire and t is time. For a uniform wire with constant M and T, this can be written as the wave equation:

$$a^2\ \partial^2 Y/\partial x^2 = \partial^2 Y/\partial t^2 \qquad a^2 = T/M$$

where a is the wave speed. During steady free vibration of a wire, one can write for each point on the wire:

$$Y = \mathbf{Y}\ \mathrm{Sin}\ \omega t$$

Substitution into the governing equation gives:

$$a^2\ d^2\mathbf{Y}/dx^2 = -\ \omega^2\ \mathbf{Y}$$
$$d^2\mathbf{Y}/dx^2 = -\ \beta^2\ \mathbf{Y} \qquad \beta^2 = \omega^2/a^2$$

A general solution is

$$\mathbf{Y} = \mathbf{Y}_o \sin \beta x$$

For a wire held at both ends, **Y** is zero at both ends. This implies that β must be $n\pi/L$, where n is any positive integer and L is the length of the wire. Substitution into the β^2 equation gives the natural frequencies:

$$\omega_n = n\pi a/L = n\pi/L \sqrt{[T/M]}$$

The corresponding natural periods are:

$$\mathbf{T}_n = 2L/n \sqrt{[M/T]}$$

The natural mode shapes are:

$$\sin [n\pi x/L]$$

For a beam free to undergo lateral motion, the governing equation is

$$- \partial^2/\partial x^2 \ (EI\ \partial^2 Y/\partial x^2) = M\ \partial^2 Y/\partial t^2$$

where E is the beam material Elastic Modulus and I is the section area moment of inertia.

During steady free vibration of a beam, one can write for each point on the beam:

$$Y = \mathbf{Y} \text{ Sin } \omega t$$

Substitution into the equation of motion gives:

$$d^2/dx^2 \ (EI \ d^2\mathbf{Y}/dx^2) = \omega^2 \ M \ \mathbf{Y}$$

For a uniform beam with constant M and EI, this becomes:

$$d^4\mathbf{Y}/dx^4 = \beta^4 \ \mathbf{Y} \qquad \beta^4 = \omega^2 \ M/[EI]$$

The general solution is:

$$\mathbf{Y} = A \text{ Sin}[\beta x] + B \text{ Cos}[\beta x] + C \text{ Sinh}[\beta x] + D \text{ Cosh}[\beta x]$$

where A and B and C and D are constants of integration. These are determined by the boundary conditions.

For a beam with pivot supports, the boundary conditions are zero deflection and zero bending moment at each end. This implies that at each end:

$$\mathbf{Y} = 0 \qquad d^2\mathbf{Y}/dx^2 = 0$$

In this case, the general solution reduces to:

$$\mathbf{Y} = \mathbf{Y}_\circ \text{ Sin } \beta x$$

As for the wire, β must be $n\pi/L$, where n is any positive integer and L is the length of the beam. Substitution into the β^4 equation gives the natural frequencies:

$$\omega_n = [n\pi/L]^2 \sqrt{[EI]/M}$$

The corresponding natural periods are:

$$\mathbf{T}_n = [L/n]^2 \ 2/\pi \ \sqrt{M/[EI]}$$

The natural mode shapes are:

$$\text{Sin}\ [n\pi x/L]$$

For a cantilever beam, the boundary conditions at the wall are zero deflection and zero slope. This implies that

$$\mathbf{Y} = 0 \qquad d\mathbf{Y}/dx = 0$$

Application of these conditions shows that:

$$C = -A \qquad D = -B$$

At the free end of the beam, the bending moment and shear are both zero. This implies that

$$d^2\mathbf{Y}/dx^2 = 0 \qquad d^3\mathbf{Y}/dx^3 = 0$$

Application of these conditions gives

$$[\mathrm{Sin}\beta L+\mathrm{Sinh}\beta L]\ A + [\mathrm{Cos}\beta L+\mathrm{Cosh}\beta L]\ B = 0$$

$$[\mathrm{Cos}\beta L+\mathrm{Cosh}\beta L]\ A - [\mathrm{Sin}\beta L-\mathrm{Sinh}\beta L]\ B = 0$$

Manipulation of these equations gives the β condition:

$$\mathrm{Cos}\beta_n L\ \mathrm{Cosh}\beta_n L + 1 = 0$$

This gives the natural frequencies of the beam. For each frequency, one gets the natural mode shape:

$$(\mathrm{Sin}[\beta_n L] - \mathrm{Sinh}[\beta_n L])\ (\mathrm{Sin}[\beta_n x] - \mathrm{Sinh}[\beta_n x])$$
$$+$$
$$(\mathrm{Cos}[\beta_n L] + \mathrm{Cosh}[\beta_n L])\ (\mathrm{Cos}[\beta_n x] - \mathrm{Cosh}[\beta_n x])$$

The first 3 natural frequencies are:

$$\omega_1 = 3.52/L^2\ \sqrt{[EI]/M}$$
$$\omega_2 = 22.03/L^2\ \sqrt{[EI]/M}$$
$$\omega_3 = 61.70/L^2\ \sqrt{[EI]/M}$$

The corresponding natural periods are:

$$\mathbf{T}_1 = 2\pi L^2/3.52\ \sqrt{M/[EI]}$$
$$\mathbf{T}_2 = 2\pi L^2/22.03\ \sqrt{M/[EI]}$$
$$\mathbf{T}_3 = 2\pi L^2/61.70\ \sqrt{M/[EI]}$$

VIBRATION MODES OF COMPLEX WIRES

The equation governing the lateral motion of a wire is:

$$- \partial/\partial x \ (T\partial Y/\partial x) \ + \ M \ \partial^2 Y/\partial t^2 \ = \ 0$$

In this equation, Y is deflection of the wire from its neutral position, T is its tension, x is location along the wire, M is the mass of the wire and t is time. During steady free vibration of a wire:

$$Y = \mathbf{Y} \ \mathrm{Sin} \ \omega t$$

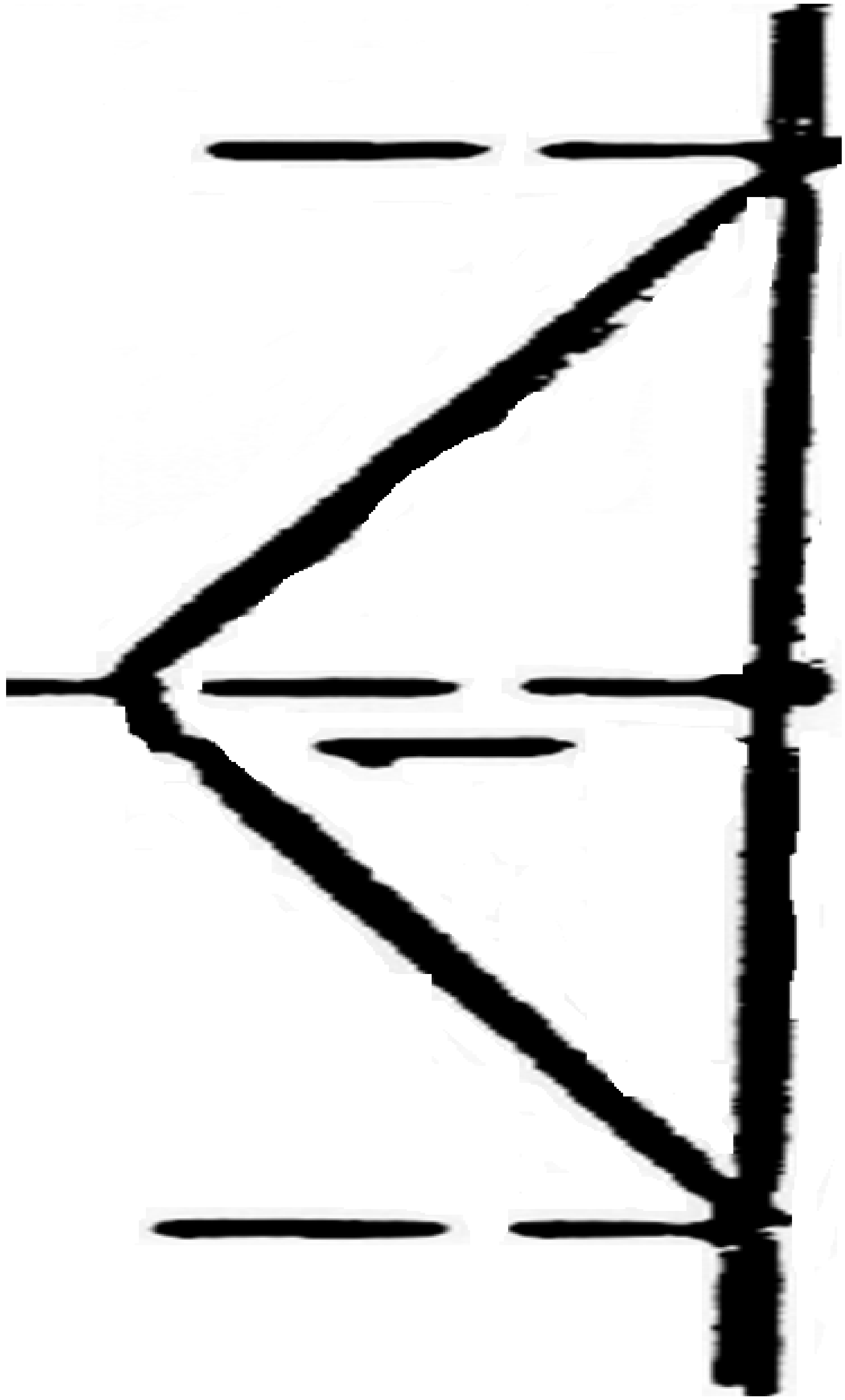
Substitution into the equation of motion gives

$$- d/dx \ (T \ d\mathbf{Y}/dx) \ - \ \omega^2 \ M \ \mathbf{Y} = \ 0$$

For a Galerkin finite element analysis, we assume that deflection along the wire can be given as a sum of scaled shape functions:

$$\mathbf{Y} \ = \ \Sigma \ A \ n$$

where n is deflection at a node and A is a shape function. For shape functions, we use piecewise linear polynomials. The sketch on the next page shows one for a typical node.



Substitution of the assumed form for **Y** into the governing equation gives a residual. In a Galerkin analysis, weighted averages of this residual along the wire are set to zero. After some manipulation, one gets

$$\int_0^L [dW/dx\ T\ d\mathbf{Y}/dx - W\ \omega^2\ M\ \mathbf{Y}]\ dx = 0$$

where L is the length of the wire and W is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon \qquad A_R = 1-\varepsilon$$

where ε is a local coordinate. The subscripts L and R indicate elements immediately to the left and right of the node. Notice the integration by parts of the space derivative term in the integral. This introduces slope end boundary conditions into the formulation. Such boundary conditions are not needed for a wire held at both ends. Application of vibration theory gives the vibration modes of the wire. A computer program was written to do this. For a uniform wire with L=10 and M=10 and T=100, theory gives ω_1=0.993. With 10 elements, Galerkin gives ω_1=0.998.

VIBRATION MODES OF COMPLEX BEAMS

The equation governing the lateral motion of a beam is:

$$\partial^2/\partial x^2 \ (EI\partial^2 Y/\partial x^2) \ + \ M \ \partial^2 Y/\partial t^2 \ = \ 0$$

In this equation, Y is deflection of the beam from its neutral position, EI is its flexural rigidity, x is location along the beam, M is the mass of the beam and t is time. During steady free vibration of a beam:

$$Y = \mathbf{Y} \ \mathrm{Sin} \ \omega t$$

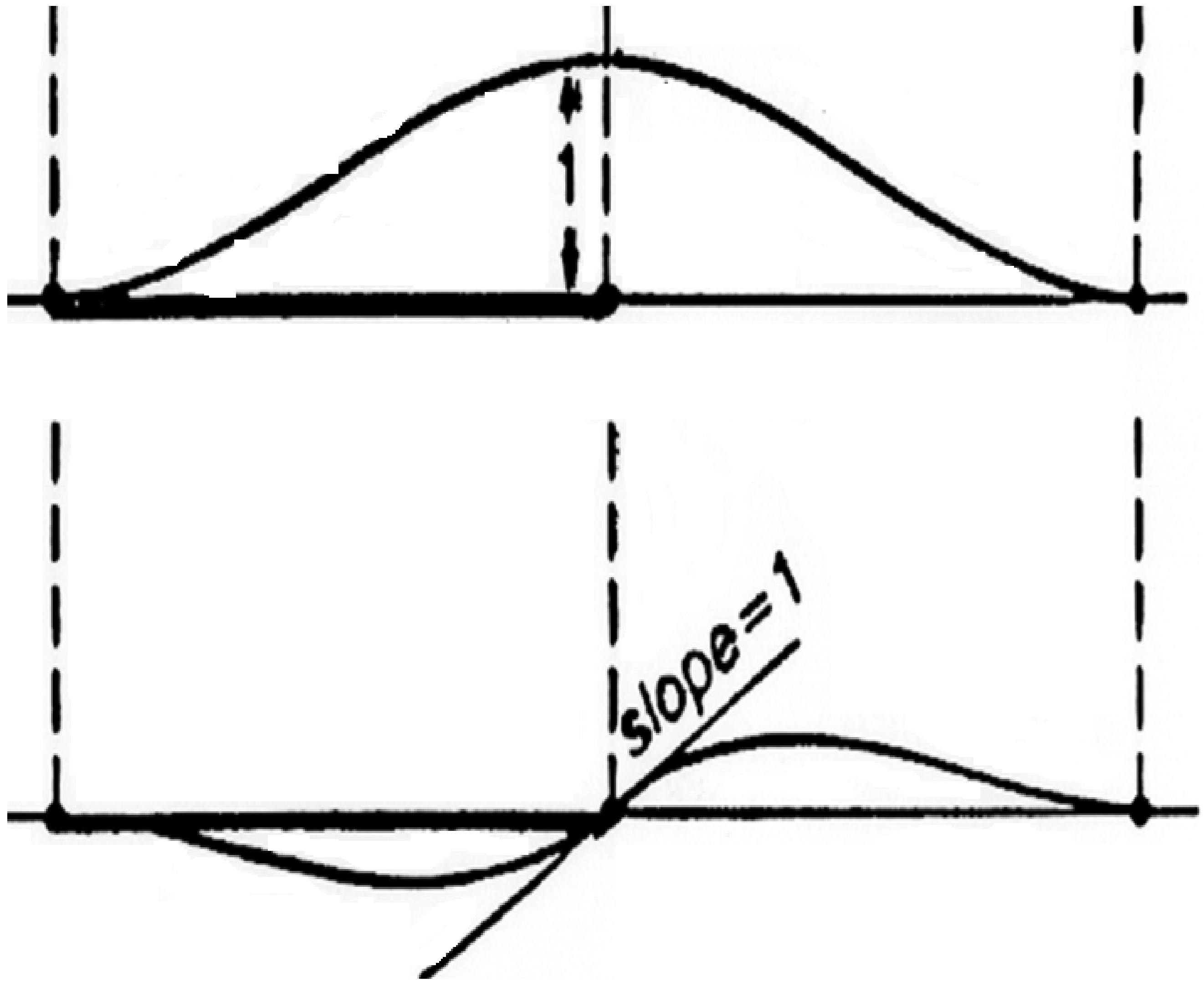
Substitution into the equation of motion gives

$$d^2/dx^2 \ (EI \ d^2\mathbf{Y}/dx^2) \ - \ \omega^2 \ M \ \mathbf{Y} \ = \ 0$$

For a Galerkin finite element analysis, we assume that deflection can be given as a sum of scaled shape functions:

$$\mathbf{Y} \ = \ \Sigma \ [A \ n \ + \ B \ m]$$

where n is the deflection at a node and m is the slope at the node. A and B are shape functions. Theory shows that these must be Hermite polynomials. Such polynomials must be used because the stiffness term is 4^{th} order. The sketch on the next page shows what they look like for a typical node.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slope = 1

Substitution of the assumed form for **Y** into the governing equation gives a residual. In a Galerkin analysis, weighted averages of this residual along the beam are set to zero. After some manipulation, one gets

$$\int_0^L [d^2W/dx^2 \ EI \ d^2\mathbf{Y}/dx^2 - W \ \omega^2 \ M \ \mathbf{Y}] \ dx = 0$$

where L is the length of the beam and W is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of the node. Notice the double integration by parts of the space derivative term in the integral. This introduces tip shear and tip bending moment boundary conditions into the formulation. These are both zero for a cantilever beam. Application of vibration theory gives the vibration modes of the beam. A computer program was written to do this. For a uniform beam with L=1 and M=10 and EI=8.33, theory gives ω_1=3.213. With 10 elements, Galerkin gives ω_1=3.210.

GOVERNING EQUATIONS FOR WIRES AND BEAMS

Sketch A shows a wire under tension. A force balance on a small segment of the wire gives:

$$- T\partial Y/\partial x \quad + \quad [T\partial Y/\partial x + \partial/\partial x \; (T\partial Y/\partial x) \; \Delta x] \quad = \quad M \; \Delta x \; \partial^2 Y/\partial t^2$$

Manipulation gives the equation of motion:

$$\partial/\partial x \; (T\partial Y/\partial x) \quad = \quad M \; \partial^2 Y/\partial t^2$$

Sketches B and C show a beam undergoing bending. A force balance on a small segment of the beam gives:

$$- \mathbf{Q} \quad + \quad (\mathbf{Q} + \partial\mathbf{Q}/\partial x \; \Delta x) \quad = \quad M \; \Delta x \; \partial^2 Y/\partial t^2$$

Manipulation gives:

$$\partial\mathbf{Q}/\partial x \quad = \quad M \; \partial^2 Y/\partial t^2$$

A moment balance on the beam segment gives:

$$- \mathbf{M} \quad + \quad (\mathbf{M} + \partial\mathbf{M}/\partial x \; \Delta x) \quad + \quad (\mathbf{Q} + \partial\mathbf{Q}/\partial x \; \Delta x) \; \Delta x \quad = \quad 0$$

Manipulation gives :

$$\mathbf{Q} \quad = \quad - \; \partial\mathbf{M}/\partial x$$

Sketch D shows how a beam is strained when bent. Inspection of the sketch shows that the strain is:

$$\varepsilon = Y/R$$

The stress is:

$$\sigma = E\ \varepsilon$$

where E is the Elastic Modulus. Geometry gives

$$R\partial\Theta = \partial s \qquad \partial\Theta/\partial s = 1/R$$

$$\partial s = \partial x \qquad \Theta = \partial Y/\partial x$$

Manipulation gives:

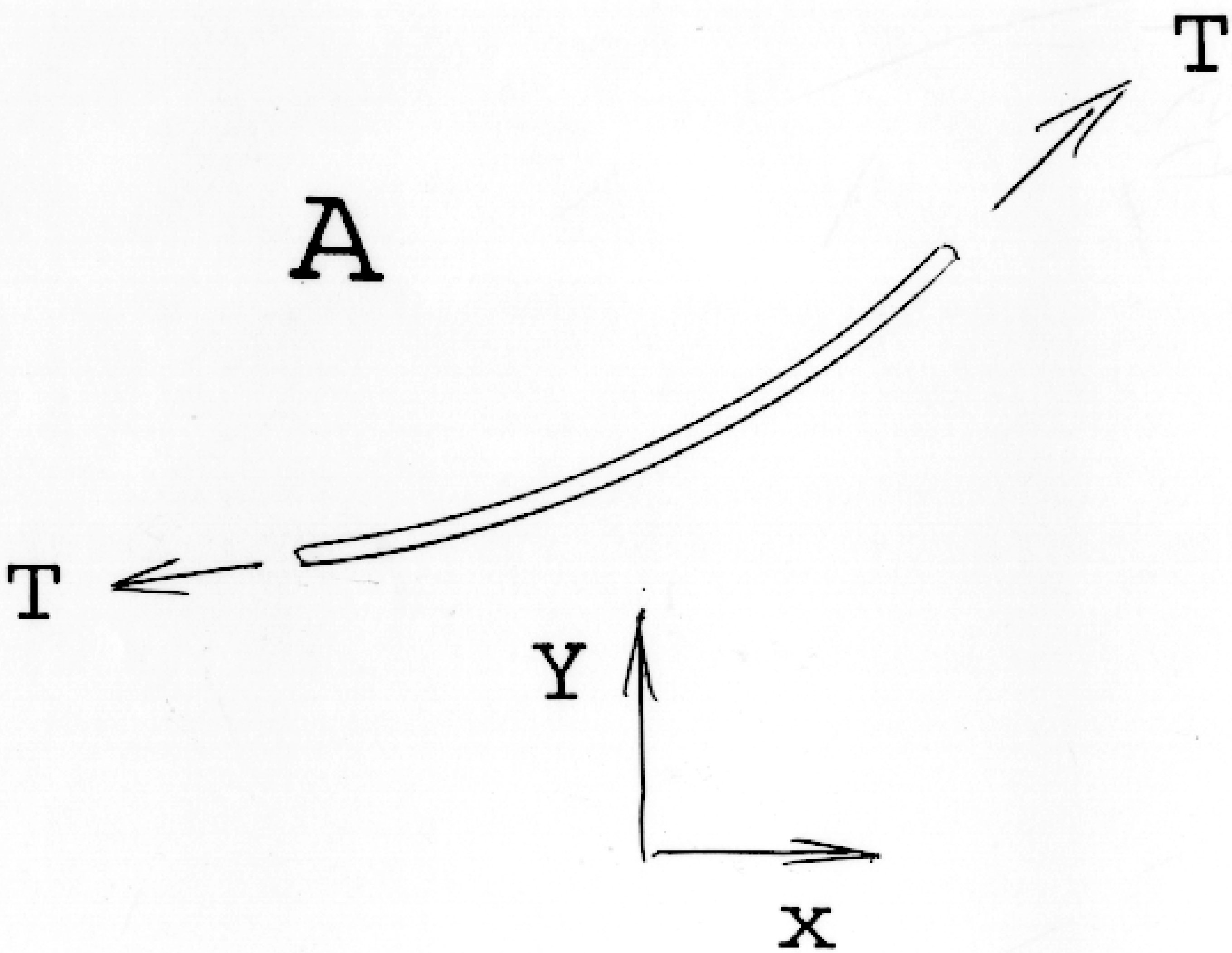
$$\partial^2 Y/\partial x^2 = 1/R$$

Moment considerations give:

$$\mathbf{M} = \int \sigma Y\ dA = E/R \int Y^2\ dA = EI/R = EI\ \partial^2 Y/\partial x^2$$

So, the equation of motion becomes

$$-\ \partial^2/\partial x^2\ \mathbf{M} = -\ \partial^2/\partial x^2\ (EI\ \partial^2 Y/\partial x^2) = M\ \partial^2 Y/\partial t^2$$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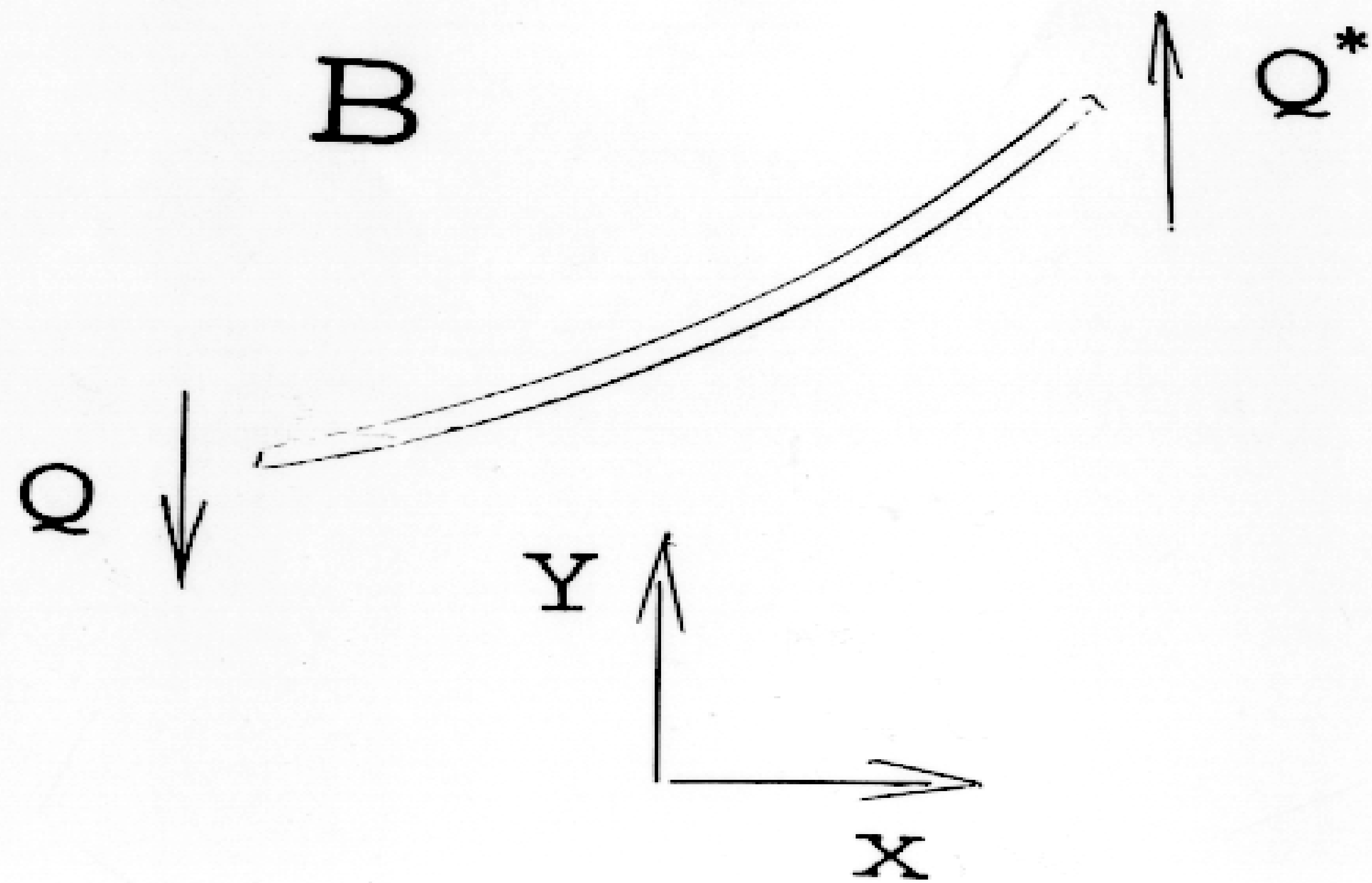

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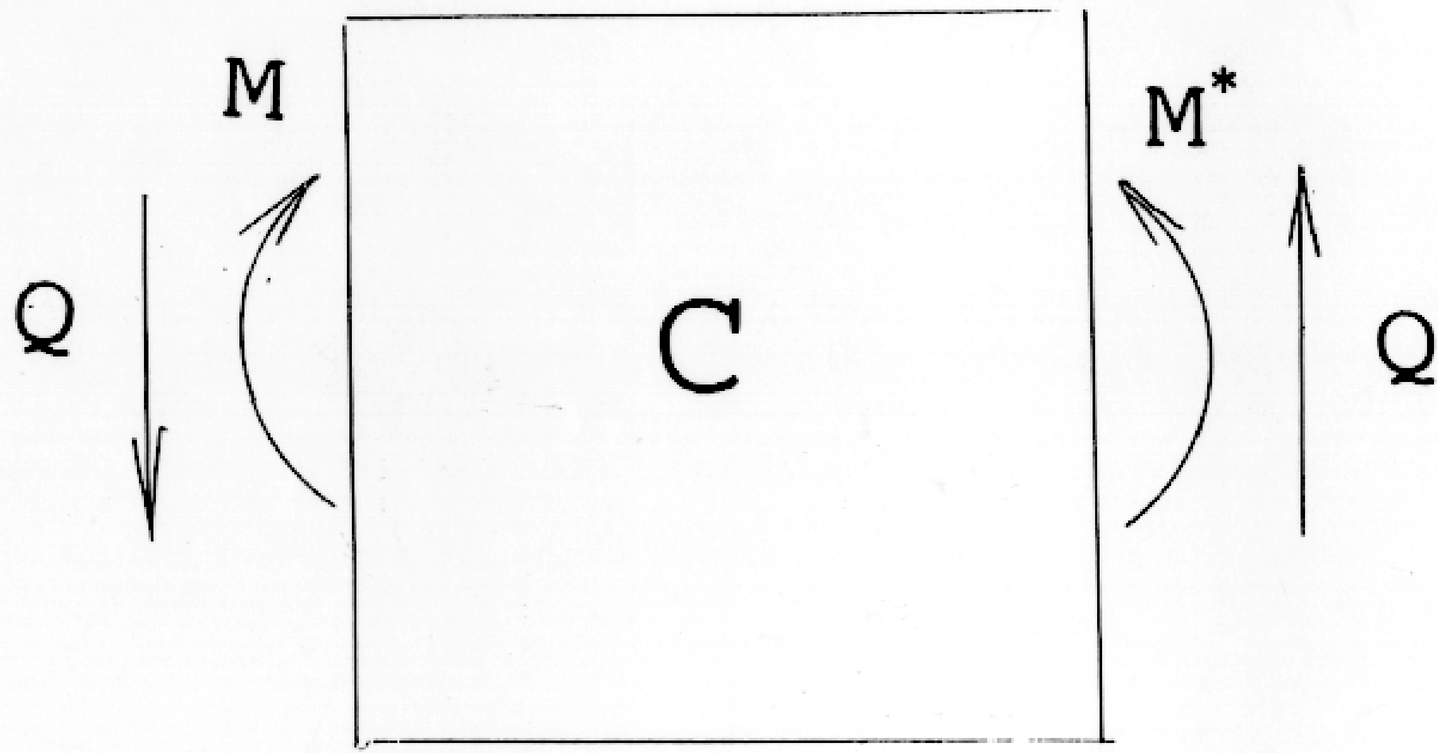
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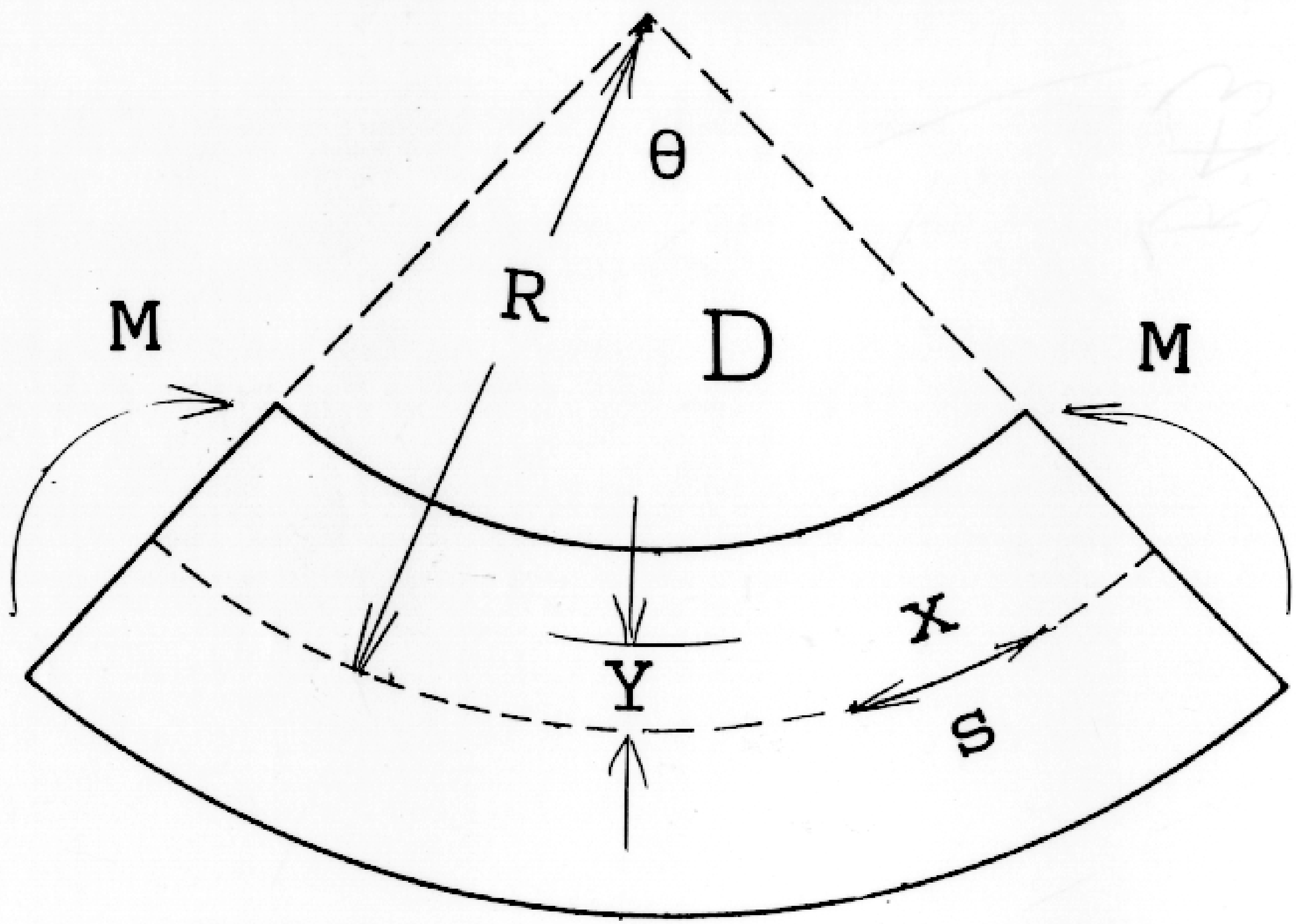
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LIFTING BODY INSTABILITIES

Flutter is a dynamic instability of a lifting body. When it occurs, the heave and pitch motions of the body are 90° out of phase. The passing stream does work on the body over an oscillation cycle. Divergence is a static instability. It occurs when the pitch moment due to fluid dynamics overcomes the moment due to the structural pitch stiffness of the body.

FLUTTER AND DIVERGENCE OF FOILS

A foil is a section of a lifting body. Here quasi steady fluid dynamics theory is used to get the loads on the foil. This ignores the fact that, when a foil is heaving and pitching, vortices are shed behind it because its circulation keeps changing. These vortices influence the loads on the foil. The equations governing motions of a foil are:

$$K h + i\, dh/dt + M\, d^2h/dt^2 + Ma\, d^2\alpha/dt^2 + L = H$$

$$k \alpha + j\, d\alpha/dt + I\, d^2\alpha/dt^2 + Ma\, d^2h/dt^2 + T = P$$

where h is the downward heave displacement of the foil, α is its upward pitch displacement, M is the mass of the foil, I is its rotary inertia, K is the heave stiffness of the foil, k is its pitch stiffness, i is the heave damping coefficient of the foil, j is its pitch damping coefficient, L is the lift on the foil, T is the pitch moment and H and P are disturbance loads. Quasi steady fluid dynamics theory gives for the fluid dynamic loads L and T:

$$L = \rho U^2/2 \; C C_P \; \beta \qquad T = \rho U^2/2 \; C^2 \; \kappa$$

where

$$\beta = \alpha + (dh/dt)/U + (3C/4-b)/U \; (d\alpha/dt)$$

$$\kappa = (C/4-b)/C \; C_P \; \beta + C\pi/[8U] \; (d\alpha/dt)$$

where U is the speed of the foil, C is its chord length, a indicates how far the center of gravity is behind the elastic axis, b is the distance between the elastic axis and the leading edge of the foil and C_P is a constant given by fluid dynamics theory: it is approximately 2π.

Note that the parameter β is the instantaneous angle of attack of the foil 3C/4 back from its leading edge. It is made up of three components. The first component is the pitch angle α. The second component is due to the change in flow direction caused by the heave rate dh/dt. The third component is due to the change in flow direction caused by the pitch rate dα/dt at the 3C/4 location. The 3C/4 location is suggested by flat plate foil theory. Theory shows that the center of pressure on a foil is at C/4 back from the leading edge. This gives rise to the first term in the pitch moment parameter κ. The second term is due to the distribution of pressure over the foil.

One can Laplace Transform the governing equations and manipulate to get a characteristic equation. Stability is dependent on the roots of this equation. One can get the roots numerically and plot them in a Root Locus Plot as a function of foil speed. This would give the critical speed corresponding to the onset of instability.

FLUTTER AND DIVERGENCE OF WINGS

Here strip theory is used to get the loads on a wing. The wing is broken into strips spanwise and quasi steady fluid dynamics theory is used to get the loads on each strip. This ignores the fact that, when a wing is heaving and pitching, vortices are shed behind it because its circulation keeps changing. These vortices influence the loads on the wing. It also ignores the fact that for a finite span wing vortices are shed along its span but mainly at its tips. These vortices create a downwash on the wing. This reduces the lift on the wing because it lowers its apparent angle of attack. It also tilts the load on the wing backwards and this gives rise to a drag. The equations governing heave and pitch motions of a wing are:

$$\partial^2/\partial y^2 \; (\mathbf{EI}\partial^2 h/\partial y^2) \; + \; M \, \partial^2 h/\partial t^2 \; + \; Ma \, \partial^2\alpha/\partial t^2$$
$$+ \; \rho U^2/2 \; CC_P \, \beta \; = \; H$$

$$- \partial/\partial y\ (\mathbf{GJ}\partial\alpha/\partial y) + I\ \partial^2\alpha/\partial t^2 + Ma\ \partial^2 h/\partial t^2$$
$$+ \rho U^2/2\ C^2\ \kappa = P$$

In these equations, h is the downward heave displacement of the wing and α is the upward pitch displacement of the wing. **EI** and **GJ** account for the stiffness of the wing per unit span. M and I are its inertias per unit span. The chord of the wing is C and its span is Q. The speed of the wing is U. The distance from the elastic axis to the center of gravity is a. The distance from the leading edge to the elastic axis is b. H and P are disturbance loads.

Fluid dynamic loads per unit span acting on the wing are determined by the β and κ parameters. These are:

$$\beta = \alpha + (\partial h/\partial t)/U + (3C/4-b)/U\ (\partial\alpha/\partial t)$$
$$\kappa = (C/4-b)/C\ C_P\ \beta + C\pi/8/U\ (\partial\alpha/\partial t)$$

For a Galerkin finite element analysis, we let h and α each be a sum of scaled shape functions as follows:

$$h = \Sigma\ [A\ n + B\ m] \qquad \alpha = \Sigma\ D\ p$$

A and B and D are the shape functions. In the equation for heave, n is the heave at a node while m is the heave slope at a node. In the equation for pitch, p is the pitch at a node. For a typical node, the shape functions are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

$$D_L = \varepsilon \qquad D_R = 1-\varepsilon$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of a node. The polynomials used for heave are known as Hermite polynomials. They must be used because the stiffness term in the heave governing equation is 4^{th} order. They are not needed for pitch because its stiffness term is only 2^{nd} order: linear shape functions are adequate for it.

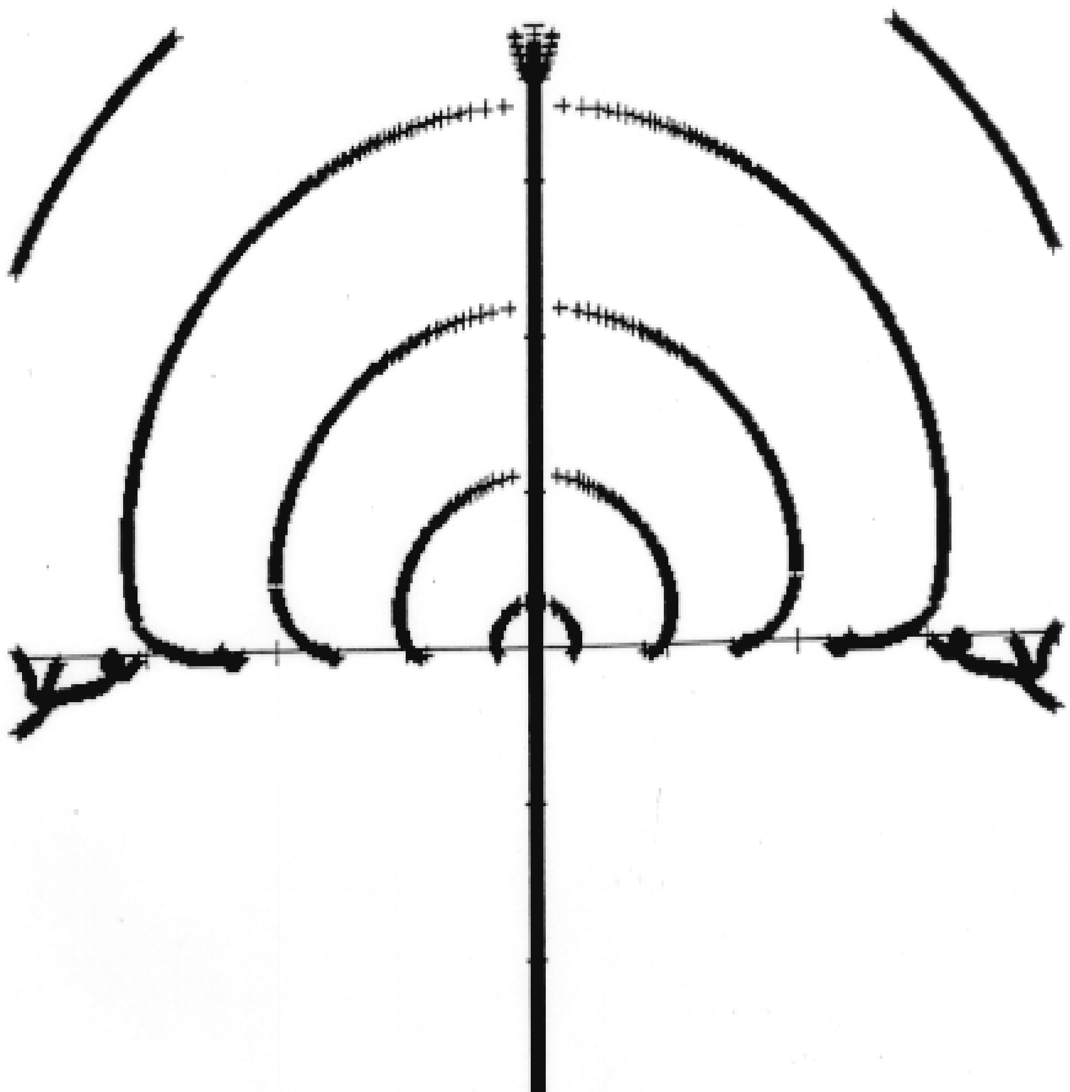
Substitution of the assumed forms for h and α into the governing equations gives residuals. In a Galerkin analysis, weighted averages of these residuals along the span of the wing are set to zero. After some manipulation, one gets

$$\int [\partial^2 W/\partial y^2 \, \mathbf{EI} \, \partial^2 h/\partial y^2 + WM \, \partial^2 h/\partial t^2 + WMa \, \partial^2\alpha/\partial t^2 + W\rho U^2/2 \, CC_P \, \beta - WH] \, dy = 0$$

$$\int [\partial \mathbf{W}/\partial y \, \mathbf{GJ} \, \partial\alpha/\partial y + \mathbf{W}I \, \partial^2\alpha/\partial t^2 + \mathbf{W}Ma \, \partial^2 h/\partial t^2 + \mathbf{W}\rho U^2/2 \, C^2 \, \kappa - \mathbf{W}P] \, dy = 0$$

where W and **W** are weighting functions. For a Galerkin analysis, these are just the shape functions used to define h and α. In other words, W is A and B for each node while **W** is D for each node. Notice the double integration by parts of the space derivative term in the heave integral. This introduces tip shear and tip bending moment boundary conditions into the formulation. Both of these are zero for a wing. Notice the single integration by parts of the space derivative term in the pitch integral. This introduces tip torsion into the formulation. Again this is zero for a wing.

After performing the integrations numerically using Gaussian Quadrature, one gets a set of Ordinary Differential Equations or ODEs in time. One can Laplace Transform these and manipulate to get a characteristic equation. Stability is dependent on the roots of this equation. Instead of using Laplace Transform approach, one can put the ODEs in a matrix form and use matrix manipulation to get the roots of the characteristic equation. One can plot them in a Root Locus Plot as a function of wing speed. This would give the critical speed corresponding to the onset of instability.



KELVIN HELMHOLTZ INSTABILITIES

Consider the flexible panel shown in Figure A. A fluid flowing over such a panel can cause it to flutter. The simplest analysis of this assumes the panel to be an infinitely long thin plate. It also assumes that the flow above and below the panel is potential flow. Conservation of mass considerations give:

$$\nabla^2\varphi_T = 0 \qquad\qquad \nabla^2\varphi_B = 0$$

where T indicates the top flow and B indicates the bottom flow. The kinematic constraints at walls are:

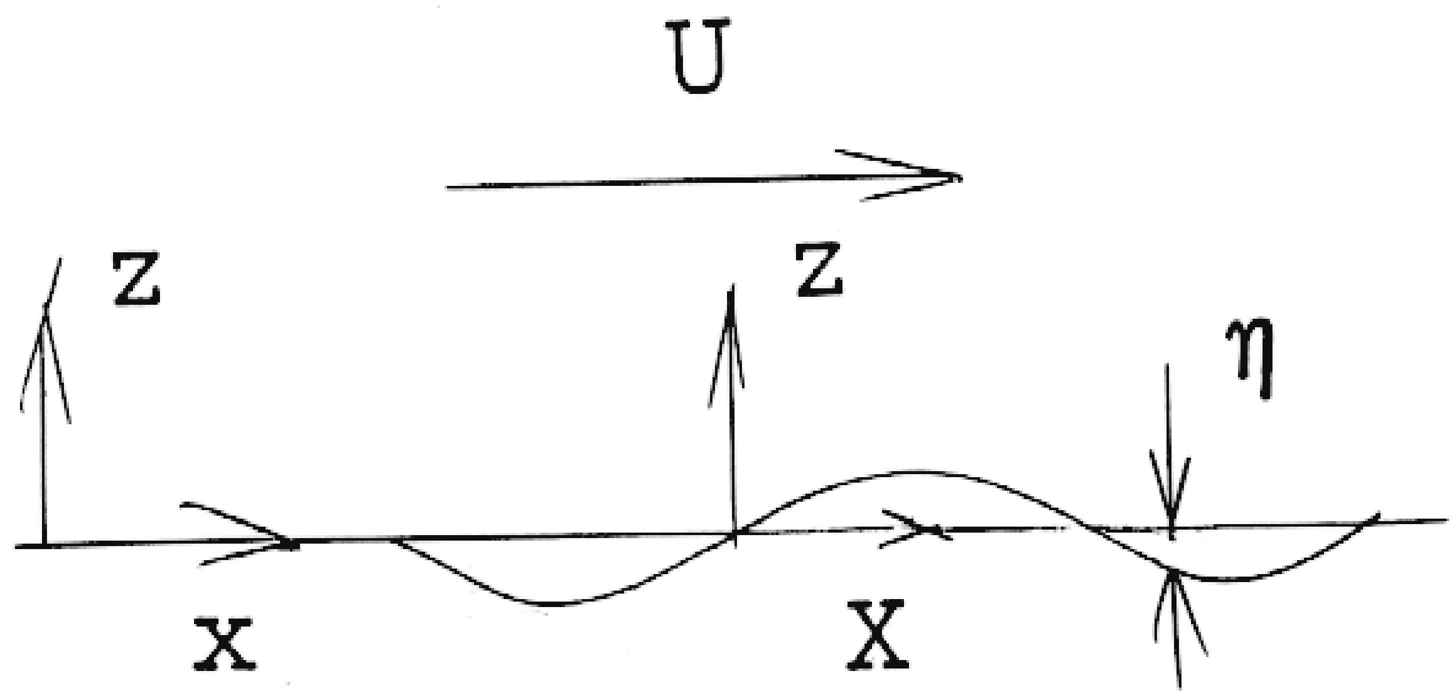
$$\partial\varphi_T/\partial z = 0 \qquad \text{at } z = +d_T$$

$$\partial\varphi_B/\partial z = 0 \qquad \text{at } z = -d_B$$

The panel kinematic constraints are based on:

$$D\eta/Dt = Dz/Dt$$

where η is the vertical deflection of the panel from its rest state. The η for a point on the panel must follow the z for



A

that point. The constraint gives for the top and bottom of the panel:

$$\partial\eta/\partial t + U\,\partial\eta/\partial x = \partial\varphi_T/\partial z \quad \text{at} \quad z = 0$$
$$\partial\eta/\partial t = \partial\varphi_B/\partial z \quad \text{at} \quad z = 0$$

The panel dynamic constraints are:

$$\partial\varphi_T/\partial t + U\,\partial\varphi_T/\partial x + P_T/\rho_T + g\eta = 0 \quad \text{at} \quad z = 0$$
$$\partial\varphi_B/\partial t + P_B/\rho_B + g\eta = 0 \quad \text{at} \quad z = 0$$

Finally, the equation of motion of the panel is:

$$\sigma\, w\, \partial^2\eta/\partial t^2 = (P_B - P_T)\,w - K\,\eta + T\,w\,\partial^2\eta/\partial x^2 - D\,w\,\partial^4\eta/\partial x^4$$

where σ is the sheet density of the panel, w is the panel width, K accounts for side support forces, T is the tension in the panel and D=EI is its flexural rigidity.

The dynamic constraints give:

$$P_T = -\rho_T\,(\partial\varphi_T/\partial t + U\partial\varphi_T/\partial x) - \rho_T\,g\eta \quad \text{at} \quad z = 0$$
$$P_B = -\rho_B\,(\partial\varphi_B/\partial t) - \rho_B\,g\eta \quad \text{at} \quad z = 0$$

Substitution into the panel equation of motion gives:

$$\sigma\,\partial^2\eta/\partial t^2 = -\rho_B\,(\partial\varphi_B/\partial t) + \rho_T\,(\partial\varphi_T/\partial t + U\partial\varphi_T/\partial x)$$
$$-\rho_B g\,\eta + \rho_T g\,\eta - K/w\,\eta + T\,\partial^2\eta/\partial x^2 - D\,\partial^4\eta/\partial x^4$$

Consider the general solution forms:

$$\varphi_T = [G \, \mathrm{Sinh}[kz] + H \, \mathrm{Cosh}[kz]] \quad e^{jkX}$$

$$\varphi_B = [I \, \mathrm{Sinh}[kz] + J \, \mathrm{Cosh}[kz]] \quad e^{jkX}$$

$$\eta = \eta_O \, e^{jkX}$$

where $kX = k(x - C_P t) = kx - \omega t$ where X is the horizontal coordinate of a wave fixed frame, x is the horizontal coordinate of an inertial frame, C_P is the wave phase speed, k is the wave number and ω is the wave frequency. The wall constraints give

$$\varphi_T = \varphi_{TO} \quad \mathrm{Cosh}[k(d_T - z)]/\mathrm{Cosh}[kd_T] \quad e^{jkX}$$

$$\varphi_B = \varphi_{BO} \quad \mathrm{Cosh}[k(d_B + z)]/\mathrm{Cosh}[kd_B] \quad e^{jkX}$$

$$\eta = \eta_O \, e^{jkX}$$

These satisfy everything except the panel kinematic constraints and the panel equation of motion. Substitution into the panel equations gives, after common terms are cancelled away:

$$-j\omega \, \eta_O + Ujk \, \eta_O = -k \, \varphi_{TO} \, \mathrm{Tanh}[kd_T]$$

$$-j\omega \, \eta_O = +k \, \varphi_{BO} \, \mathrm{Tanh}[kd_B]$$

$$\rho_T \, [-j\omega + Ujk] \, \varphi_{TO} - \rho_B \, [-j\omega] \, \varphi_{BO} + \rho_T g \, \eta_O - \rho_B g \, \eta_O$$

$$- Tk^2 \, \eta_O - Dk^4 \, \eta_O - K/w \, \eta_O - \sigma[-j\omega]^2 \, \eta_O = 0$$

Substitution into the last equation gives:

$$\rho_T[-j\omega+Ujk][+j\omega\eta_O-Ujk\eta_O]/[kTanh[kd_T]]$$
$$-\ \rho_B[-j\omega][-j\omega\eta_O]/[kTanh[kd_B]]\ +\ \rho_T g\ \eta_O$$
$$-\rho_B g\ \eta_O\ -\ Tk^2\ \eta_O\ -\ Dk^4\ \eta_O\ -K/w\ \eta_O\ -\sigma[-j\omega]^2\ \eta_O\ =\ 0$$

Manipulation of this gives an equation of the form:

$$A\ \omega^2\ +\ B\ \omega\ +\ C\ =\ 0$$

$$A\ =\ \rho_T/[kTanh[kd_T]]\ +\ \rho_B/[kTanh[kd_B]]\ +\ \sigma$$

$$B\ =\ -\ 2U\rho_T/Tanh[kd_T]$$

$$C\ =\ -\ S\ +\ U^2k\rho_T/Tanh[kd_T]$$

$$S = +\ Tk^2\ +\ Dk^4\ +\ K/w\ -\ \rho_T g\ +\ \rho_B g$$

When $B^2 - 4AC$ is negative, the roots of the quadratic for ω form a complex conjugate pair:

$$\omega_1 = \alpha + \beta j \qquad \omega_2 = \alpha - \beta j$$
$$\alpha = -B/2A \qquad \beta = \sqrt{[4AC-B^2]}/2A$$

Substitution of ω_1 into the wave profile equation gives:

$$\eta_O \, e^{jkX} \;=\; (\Delta_R+\Delta_I j)\; e^{j[kx-(\alpha+\beta j)t]}$$

$$=\; (\Delta_R+\Delta_I j)\; e^{j[kx-\alpha t]}\; e^{j[-\beta j t]} \;=\; (\Delta_R+\Delta_I j)\; e^{+\beta t}\; e^{j[kx-\alpha t]}$$

$$=\; (\Delta_R+\Delta_I j)\; e^{+\beta t}\; [\mathrm{Cos}(kx-\alpha t)+j\mathrm{Sin}(kx-\alpha t)]$$

The real part of this is:

$$[\; \Delta_R \,\mathrm{Cos}(kx-\alpha t) - \Delta_I \,\mathrm{Sin}(kx-\alpha t)\;]\; e^{+\beta t}$$

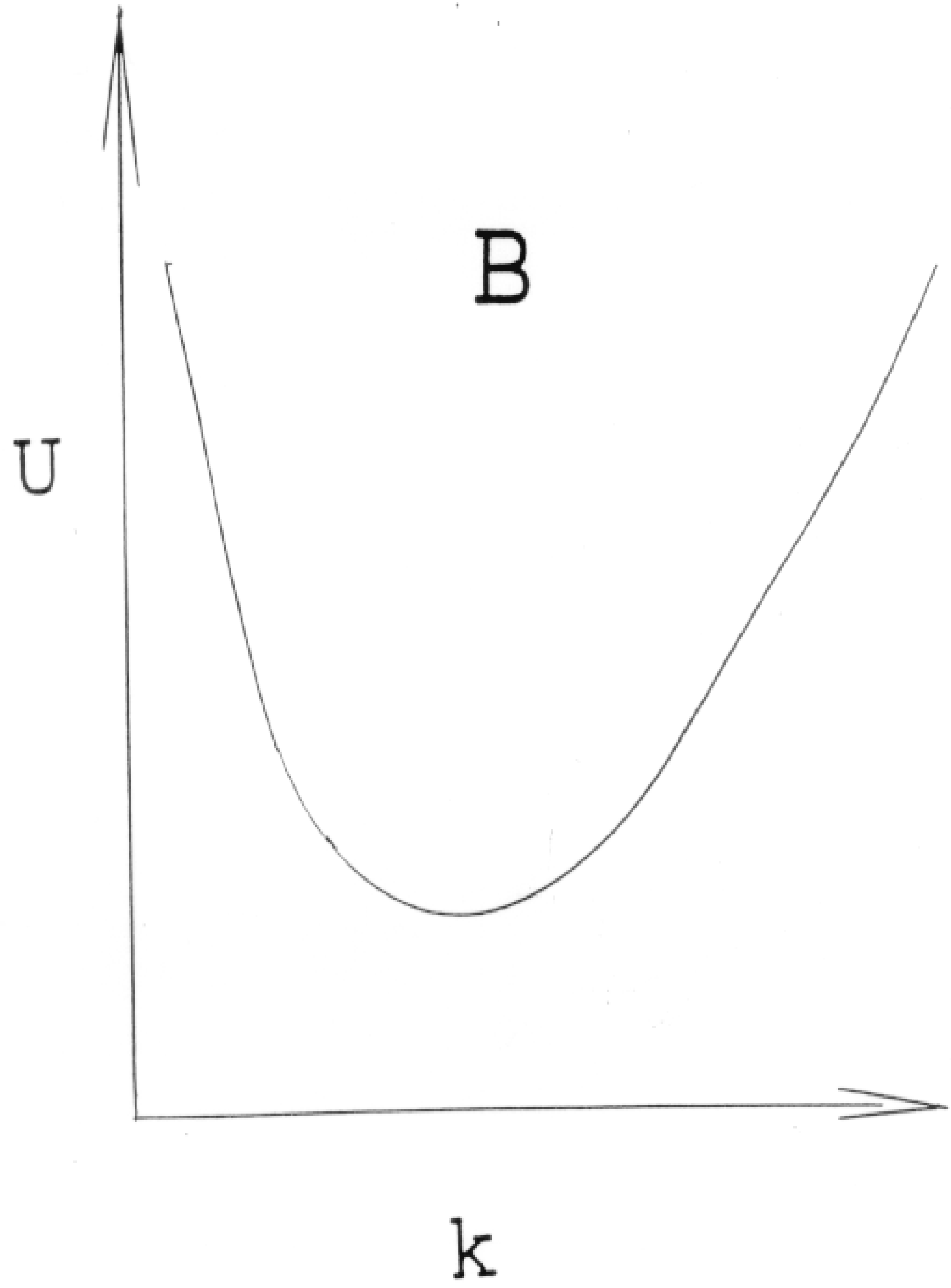
$$=\; \Delta \;\; e^{+\beta t}\, \mathrm{Sin}[(kx-\alpha t) + \varepsilon]$$

This shows that, when B^2-4AC is negative, the ω_1 wave grows. Similarly, one can show that the ω_2 wave decays. Substitution into $B^2 - 4AC = 0$ gives the critical speed:

$$U^2 \;=\; S\; V/W$$

$$V \;=\; \rho_T/[k\mathrm{Tanh}[kd_T]] + \rho_B/[k\mathrm{Tanh}[kd_B]] + \sigma$$
$$W = \rho_B\rho_T/[\mathrm{Tanh}[kd_T]\mathrm{Tanh}[kd_B]] + k\sigma\rho_{T/}\mathrm{Tanh}[kd_T]$$

This is sketched in Figure B. The plot shows that, if U is below a certain level, the panel does not flutter. For U beyond this level, it flutters for a range of k.


B
U
k

For a membrane under uniform pressure load

$$\mathbf{T}\, d^2\Delta/dx^2 = P$$

Integration shows that the mean deflection is:

$$\boldsymbol{\Delta} = P\, w^2 / [12\, \mathbf{T}]$$

This gives the side support stiffness

$$K^* = [12\, \mathbf{T}] / w^2$$

For a beam under uniform pressure load

$$\mathbf{EI}\, d^4\Delta/dx^4 = P$$

Integration shows that the mean deflection is:

$$\boldsymbol{\Delta} = P\, w^4 / [120\, \mathbf{EI}]$$

This gives the side support stiffness

$$K^* = [120\, \mathbf{EI}] / w^4$$

PIPE INSTABILITIES DUE TO INTERNAL FLOW

The equation governing the lateral vibration of a pipe containing an internal flow is

$$M\ \partial^2Y/\partial t^2 = -\ \partial^2/\partial x^2\ (EI\ \partial^2Y/\partial x^2) + T\ \partial^2Y/\partial x^2$$

$$-\ PA\ \partial^2Y/\partial x^2 - \rho AU^2\ \partial^2Y/\partial x^2 - 2\rho AU\ \partial^2Y/\partial x\partial t$$

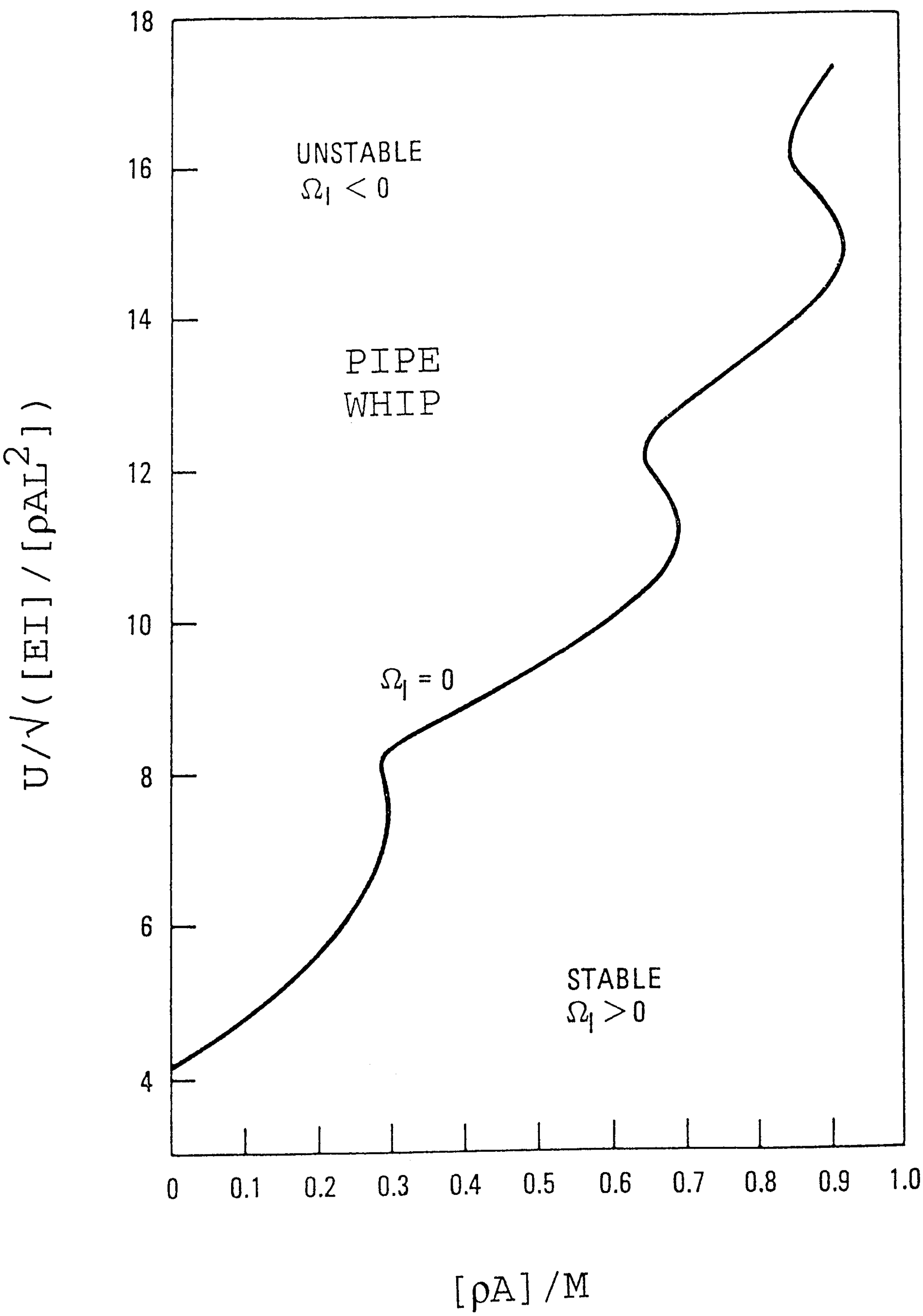
For a pipe pivoted at both ends, a static force balance shows that centrifugal forces generated by fluid motion can cause buckling when U is greater than

$$U^2 = [\ EI/[\rho A]\ \pi^2/L^2 + T/[\rho A] - P/\rho\ ]$$

For a pipe clamped at one end and open and free at the other end, a stability analysis shows that the pipe can undergo a flutter like phenomenon known as pipe whip. The critical speed U can be obtained from the sketch on the next page. A straight line fit to the wavy curve there is

$$U = [4 + 14\ M_o/M]\ U_o$$

$$U_o = \sqrt{[EI]/[M_oL^2]} \qquad M_o = \rho A$$



UNSTABLE
$\Omega_I < 0$
PIPE
WHIP
$\Omega_I = 0$
STABLE
$\Omega_I > 0$
18
16
14
12
10
8
6
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$U/\sqrt{([EI]/[\rho AL^2])}$
$[\rho A]/M$

PIPE WHIP INSTABILTY

The equation governing the lateral vibration of a pipe containing an internal flow is

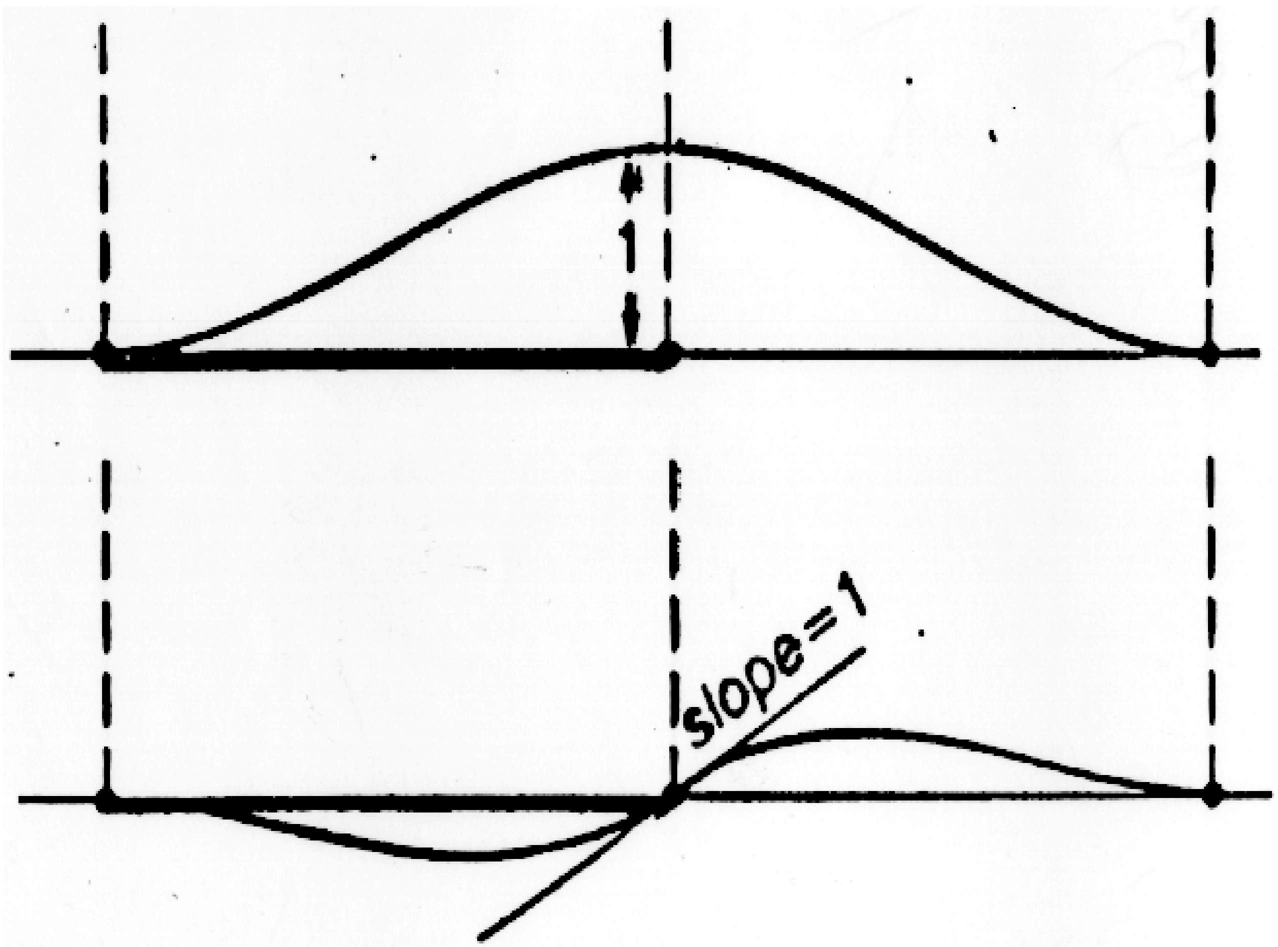
$$0 = M\, \partial^2 Y/\partial t^2 + \partial^2/\partial x^2\, (EI\, \partial^2 Y/\partial x^2) - T\, \partial^2 Y/\partial x^2$$

$$+ PA\, \partial^2 Y/\partial x^2 + \rho A U^2\, \partial^2 Y/\partial x^2 + 2\rho A U\, \partial^2 Y/\partial x \partial t$$

In this equation, Y is the lateral deflection of the pipe from its neutral position, M is the total mass of the pipe per unit length, EI is its flexural rigidity, T is tension, P is pressure, U is flow speed, A is pipe area, x is location along the pipe and t is time. For a Galerkin finite element analysis, we assume that the deflection of the pipe can be given as a sum of scaled shape functions:

$$\mathbf{Y} = \Sigma\, [A\ n + B\ m]$$

where n is the deflection at a node and m is the slope at the node. A and B are shape functions. Theory shows that these must be Hermite polynomials. Such polynomials must be used because the EI term is 4^{th} order. The sketch on the next page shows what they look like for a typical node.


1
slope=1

Substitution of the assumed form for **Y** into the governing equation gives a residual **R**. In a Galerkin analysis, weighted averages of this residual along the pipe are set to zero:

$$\int_0^L \mathbf{W}\,\mathbf{R}\,dx = 0$$

where L is the length of the pipe and **W** is a weighting function. For a Galerkin analysis, shape functions are used as weighting functions. For a typical node, these are:

$$A_L = \varepsilon^2(3-2\varepsilon) \qquad A_R = 1-3\varepsilon^2+2\varepsilon^3$$

$$B_L = S\varepsilon^2(\varepsilon-1) \qquad B_R = S\varepsilon(\varepsilon-1)^2$$

where ε is a local coordinate and S is an element length. The subscripts L and R indicate elements immediately to the left and right of the node. After performing the integrations and applying boundary conditions, one gets a set of ODEs in time. One can put them in a matrix form and use matrix manipulation to get the roots **λ** of the system characteristic equation.

$$[GI]\ |d\Phi/dt| + [GS]\ |\Phi| = |0|$$

$$[GI]\ \lambda\ |\Phi_o| + [GS]\ |\Phi_o| = |0|$$

One can plot the roots in a Root Locus Plot to get the critical speed corresponding to the onset of instability.