FLUID STRUCTURE INTERACTIONS

WATER WAVE INTERACTION

WITH STRUCTURES

PREAMBLE

Most water waves are generated by storms at sea. Many waves are present in a storm sea state: each has a different wavelength and period. Theory shows that the speed of propagation of a wave or its phase speed is a function of water depth. It travels faster in deeper water. Theory also shows that the speed of a wave is a function of its wavelength. Long wavelength waves travel faster than short wavelength waves. This explains why storm generated waves, which approach shore, are generally a single wavelength. Because waves travel at different speeds, they tend to separate or disperse. When waves approach shore, they are influenced by the seabed by a process known as refraction. This can focus or spread out wave energy onto a site. Close to shore water depth is not the same everywhere: so points on wave crests move at different speeds and crests become bent. This explains why crests which approach a shore line tend to line up with it: points in deep water travel faster than points in shallow water and overtake them. Wave energy travels at a speed known as the group speed. This is generally not the same as the phase speed. However for shallow water both speeds are the same and they depend only on the water depth. A large low pressure system moving over shallow water would generate an enormous wave if the system speed and the wave energy speed were the same. Basically wave energy gets trapped in the system frame when the system speed matches the wave energy speed. Tides are basically shallow water waves. Here the pull of the Moon mimics a low pressure system. Theory shows that if water depth was 22km everywhere on Earth the Moon pull would produce gigantic tides. They would probably drain the oceans and swamp the continents everyday. Fortunately the average water depth is only 3km.

Water waves can interact with structures and cause them to move or experience loads. For wave structure interaction, an important parameter is $5D/\lambda$ where D is the characteristic dimension of the structure and λ is the wavelength. Structures are considered large if $5D/\lambda$ is much greater than unity: they are considered small if $5D/\lambda$ is much less than unity. Small structures are transparent to waves. Large structures scatter waves.

These notes start with a description of water wave theory. Then interaction of waves with small structures is considered. Finally interaction with large structures is considered.

WATER WAVES

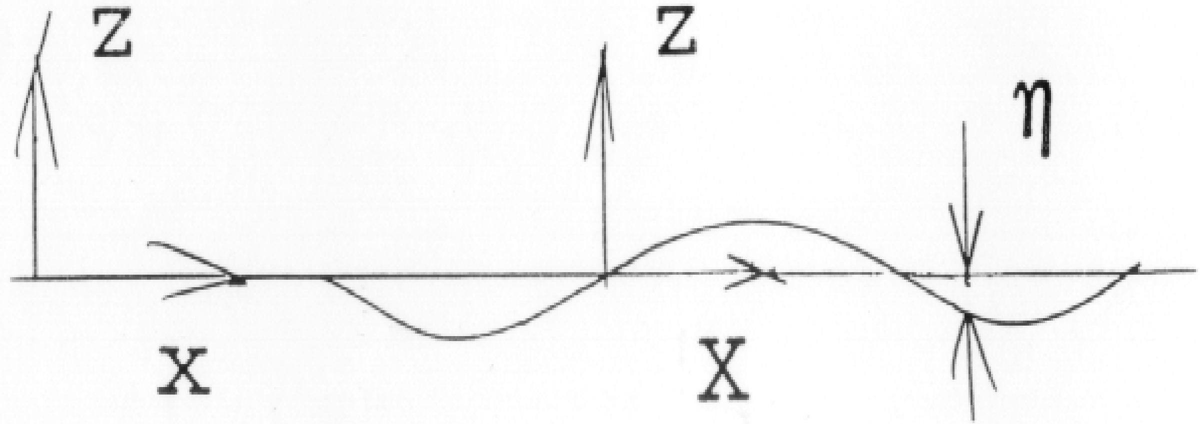
To calculate wave interactions with structures, one needs a detailed knowledge of the wave field. Water wave theory provides this. It will be assumed for much of what is given below that wave amplitudes are very small. It turns out that this is good even for waves not far from breaking. Water waves in deep water propagate for long distances with little loss of energy. They lose energy in shallow water due to interaction with the seabed. They also lose energy when they move pass small structures and when they break on beaches. Water wave theory ignores these energy losses. It assumes that water has zero viscosity and it is incompressible. It also assumes that its motion is irrotational. This means that water particles do not spin. With these assumptions, the conservation laws reduce to potential flow forms.

Conservation of mass considerations give:

$$\nabla . \mathbf{v} = 0 \qquad \nabla^2 \varphi = 0 \quad .$$

The velocity vector **v** in terms of the potential φ is

$$\mathbf{v} = \nabla \varphi = U\mathbf{i} + V\mathbf{j} + W\mathbf{k} \quad .$$


z
Z
η
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X

Conservation of momentum considerations give:

$$\rho\ \partial \mathbf{v}/\partial t + \rho\ \nabla\ \mathbf{v}.\mathbf{v}/2 + \nabla\ P + \nabla\ \rho g z = 0$$

$$\partial\varphi/\partial t + \nabla\varphi.\nabla\varphi/2 + P/\rho + gz = C \quad .$$

It turns out that, for water waves, mass is the main governing equation: momentum is used as a boundary condition. The kinematic or motion constraint at the seabed is:

$$\partial\varphi/\partial z = 0 \qquad \text{at } z = -h \quad .$$

where h is the water depth. The kinematic or motion constraint at the water surface is based on:

$$D\eta/Dt = Dz/Dt$$

where η is the vertical deflection of the water from the still water line. The η for a point on the water must follow the z for that point. The constraint gives:

$$\partial\eta/\partial t + \partial\varphi/\partial x\ \partial\eta/\partial x = \partial\varphi/\partial z \qquad \text{at } z = \eta \quad .$$

For small amplitude waves, this becomes:

$$\partial\eta/\partial t = \partial\varphi/\partial z \qquad \text{at } z = 0 \quad .$$

The dynamic or load constraint at the water surface is:

$$\partial\varphi/\partial t + \nabla\varphi.\nabla\varphi/2 + P/\rho + g\eta = 0 \quad \text{at} \quad z = \eta \quad .$$

For small amplitude waves, this becomes:

$$\partial\varphi/\partial t + g\eta = 0 \quad \text{at} \quad z = 0 \quad .$$

Manipulation of the water surface constraints allows one to eliminate η from the formulation. One gets:

$$\partial^2\varphi/\partial t^2 + g\partial\varphi/\partial z = 0 \quad \text{at} \quad z = 0 \quad .$$

The Separation of Variables solution procedure gives:

$$\varphi = \varphi_O \, \text{Cosh}[k(z+h)]/\text{Cosh}[kh] \, \text{Cos}(kX)$$

where $kX = k(x - C_P t) = kx - \omega t$ where X is the horizontal coordinate of a wave fixed frame, x is the horizontal coordinate of an inertial frame, C_P is the wave phase speed, k is the wave number and ω is the wave frequency. The wave number k is related to the wave length λ as follows: $k = 2\pi/\lambda$.

The wave profile equation has the form:

$$\eta = \eta_O \, \text{Sin}(kX) \quad .$$

Substitution into the combined water surface constraint gives the dispersion relationships:

$$C_P = \sqrt{(g/k \; Tanh[kh])}$$
$$\omega = \sqrt{(gk \; Tanh[kh])} \quad .$$

These show that deep water waves travel faster than shallow water waves. They also show that long wave length waves travel faster than short wave length waves.

Substitution into the water surface constraints gives the connection between potential amplitude and wave amplitude:

$$\varphi_O = - gT/[2\pi] \; H/2$$

where T is the wave period and H is the wave height.

Differentiation gives the water particle velocities:

$$U = \partial\varphi/\partial x = - \varphi_O \; k \; Cosh[k(z+h)]/Cosh[kh] \; Sin(kX)$$
$$= + H/2 \; 2\pi/T \; Cosh[k(z+h)]/Sinh[kh] \; Sin(kX)$$

$$W = \partial\varphi/\partial z = + \varphi_O \; k \; Sinh[k(z+h)]/Cosh[kh] \; Cos(kX)$$
$$= - H/2 \; 2\pi/T \; Sinh[k(z+h)]/Sinh[kh] \; Cos(kX) \quad .$$

These can be used to get drag loads on small structures.

More differentiation gives the particle accelerations:

$$dU/dt = - H/2\ (2\pi/T)^2\ Cosh[k(z+h)]/Sinh[kh]\ Cos(kX)$$

$$dW/dt = - H/2\ (2\pi/T)^2\ Sinh[k(z+h)]/Sinh[kh]\ Sin(kX) \quad .$$

These can be used to get inertia loads on small structures.

The particle positions are:

$$x_p = x_o + H/2\ Cosh[k(z+h)]/Sinh[kh]\ Cos(kX)$$

$$Z_p = z_o + H/2\ Sinh[k(z+h)]/Sinh[kh]\ Sin(kX) \quad .$$

These give the water particle orbit size.

The momentum equation gives the wave pressure

$$\Delta P = \rho g\ \eta\ Cosh[k(z+h)]/Cosh[kh] \quad .$$

This can be used to get pressure loads on structures.

For deep water, the solution becomes:

$$\varphi = \varphi_O\ e^{-kz}\ Cos(kX) \qquad \eta = \eta_O\ Sin(kX) \quad .$$

With this, the dispersion relationships become:

$$C_P = \sqrt{g/k} \qquad \omega = \sqrt{gk} \quad .$$

Other wave parameters become:

$$U = + H/2 \; 2\pi/T \; e^{kz} \; Sin(kX)$$

$$W = - H/2 \; 2\pi/T \; e^{kz} \; Cos(kX)$$

$$dU/dt = - H/2 \; (2\pi/T)^2 \; e^{kz} \; Cos(kX)$$

$$dW/dt = - H/2 \; (2\pi/T)^2 \; e^{kz} \; Sin(kX)$$

$$x_p = x_o + H/2 \; e^{kz} \; Cos(kX)$$

$$Z_p = z_o + H/2 \; e^{kz} \; Sin(kX)$$

$$\Delta P = \rho g \; \eta \; e^{kz} \quad .$$

Wave energy travels at a speed known as the group speed. This is generally not the same as the phase speed of a wave. One can show that the group speed is given by:

$$C_G = d\omega/dk = C_P \; (1/2 + [kh]/Sinh[2kh]) \quad .$$

The wave energy density is:

$$\mathbf{E} = 1/8 \; \rho g \; H^2 \quad .$$

One can show that wave energy flux is:

$$\mathbf{P} = C_G \; \mathbf{E} \quad .$$

Group speed is responsible for many important phenomena. Some of these were mentioned earlier.

Waves at sea after a storm are random. They are made up of an infinite number of frequencies. A spectrum shows how the energy in a wave field is spread out over a range of frequencies. A popular 2 parameter fit to a wave amplitude spectrum is the ITTC fit:

$$S_\eta = A/\omega^5 \, e^{-B/\omega 4}$$

$$A=346H^2/T^4 \qquad B=691/T^4$$

where H is significant wave height and T is significant wave period. JONSWAP is a popular 3 parameter fit.

A Response Amplitude Operator or RAO can be used to connect a wave spectrum to a body motion or load response spectrum

$$S_R = RAO^2 \, S_\eta \quad .$$

An RAO is basically a Magnitude Ratio. For a specific wave period, it is the amplitude of body response divided by the wave amplitude. For small structures, Morisons Equation can be used to get RAOs. For large structures, they can be obtained using the CFD procedure known as the Panel Method. One can also get RAOs from experiments.

All sorts of statistical and probabilistic information can be obtained from spectra. For bodies, the analysis makes use of the following moments of the spectrum:

$$M_n = 1/2 \int_0^\infty S_R(\omega)\ \omega^n\ d\omega \qquad .$$

One can show that the significant response height and period of a body motion or load are:

$$2\ R_S = 4\ \sqrt{M_0} \qquad T_S = 2\pi\ M_0/M_1\ .$$

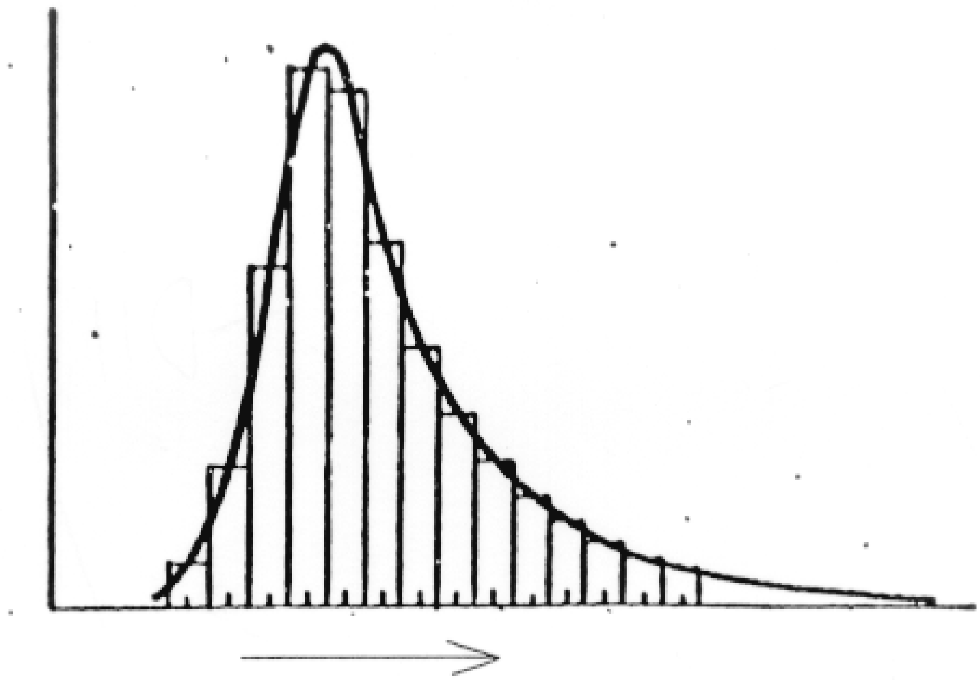
The probability of a response exceeding a certain level is:

$$P(R_o > R_\bullet) = e^{-X} \qquad X = R_\bullet R_\bullet/[2M_0] \qquad .$$

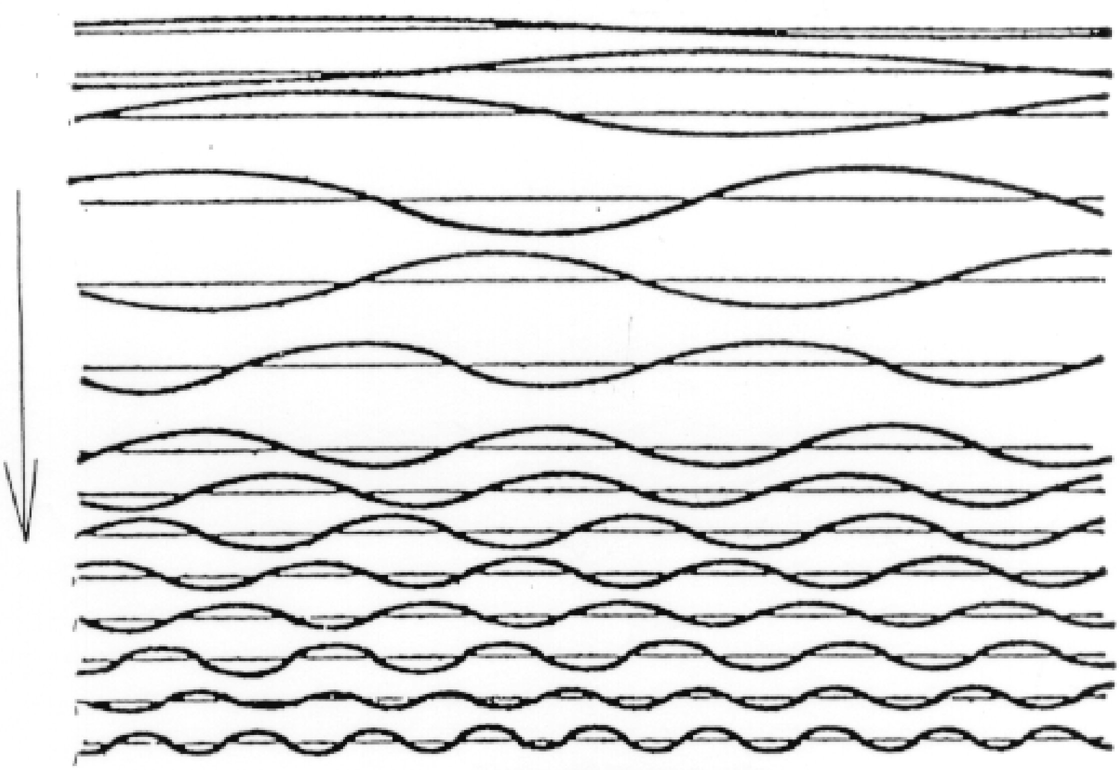
The theory assumes that spectra follow a Rayleigh Distribution. Actual spectra deviate from this and predictions must often be corrected. A correction factor based on moments is:

$$CF = \sqrt{1-\varepsilon\varepsilon} \qquad \varepsilon = [M_0M_4-M_2M_2]/[M_0M_4]$$

where ε is known as the broadness parameter.



SPECTRUM



TIME OR DISTANCE

COMPONENT WAVES

WAVE INTERACTION WITH BODIES

REAL FLUID FORMULATION

PREAMBLE

At low speeds, fluid particles move along smooth paths: motion has a laminar or layered structure. At high speeds, particles have superimposed onto their basic streamwise observable motion a random walk or chaotic motion. Particles move as groups in small spinning bodies known as eddies. The flow pattern is said to be turbulent. A turbulent wake flow is one that contains some large eddies together with a lot of small ones. Such a flow could be found around the GBS on a stormy day. The large eddies generally stay roughly in one place. Fluid in them swirls around and around or recirculates in roughly closed orbits. The smaller eddies are associated with turbulence and are carried along by the local flow. The large eddies can usually be found inside wakes. Most of the smaller ones can be found near wake boundaries. They are generated in regions where velocity gradients are high like at the edges of wakes or in the boundary layers close to structures. They are dissipated in regions where gradients are low like in sheltered areas like corners. Turbulent wake flows are governed by the basic conservation laws. However, they are so complex that analytical solutions are impossible.

One could develop computational fluid dynamics or CFD codes based on the conservation law equations. Unfortunately, the small eddies are so small that an extremely fine grid spacing and a very small time step would be needed to follow individual eddies in a flow. Small eddies are typically around 1mm in diameter. One would need a grid spacing smaller than 0.1mm to follow such eddies. CFD converts each governing equation into a set of algebraic equations or AEs: one AE for each PDE for each xyz grid point. Workable CFD is not possible because computers cannot handle the extremely large number of AEs generated. For example, a 100m x 100m x 100m volume of water near a structure like the GBS would need 10^6 x 10^6 x 10^6 or 10^{18} grid points if the grid spacing was 0.1mm. Also very many time steps would be needed to complete a simulation run. No computer currently exists that can handle so many grid points and so many time steps. The random motions of molecules in a gas diffuse momentum: they give gas its viscosity. Small eddies in a turbulent flow also diffuse momentum: they make fluid appear more viscous than it really is. This apparent increase in viscosity controls overall flow patterns and loads on structures. Models which account for this apparent increase in viscosity are known as eddy viscosity models. They can be obtained from the momentum equations by a complex time averaging process. The time averaging introduces the so called Reynolds Stresses into the momentum equations, and these are modelled using the eddy viscosity concept. Models have been developed which can estimate how eddy viscosity varies throughout a flow. Workable CFD is now possible because one can now use much larger grid spacing and time steps: it is no longer necessary to follow individual eddies around in a flow. When small eddies are accounted for in this way, they no longer show up in flow: they are suppressed by eddy viscosity. For the GBS case, a grid spacing around 1m would now be adequate. This means a 100m x 100m x 100m volume of water near the GBS would now need only 10^2 x 10^2 x 10^2 or 10^6 grid points.

CONSERVATION LAWS FOR HYDRODYNAMICS FLOWS

Hydrodynamics flows are often turbulent. Conservation of momentum considerations for such flows give:

$$\rho \left(\partial U/\partial t + U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z \right) + A = - \partial P/\partial x + \left[\partial/\partial x \, (\mu \, \partial U/\partial x) + \partial/\partial y \, (\mu \, \partial U/\partial y) + \partial/\partial z \, (\mu \, \partial U/\partial z) \right]$$

$$\rho \left(\partial V/\partial t + U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z \right) + B = - \partial P/\partial y + \left[\partial/\partial x \, (\mu \, \partial V/\partial x) + \partial/\partial y \, (\mu \, \partial V/\partial y) + \partial/\partial z \, (\mu \, \partial V/\partial z) \right]$$

$$\rho \left(\partial W/\partial t + U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z \right) + C = - \partial P/\partial z - \rho g + \left[\partial/\partial x \, (\mu \, \partial W/\partial x) + \partial/\partial y \, (\mu \, \partial W/\partial y) + \partial/\partial z \, (\mu \, \partial W/\partial z) \right]$$

where U V W are respectively the velocity components in the x y z directions, P is pressure, ρ is the density of water and μ is its effective viscosity. The time averaging process introduces source like terms A B C into the momentum equations. Each is a complex function of velocity and viscosity gradients as indicated below:

$$A = \partial\mu/\partial y \; \partial V/\partial x - \partial\mu/\partial x \; \partial V/\partial y + \partial\mu/\partial z \; \partial W/\partial x - \partial\mu/\partial x \; \partial W/\partial z$$

$$B = \partial\mu/\partial x \; \partial U/\partial y - \partial\mu/\partial y \; \partial U/\partial x + \partial\mu/\partial z \; \partial W/\partial y - \partial\mu/\partial y \; \partial W/\partial z$$

$$C = \partial\mu/\partial y \; \partial V/\partial z - \partial\mu/\partial z \; \partial V/\partial y + \partial\mu/\partial x \; \partial U/\partial z - \partial\mu/\partial z \; \partial U/\partial x$$

Conservation of mass considerations give:

$$\partial P/\partial t + \rho \; c^2 \; (\partial U/\partial x + \partial V/\partial y + \partial W/\partial z) = 0$$

where c is the speed of sound in water. Although water is basically incompressible, CFD takes it to be compressible. Mass is used to adjust pressure at points in the grid when the divergence of the velocity vector is not zero.

A special function F known as the volume of fluid or VOF function is used to locate the water surface. For water, F is taken to be unity: for air, it is taken to be zero. Regions with F between unity and zero must contain the water surface. Material volume considerations give:

$$\partial F/\partial t + U\partial F/\partial x + V\partial F/\partial y + W\partial F/\partial z = 0 \quad .$$

TURBULENCE MODEL

Engineers are usually not interested in the details of the eddy motion. Instead they need models which account for the diffusive character of turbulence. One such model is the k-ε model, where k is the local intensity of turbulence and ε is its local dissipation rate. Its governing equations are:

$$\partial k/\partial t + U\partial k/\partial x + V\partial k/\partial y + W\partial k/\partial z = T_P - T_D$$
$$+ \ [\ \partial/\partial x\ (\mu/a\ \partial k/\partial x) + \partial/\partial y\ (\mu/a\ \partial k/\partial y) + \partial/\partial z\ (\mu/a\ \partial k/\partial z)\ ]$$

$$\partial \varepsilon/\partial t + U\partial \varepsilon/\partial x + V\partial \varepsilon/\partial y + W\partial \varepsilon/\partial z = D_P - D_D$$
$$+ \ [\ \partial/\partial x\ (\mu/b\ \partial \varepsilon/\partial x) + \partial/\partial y\ (\mu/b\ \partial \varepsilon/\partial y) + \partial/\partial z\ (\mu/b\ \partial \varepsilon/\partial z)\ ]$$

where

$$T_P = G\ \mu_t\ /\ \rho \qquad D_P = T_P\ C_1\ \varepsilon\ /\ k$$

$$T_D = C_D\ \varepsilon \qquad D_D = C_2\ \varepsilon^2\ /\ k$$

$$\mu_t = C_3\ k^2\ /\ \varepsilon \qquad \mu = \mu_t + \mu_l$$

where

$$G = 2 [(\partial U/\partial x)^2 + (\partial V/\partial y)^2 + (\partial W/\partial z)^2] + [\partial U/\partial y + \partial V/\partial x]^2 + [\partial U/\partial z + \partial W/\partial x]^2 + [\partial W/\partial y + \partial V/\partial z]^2$$

where C_D=1.0 C_1=1.44 C_2=1.92 C_3=0.9 a=1.0 b=1.3 are constants based on data from geometrically simple experiments, μ_l is the laminar viscosity, μ_t is extra viscosity due to eddy motion and G is a production function. The k-ε equations account for the convection, diffusion, production and dissipation of turbulence. Special wall functions are used to simplify consideration of the sharp normal gradients in velocity and turbulence near walls.

COMPUTATIONAL FLUID DYNAMICS

For CFD, the flow field is discretized by a Cartesian or xyz system of grid lines. Small volumes or cells surround points where grid lines cross. Flow is not allowed in cells occupied by fixed bodies. Ways to handle moving bodies are still under development. Flow can enter or leave the region of interest through its boundaries. For hydrodynamics problems, an oscillating pressure over a patch of the water surface could be used to generate waves. An oscillating flow at a vertical wall could also be used for this. For CFD, each governing equation is put into the form:

$$\partial M/\partial t = N \quad .$$

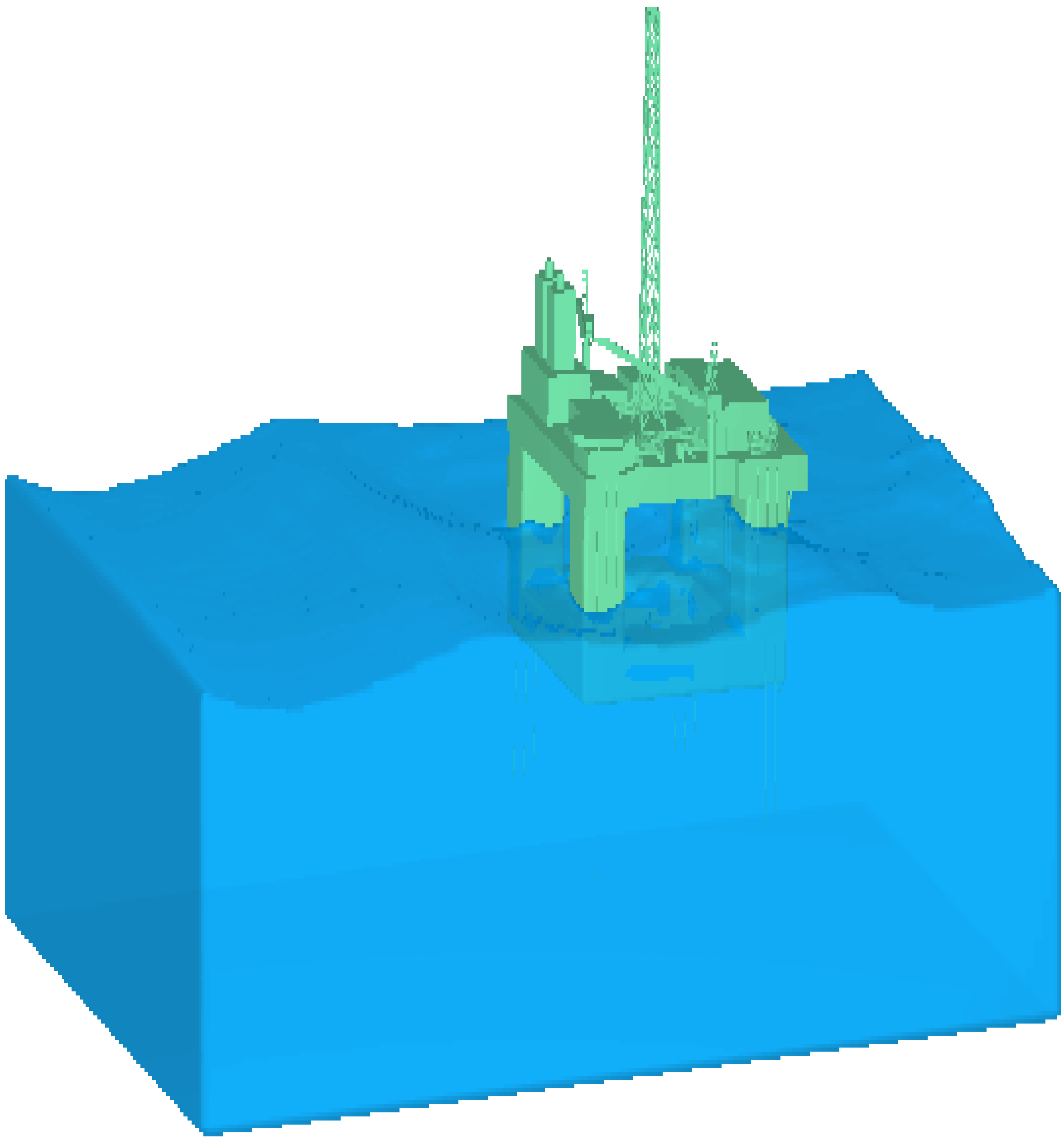
At points within the CFD grid, each governing equation is integrated numerically across a time step to get:

$$M(t+\Delta t) = M(t) + \Delta t\ N(t)$$

where the various derivatives in N are discretized using finite difference approximations. The discretization gives algebraic equations for the scalars P F k ε at points where grid lines cross and equations for the velocity components at staggered positions between the grid points. Central differences are used to discretize the viscous terms in the momentum and turbulence equations. To ensure numerical stability, a combination of central and upwind differences is used for the convective terms. Collocation or lumping is used for the T and D terms. To march the unknowns forward in time, the momentum equations are used to update U V W, the mass equation is used to update P and correct U V W, the VOF equation is used to update F and the location of the water surface and the turbulence equations are used to update k ε.

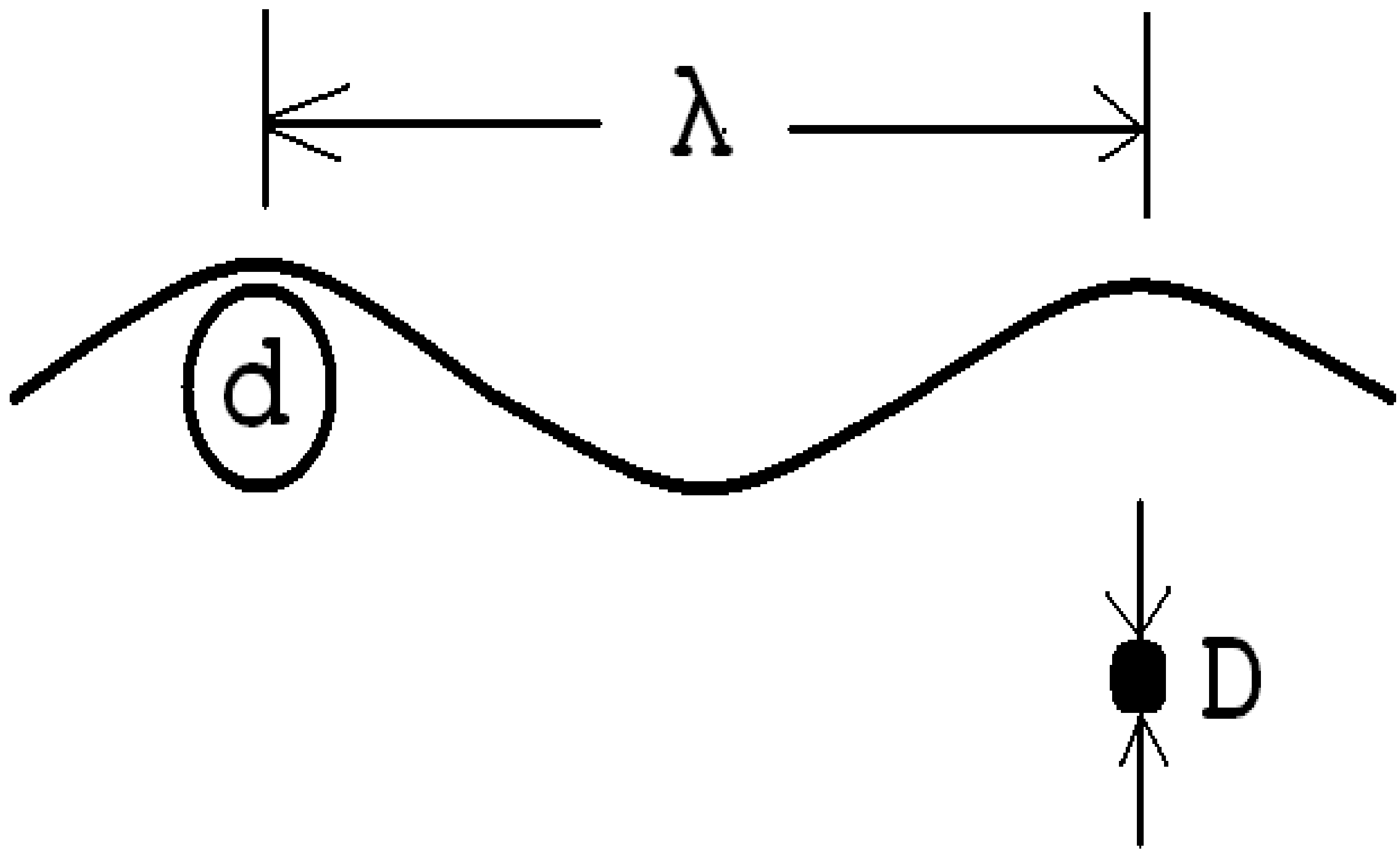
APPLICATIONS OF FLOW-3D CODE

FLOW-3D is a CFD software package for hydrodynamics and other flows <www.flow3d.com>. It can handle all sorts of complex phenomena such as wave breaking and phase changes such as vaporization and solidification. No other CFD package can handle these phenomena. A new feature known as the General Moving Object or GMO can simulate the complex motions of floating bodies in steep waves. The motions of the bodies can be prescribed or they can be coupled to the motion of the fluid. It allows for extremely complicated motions and flows. One can think of a GMO as a bubble in a flow where the pressure on the inside surface of the bubble is adjusted in such a way that its boundary matches the shape of a body. FLOW 3D uses a complex interpolation scheme to fit the body into the Cartesian grid. The sketch on the next page shows a FLOW-3D simulation of an oil rig in waves.



WAVE INTERACTION WITH SMALL STRUCTURES

When a wave passes a small structure, there can be two kinds of loads on the structure: wake load due to the formation of wakes back of the structure and inertia load due to pressures in the water caused by acceleration and deceleration of water particles in the wave. In deep water, water particles move in circular orbits. In finite depth water, the orbits are ellipses. Let the orbit dimension normal to the structure be d and let the characteristic dimension of the structure be D. When 5D<<d, a well defined wake forms behind the structure. When 5D>>d, such a wake does not form. When 5D is approximately equal to d, flows are extremely complex. Let T be the wave period and let T be the time it takes a water particle to move pass the structure. It turns out that 5T<<T corresponds to 5D<<d while 5T>>T corresponds to 5D>>d. When 5D<<d, wakes form because transit time is short relative to wave period. So, water is moving sufficiently long in one direction to pass the structure. When 5D>>d, wakes do not form because transit time is long relative to wave period. So, before water particles can pass the structure, they reverse direction.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For a small structure like an underwater vehicle or a cable float, the drag load is

$$C_D \, A \, \rho \, \mathbf{S.S}/2 \, \mathbf{s}$$

while the inertia load is

$$C_M \, \rho \, B \, d\mathbf{S}/dt$$

where **S** is the water particle velocity and d**S**/dt is the water particle acceleration. The frontal area of the structure is A and its volume is B. The drag and inertia loads can be combined to get Morisons equation:

$$\mathbf{F} \;=\; C_D \, A \, \rho \, \mathbf{S.S}/2 \, \mathbf{s} \;+\; C_M \, \rho \, B \, d\mathbf{S}/dt \quad .$$

The drag and inertia coefficients depend on the shape of the structure. For 5D much less than d, the drag coefficient C_D for a sphere is around 0.5. For 5D much greater than d, the inertia coefficient C_M for a sphere is around 0.5. In the reverse limits, each coefficient for a sphere is approximately zero.

For a long cylindrical structure like a tether for an underwater vehicle or a mooring cable, the drag load is

$$C_D \, D \, \rho \, \mathbf{S.S}/2 \, \mathbf{s} \, dc$$

while the inertia load is

$$C_M \; \rho \; \pi D^2/4 \; d\mathbf{S}/dt \; dc$$

where in this case **S** is the normal water particle velocity and d**S**/dt is the normal water particle acceleration. Only normal components of flow contribute to loads. These are:

$$\mathbf{S} = \mathbf{M} - \mathbf{N} \qquad d\mathbf{S}/dt = d\mathbf{M}/dt - d\mathbf{N}/dt$$

where

$$\mathbf{M} = U\mathbf{i} + W\mathbf{k} \qquad \mathbf{N} = \mathbf{M.n\;n} \quad .$$

where **M** is the flow velocity due the wave and **N** is the component of **M** along the structure. Again the loads can be combined to get Morisons equation:

$$d\mathbf{F} = C_D \; D \; \rho \; \mathbf{S.S}/2 \; \mathbf{s} \; dc \; + \; C_M \; \rho \; \pi D^2/4 \; d\mathbf{S}/dt \; dc \quad .$$

For 5D much less than d, the drag coefficient C_D in this case is around 1 and for 5D much greater than d, the inertia coefficient C_M is around 1. Again, in the reverse limits, each coefficient is approximately zero. An integration of d**F** along the length of the cylinder would give the total load **F**.

Generally, one would look for the maximum values of **S** and d**S**/dt to get upper limits on loads. Assume that you know the wave height H and the wave period T. At Hibernia following a storm, H would be around 5m while T would be around 10s. How do you find maximum values of **S** and d**S**/dt? How do you get the orbit size d? Wave theory gives the water particle velocities:

$$U = \partial\varphi/\partial x = + H/2\ 2\pi/T\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX)$$

$$W = \partial\varphi/\partial z = - H/2\ 2\pi/T\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

and the water particle accelerations:

$$dU/dt = - H/2\ (2\pi/T)^2\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

$$dW/dt = - H/2\ (2\pi/T)^2\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX)$$

and the water particle positions:

$$x_p = x_o + H/2\ \mathrm{Cosh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Cos}(kX)$$

$$Z_p = z_o + H/2\ \mathrm{Sinh}[k(z+h)]/\mathrm{Sinh}[kh]\ \mathrm{Sin}(kX) \quad .$$

Wave theory also gives the dispersion relationships:

$$C_P = \sqrt{(g/k \, Tanh[kh])}$$

$$\omega = \sqrt{(gk \, Tanh[kh])} \quad .$$

These equations allow us to find the wave number k given a wave period T. This in turn allows us to find velocities and accelerations. The particle position equations allow us to determine the orbit size d.

When the structure can move we must use relative velocities and accelerations to get loads. For a small structure Morisons equation becomes

$$\mathbf{F} = C_D \, A \, \rho \, \mathbf{U}.\mathbf{U}/2 \, \mathbf{u} + C_M \, \rho \, B \, d\mathbf{U}/dt \quad .$$

where $\mathbf{U}$ is the relative velocity and $d\mathbf{U}/dt$ is the relative acceleration. These are

$$\mathbf{U} = \mathbf{S} - \mathbf{V} \qquad d\mathbf{U}/dt = d\mathbf{S}/dt - d\mathbf{V}/dt \quad .$$

For a long cylindrical structure Morisons equation becomes

$$d\mathbf{F} = C_D \, D \, \rho \, \mathbf{U}.\mathbf{U}/2 \, \mathbf{u} \, dc + C_M \, \rho \, \pi D^2/4 \, d\mathbf{U}/dt \, dc \quad .$$

WAVE INTERACTIONS WITH LARGE BODIES

IDEAL FLUID FORMULATION

For an ideal fluid formulation, we assume that water is incompressible and it has zero viscosity. With these assumptions, conservation of mass for water is

$$\nabla . \mathbf{v} = 0$$

while conservation of momentum is

$$\rho \partial \mathbf{v} / \partial t + \rho \nabla \mathbf{v} . \mathbf{v} / 2 + \nabla P + \nabla \rho g z = 0$$

where $\mathbf{v} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}$. For an ideal fluid formulation, we also assume that water motion is irrotational. This means that water particles do not spin on internal axes: mathematically this means that the spin vector $\mathbf{\Omega}$ is zero. One can show that the spin vector $\mathbf{\Omega}$ is half the vorticity vector $\boldsymbol{\omega}$. So, for an irrotational flow, the vorticity vector is zero. One can write this as:

$$\boldsymbol{\omega} = 2\mathbf{\Omega} = \nabla x \mathbf{v} = 0$$

For any scalar φ, one can show that $\nabla x \nabla\varphi$. This suggests that for an irrotational flow $\mathbf{v}=\nabla\varphi$. Substitution into the conservation laws gives after some manipulation:

$$\nabla^2\varphi = 0$$

$$\partial\varphi/\partial t + (\nabla\varphi.\nabla\varphi)/2 + P/\rho + gz = C$$

For a body in water, the potential φ is made up of two components. One is the incident wave potential φ_w and the other is the scattered potential φ_s generated by the body. Both potentials must satisfy the seabed constraint:

$$\partial\varphi/\partial z = 0 \qquad \text{at} \quad z = -h$$

They must also satisfy the water surface constraints:

$$\partial\eta/\partial t = \partial\varphi/\partial z \qquad \partial\varphi/\partial t + g\eta = 0 \qquad \text{at} \quad z = 0$$

where η is the deflection of the water surface from the still water line. These constraints can be combined to get:

$$\partial^2\varphi/\partial t^2 + g\ \partial\varphi/\partial z = 0 \qquad \text{at} \quad z = 0$$

For a fixed body, they must also satisfy the constraint:

$$\partial\varphi/\partial n = \partial\varphi_S/\partial n + \partial\varphi_w/\partial n = 0 \qquad \text{on} \quad S$$

where S is the body surface. Finally, far from the body, the scattered potential must satisfy the radiation condition:

$$\partial\varphi_S/\partial t + C_p\, \partial\varphi_S/\partial R = 0 \qquad \text{at} \quad R = \infty$$

where C_p is the phase speed of outgoing waves. This ensures that far from the body scattered waves move radially away from it. Mathematically, they could move radially inward and be absorbed by the body but this is not realistic.

For a differential equation formulation, $\nabla^2\varphi$ must be zero everywhere within the water. One can show that in an integral formulation the same potential must satisfy the following integral at every point on the surface which surrounds the water:

$$\varphi(P) = 1/[2\pi] \int_S [\, 1/r\ \partial\varphi(Q)/\partial n - \varphi(Q)\ \partial(1/r)/\partial n \,]\, dS$$

where P and Q are points on the surface. Derivation of this integral starts with the following integral:

$$\int_S [\varphi\ \partial(1/r)/\partial n - 1/r\ \partial\varphi/\partial n]\ dS$$

Manipulation gives:

$$\int_S [\varphi\ \nabla(1/r) - (1/r)\ \nabla\varphi]\ .\ \mathbf{n}dS$$

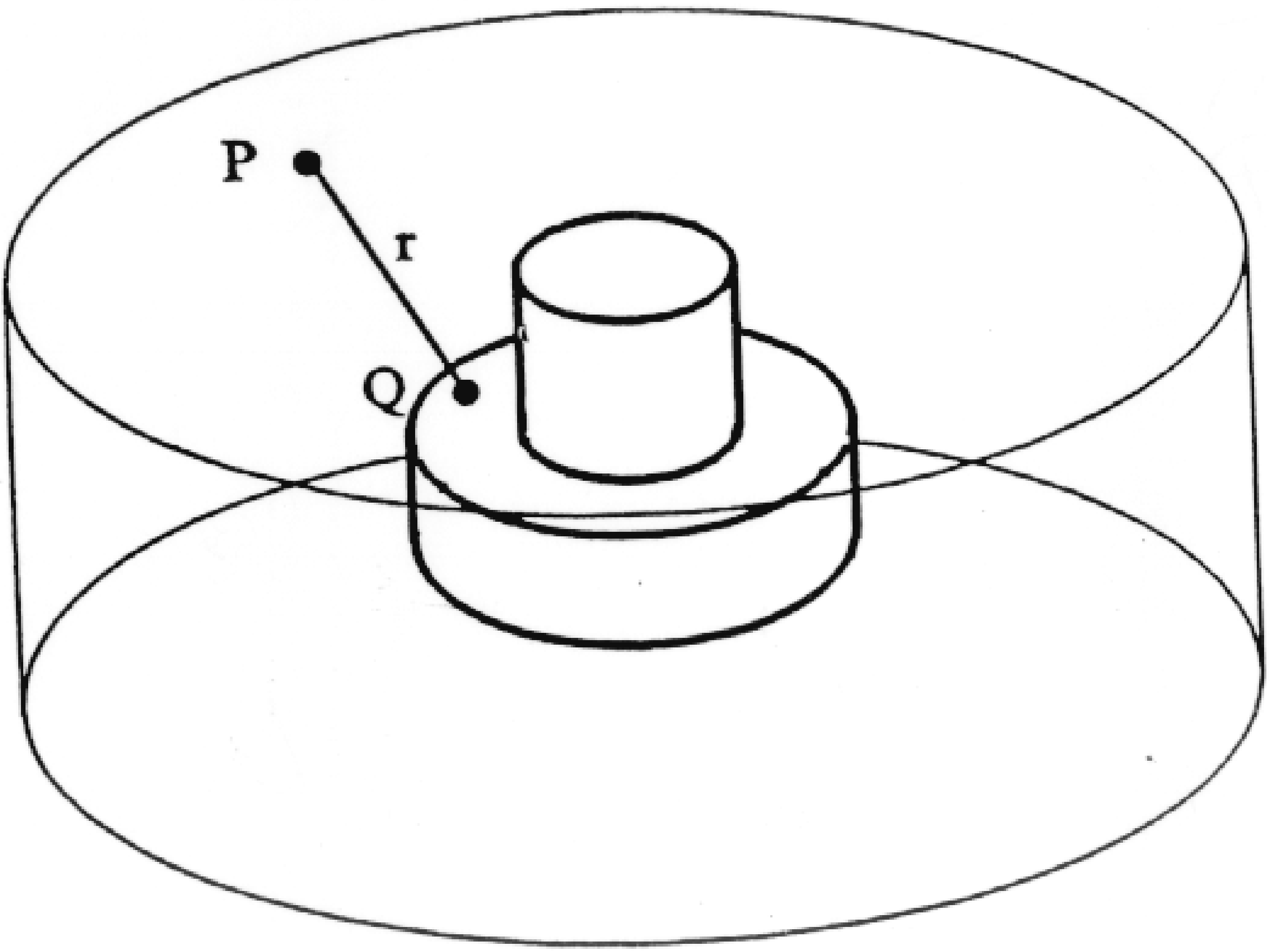
We can rewrite this as:

$$\int_V \nabla\ [\varphi\ \nabla(1/r) - (1/r)\ \nabla\varphi]\ dV$$

Expansion shows that this is zero. So, the starting surface integral must be zero. To evaluate this integral, we start by picking two points P and Q on the surface: we then set P and let Q move over the surface. Special care is required when Q approaches P because (1/r) tends to infinity when Q approaches P. We avoid this by indenting the surface with an infinitesimal radius hemisphere which makes P external. We then let Q move over this hemisphere and evaluate the integral. Afterwards we let the radius of the hemisphere tend to zero to get the Q equal to P contribution to the starting integral. The end result is the equation for $\varphi(P)$.

The incident wave is known to be:

$$\varphi_W = \varphi_o \text{ Cosh } [\ k(z+h)] \ / \text{ Cosh } [kh] \ \ \text{Cos } (kx-\omega t)$$


P
r
Q

At each point on the water surface, this has the form:

$$\varphi_W = A \sin\omega t + B \cos\omega t$$

We need to find the corresponding scattered potential:

$$\varphi_S = a \sin\omega t + b \cos\omega t$$

Once, we find the complete or total potential, we can get pressure from the unsteady Bernoulli equation:

$$\partial\varphi/\partial t + P/\rho + gz = 0$$

Once pressure is known, we can get loads from the following integrations over the surface of the body:

$$\mathbf{F} = -\int_S P\, \mathbf{n}\, dS \qquad \mathbf{M} = -\int_S P\, (\mathbf{r} \times \mathbf{n})\, dS$$

Analytical solutions are possible only for simple shapes like vertical circular cylinders. For complex shapes, one must use CFD. A popular CFD method is the Panel Method. For this, one must first discretize the surface surrounding the water with a number of facets or panels which do not overlap. Then, one uses the constraint equations to get rid of the $\partial\varphi(Q)/\partial n$ terms in the integral leaving only the φ terms. Then, one assumes that over each panel the integrand is constant. This allows us to replace the integral with the following sum:

$$\varphi(P) = 1/[2\pi] \sum_{S} [\ 1/r\ \partial\varphi(Q)/\partial n - \varphi(Q)\ \Delta(1/r)/\Delta n\ ]\ \Delta S$$

For each panel, we substitute the equations for φ_w and φ_s into the summation to get an equation of the form:

$$I\ \mathrm{Sin}\omega t + J\ \mathrm{Cos}\omega t = 0$$

This equation implies that:

$$I = 0 \qquad J = 0$$

Next, we solve the equation system to get the a and b for each panel. This gives us the complete potential on the water. With it, we can get pressure and loads on the body.

A moving body generates loads on itself. It creates another potential besides the incident and scattered potentials. For a body with a single degree of freedom, the equation of motion is of the form

$$X\ d^2R/dt^2 + Y\ dR/dt + Z\ R = W + D$$

where R is the body displacement, X is its inertia, Y is its drag, Z is its buoyancy spring, W is the load due to a wave field and D is the load due to body motion. The motion of the body would be of the form:

$$R = N \sin\omega t + M \cos\omega t$$

We want to find N and M. We can get D by first assuming that the motion is $R=\sin\omega t$. Differentiation gives dR/dt and thus $\partial\varphi/\partial n$ at points on the surface of the body. Application of the Panel Method gives a load of the form:

$$G \sin\omega t + H \cos\omega t$$

Next, we assume that the motion is $R=\cos\omega t$. Application of the Panel Method gives a load of the form:

$$E \sin\omega t + F \cos\omega t$$

The load due to an actual motion would be:

$$D = N (G \sin\omega t + H \cos\omega t) + M (E \sin\omega t + F \cos\omega t)$$

The load due to the wave would be of the form:

$$W = U \sin\omega t + V \cos\omega t$$

Substitution into the equation of motion gives:

$$- X\omega^2 (N \sin\omega t + M \cos\omega t) + Y\omega (-M \sin\omega t + N \cos\omega t)$$
$$+ Z (N \sin\omega t + M \cos\omega t) = U \sin\omega t + V \cos\omega t$$
$$+ N (G \sin\omega t + H \cos\omega t) + M (E \sin\omega t + F \cos\omega t)$$

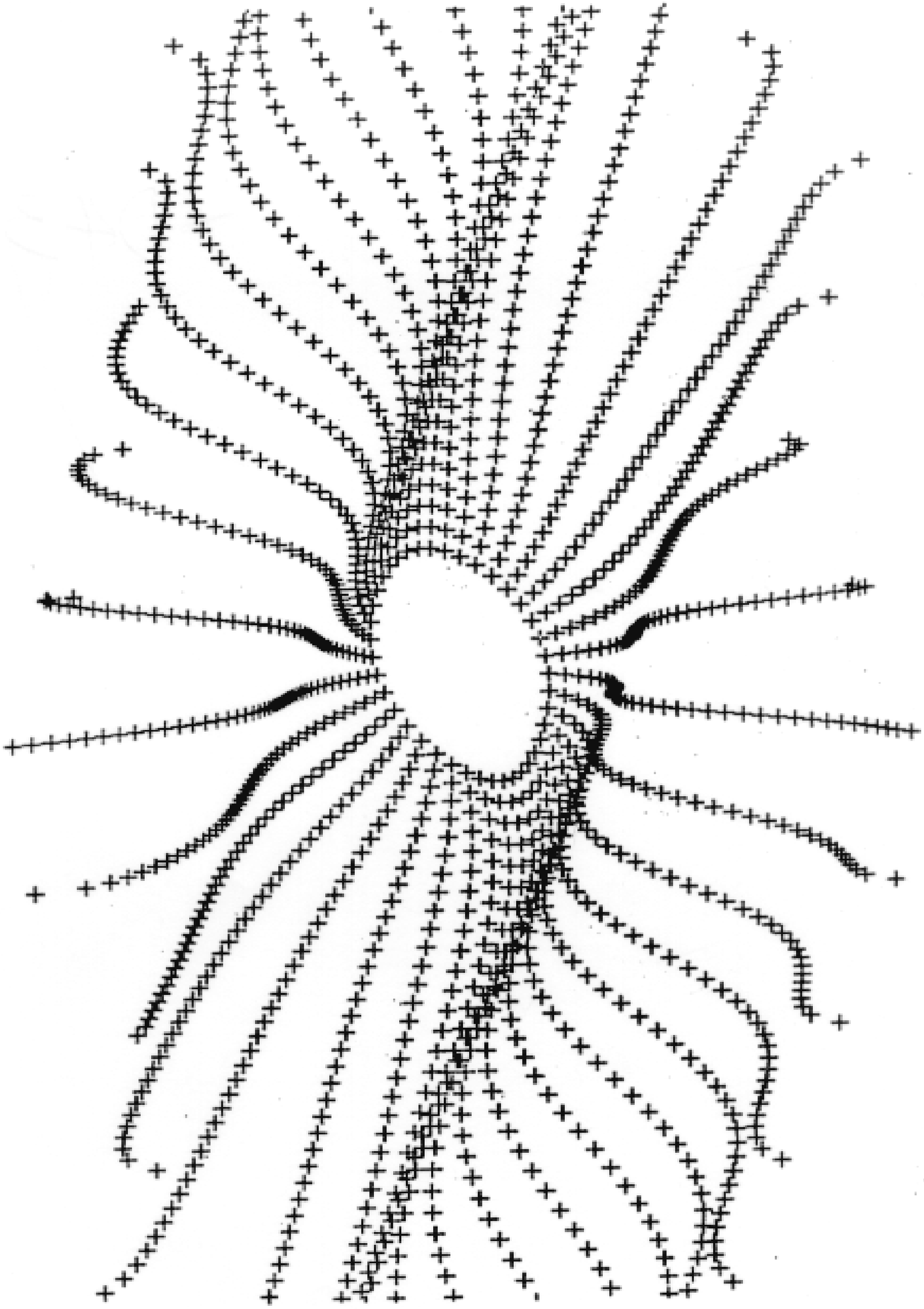
Manipulation gives an equation of the form:

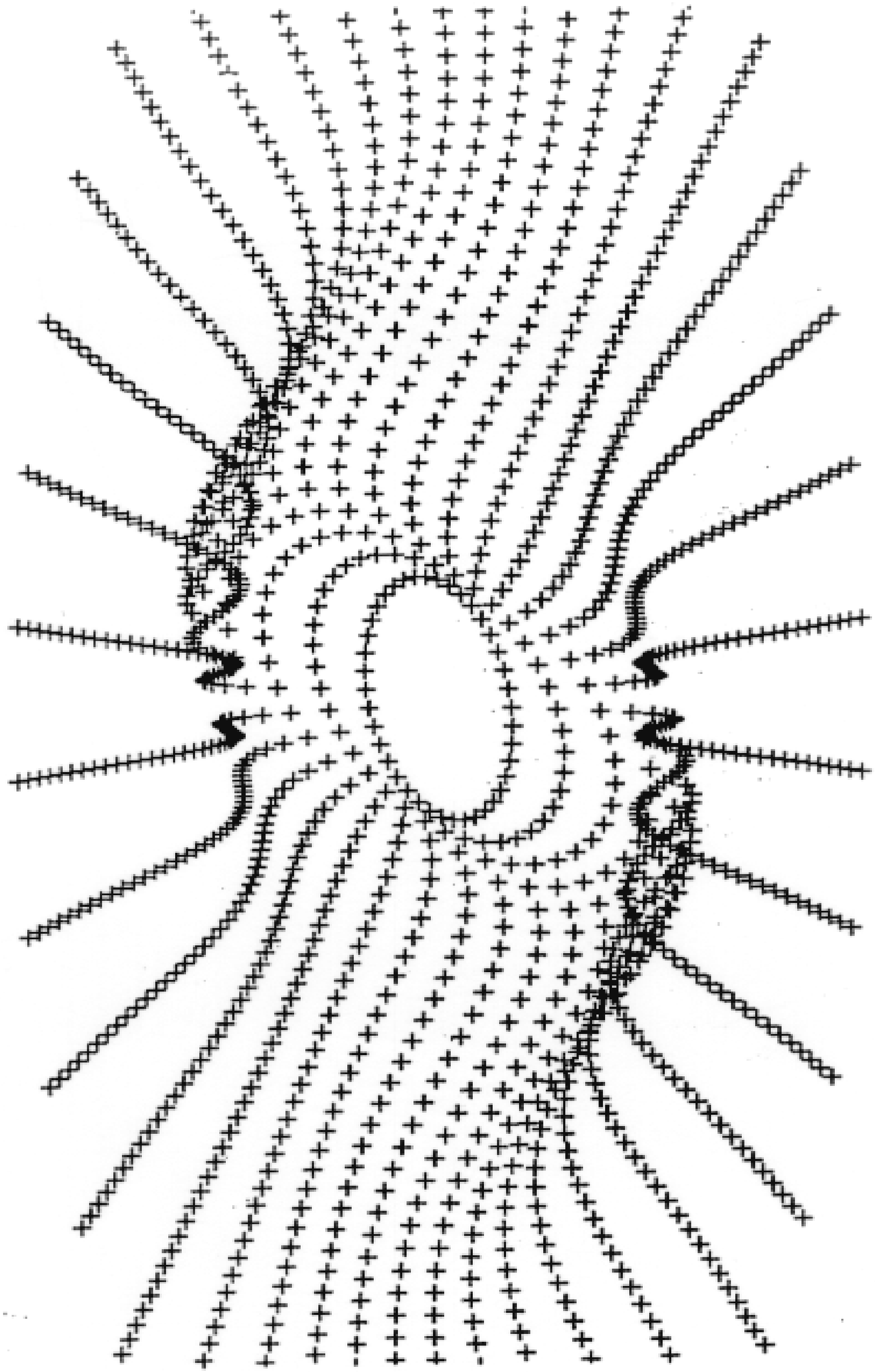
$$i \sin\omega t + j \cos\omega t = 0$$

This equation implies that:

$$i = 0 \qquad j = 0$$

These two equations allow us to find the body N and M. For each degree of freedom, we could find the N and M for a range of wave periods. These could be used to construct magnitude ratio or response amplitude operator plots. Such plot could be used to determine whether or not the body has trouble with resonance. Together with wave spectra they could be used to study motions in random waves.





Another integral formulation distributes complex oscillation sources G(P,Q) over the surface of the body:

$$\varphi_s(P) = 1/[4\pi] \int_S f(Q)\ G(P,Q)\ dS$$

The details of this formulation are beyond the scope of this note. The Panel Method in this case adjusts the strengths f(Q) so that there is no flow through the surface of the body. The boundary condition is:

$$\partial\varphi_b/\partial n = d\mathbf{R}/dt$$

With this the integral becomes

$$\partial\varphi_S(P)/\partial n = 1/[4\pi] \int_S f(Q)\ \partial G(P,Q)/\partial n\ dS$$

The Panel Method replaces the integral the summation:

$$\Delta\varphi_S(P)/\Delta n = 1/[4\pi] \sum_S f(Q)\ \Delta G(P,Q)/\Delta n\ \Delta S$$

One gets for the strengths:

$$A_{ij} f_j = B_i$$

The complex oscillating source formulation is good for solid bodies. Another integral formulation distributes complex oscillating dipoles over the surface of the body. This formulation is good for thin wall bodies exposed to waves inside and outside.

The formulations described above assume that wave amplitudes are small. Recently, formulations have been developed that can handle large motions of bodies in steep waves. These are beyond the scope of this note.