Group Speed Phenomena

Wave energy travels at a speed known as the group speed. This speed is generally not the same as the phase speed. Consider a group of waves made up of two nearly identical waves.

$$\eta_I = \eta_0 \sin\left[(k+\delta k)x - (\omega+\delta\omega)t\right]$$

$$\eta_{II} = \eta_0 \sin\left[(k-\delta k)x - (\omega-\delta\omega)t\right]$$

Superposition gives

$$\eta = \eta_I + \eta_{II} =$$

$$\eta_0 \sin(kx-\omega t)\cos(\delta k x - \delta\omega t)$$

$$+ \eta_0 \cos(kx-\omega t)\sin(\delta k x - \delta\omega t)$$

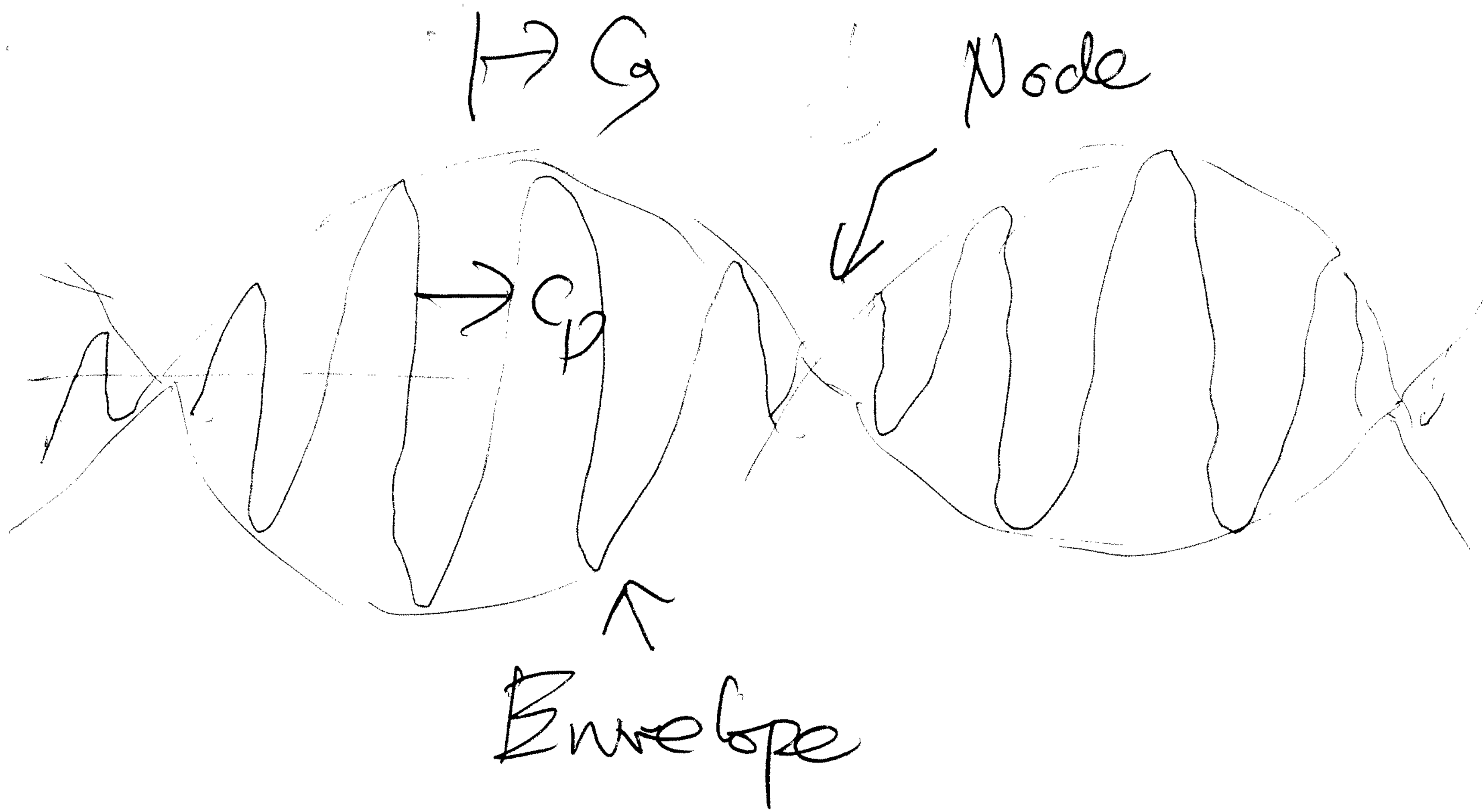
$$+ \eta_0 \sin(kx-\omega t)\cos(-\delta k x + \delta\omega t)$$

$$+ \eta_0 \cos(kx-\omega t)\sin(-\delta k x + \delta\omega t)$$

$$\eta = A \sin(kx - \omega t)$$

$$A = 2\eta_0 \cos(\delta k\, x - \delta\omega\, t)$$

The amplitude A varies slowly in space and time and generates an envelope within which η propagates.


C_g
C_p
Node
Envelope

The phase speed of an individual crest or trough moving within the envelope is $C_p = \frac{\omega}{k}$.

By analogy the phase speed of the envelope is $C_g = \frac{\delta\omega}{\delta k}$. In the limit as δk tends to zero this reduces to $C_g = \frac{d\omega}{dk}$.

In this limit the two superimposed waves become identical and the envelope wavelength tends to infinity.

Important: Node of an envelope act as energy barriers. This implies that wave energy travels at the envelope or group speed.

For open water

$$\omega^2 = gk \tanh kh$$

Differentiation gives

$$2\omega \frac{d\omega}{dk} = g \tanh kh + \frac{gkh}{\cosh^2 kh}$$

$$C_g = \frac{g \tanh kh}{2\omega} + \frac{gkh}{2\omega \cosh^2 kh}$$

Manipulation g ωs

$$C_g = \frac{\omega g \tanh kh}{2\omega^2} + \frac{\omega g kh}{2\omega^2 \cosh^2 kh}$$

$$= \frac{\omega g \tanh kh}{2 g k \tanh kh} + \frac{\omega g kh}{2 g k \tanh kh \cosh^2 kh}$$

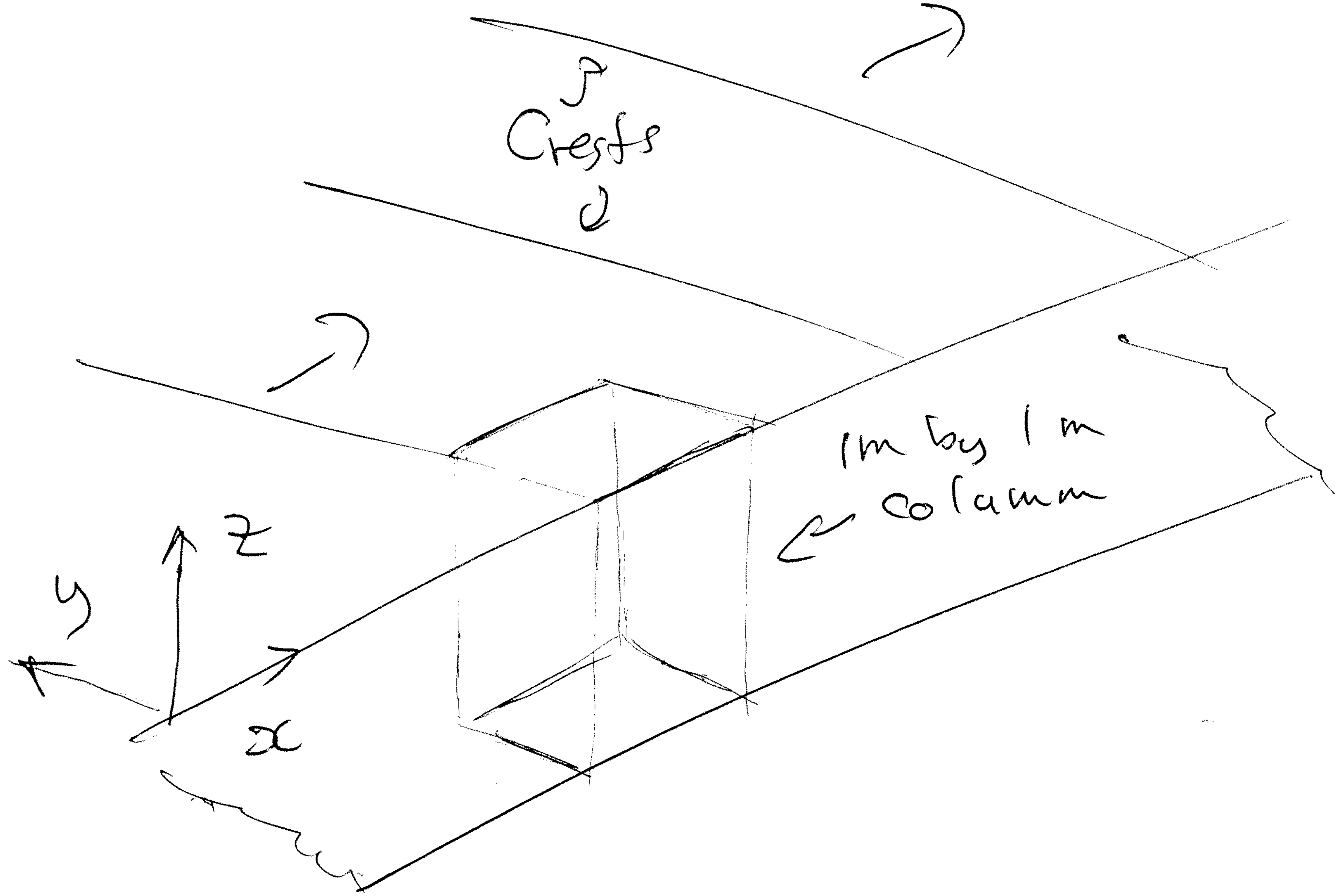
$$= \frac{\omega}{2k} + \frac{\omega}{k}\,\frac{kh}{2 \sinh kh \cosh kh}$$

$$= \frac{\omega}{2k} + \frac{\omega}{k} \frac{kh}{\sinh 2kh}$$

$$= C_p \left(\frac{1}{2} + \frac{kh}{\sinh 2kh} \right)$$

Deep Water Limit $C_g = \frac{C_p}{2}$

Shallow Water Limit $C_g = C_p$


Crests
1m by 1m
column
z
y
x

The wave energy density $\bar{E}$ is a cycle average of the wave energy in a 1m by 1m column of water extending down from the water surface to the seabed. It has two components:

Potential : $\bar{E}_p$

Kinetic : $\bar{E}_k$

$$\overline{E}_p = \frac{1}{\lambda}\int_{x_0}^{x_0+\lambda} \rho g \frac{(h+\eta)^2}{2}\,dx - \rho g \frac{h^2}{2}$$

$$= \frac{1}{4}\rho g \eta_0^2 = \frac{1}{16}\rho g H^2$$

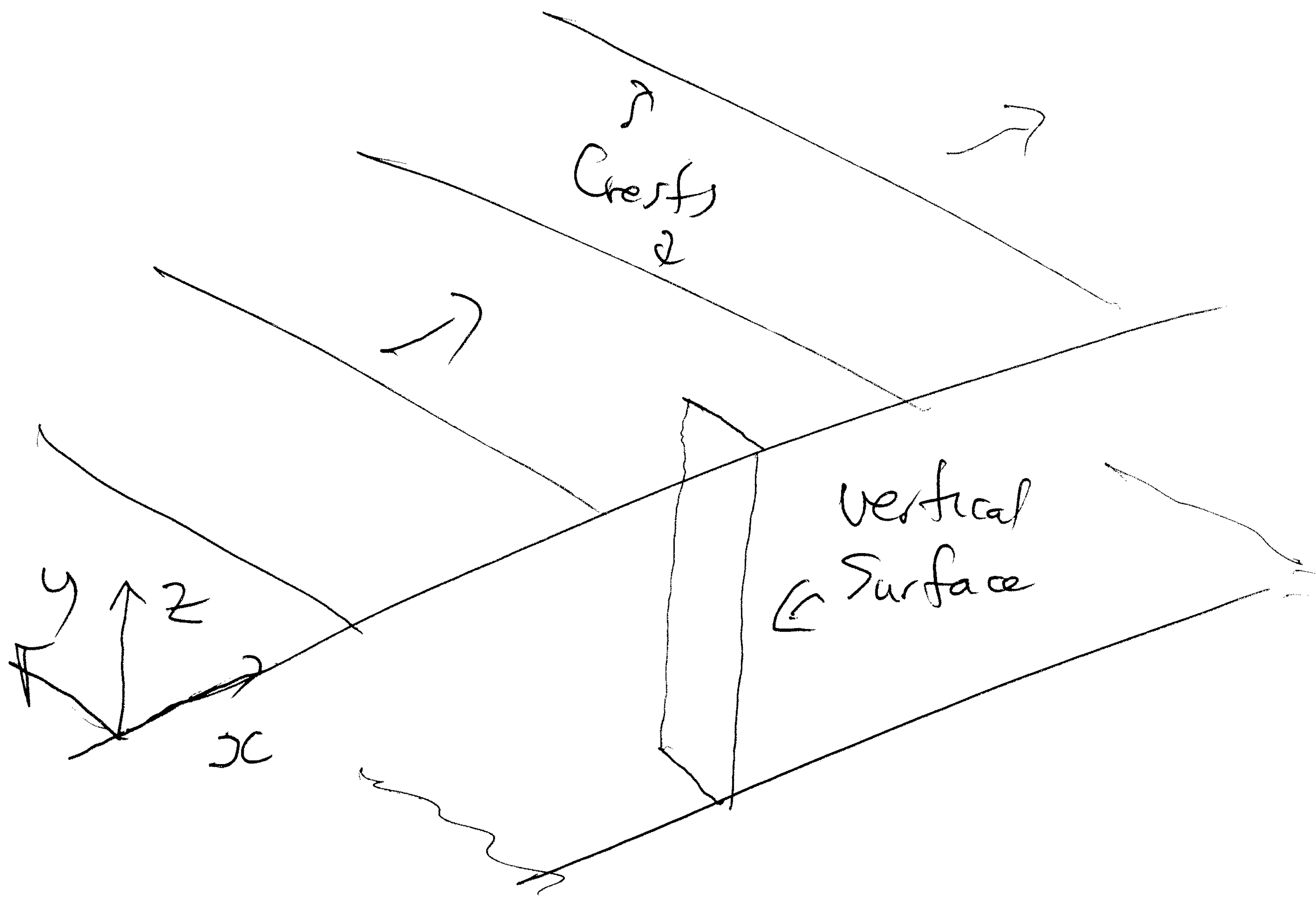
$$\overline{E}_k = \frac{1}{\lambda}\int_{x_0}^{x_0+\lambda}\int_{-h}^{\eta} \rho\left(\frac{U^2+W^2}{2}\right)dz\,dx$$

$$= \frac{1}{4}\rho g \eta_0^2 = \frac{1}{16}\rho g H^2$$

$$\overline{E} = \overline{E_p} + \overline{E_k}$$

$$= \frac{1}{16}\rho g H^2 + \frac{1}{16}\rho g H^2$$

$$= \frac{1}{8}\rho g H^2$$


Crests
Vertical Surface
y
z
x

Consider a vertical surface extending down from a water surface aligned such that waves approach normal to it. As waves cross this surface they do work on the water downstream at a certain rate. A cycle average of this work rate or power is known as the wave energy flux $\mathbb{P}$:

$$\mathbb{P} = \frac{1}{T}\int_{t_0}^{t_0+T}\int_{-h}^{\eta} \Delta P\, U\, dz\, dt$$

It turns out that

$$\mathbb{P} = C_g \bar{E}$$