BASIC CONSERVATION LAWS

REAL FLUID

$$\nabla \cdot \vec{v} = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p - \nabla \rho g z + \mu \nabla^2 \vec{v}$$

Unknown U V W P

IDEAL FLUID

$$\nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla\phi \cdot \nabla\phi + \frac{P}{\rho} + gz = C$$

Unknowns ϕ P

CONSERVATION OF MASS

Consider an arbitrary specific group of fluid particles contained within a volume $\forall$ anywhere within a flow. A differential volume $d\forall$ within $\forall$ would contain differential mass $dm = \rho\, d\forall$ where ρ is the density or mass per unit volume at points within $\forall$. The total mass M within $\forall$ is:

$$M = \int_{\forall} dm = \int_{\forall} \rho \, d\forall .$$

For a specific group of fluid particles M is constant. This implies

$$\frac{D}{Dt} \int_{\forall} \rho \, d\forall = 0 ,$$

Application of the Transport Theorem gives:

$$\int_{V} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} \right) dV = 0 .$$

As we are dealing with an arbitrary specific group of fluid particles this implies:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Expansion gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + V\frac{\partial \rho}{\partial y} + W\frac{\partial \rho}{\partial z}$$

$$+ \rho\frac{\partial U}{\partial x} + \rho\frac{\partial V}{\partial y} + \rho\frac{\partial W}{\partial z}$$

$$= 0$$

For hydrodynamic flows ρ is constant! So mass reduces to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\nabla \cdot \vec{v} = 0$$

CONTINUITY EQUATION

CONSERVATION OF MOMENTUM

Consider again an arbitrary specific group of fluid particles contained within a volume $\forall$ anywhere within a flow. A differential volume $d\forall$ within $\forall$ would contain differential momentum:

$$dm\vec{v} = \rho\, d\forall\, \vec{v} = \rho \vec{v}\, d\forall \; .$$

The total momentum is:

$$\int_{V} \rho \vec{v}\, dV$$

According to Conservation of Momentum the time rate of change of the momentum of the group of particles must balance with the net force acting on the group. Forces can be of two types: surface forces and body forces! So one can write

$$\frac{D}{Dt}\int_{\forall} \rho \vec{v}\, d\forall = \int_{S} \vec{\sigma}\, ds + \int_{\forall} \rho \vec{b}\, d\forall$$

Where $\vec{\sigma}$ is the surface

force per unit area at points on S and $\rho\tilde{b}$ is the body force per unit volume at points within $\forall$. Surface forces can be of two types; forces due to pressure and forces due to viscous traction. Body forces are generally due only to gravity.

Application of the Transport Theorem gives:

$$\int_{\forall} \left(\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v}\, \vec{v} \right) d\forall$$

$$=$$

$$\int_{S} \vec{\tau}\, ds + \int_{\forall} \rho \vec{b}\, d\forall$$

For some hydrodynamic flows pressure is the only surface force and gravity is the only body force. In this case momentum becomes

$$\int_{\forall}\left(\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v}\,\vec{v}\right) d\forall$$

$$=$$

$$\int_{S} -P\vec{n}\,dS + \int_{\forall} -\rho g \vec{k}\,d\forall$$

This can be rewritten as

$$\int_{\forall} \left(\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v} \vec{v} \right) d\forall$$

$$=$$

$$\int_{\forall} -\nabla P \, d\forall + \int_{\forall} -\rho g \nabla z \, d\forall$$

For an arbitrary specific group of particles this implies

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v} \vec{v}$$

$$=$$

$$-\nabla p - \rho g \nabla z$$

Expansion gives

$$\rho \frac{\partial \vec{v}}{\partial t} + \vec{v} \frac{\partial \rho}{\partial t}$$

$$+ \rho \vec{v} \cdot \nabla \vec{v} + \vec{v} \cdot \nabla \rho \vec{v}$$

$$=$$

$$-\nabla p - \rho g \nabla z$$

Recall Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

With this Momentum becomes

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v}$$

$$=$$

$$-\nabla P - \rho g \nabla z$$

Expansion gives

$$\rho\left[\frac{\partial u}{\partial t}\vec{i}+\frac{\partial v}{\partial t}\vec{j}+\frac{\partial w}{\partial t}\vec{k}\right]$$

$$+\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]\vec{i}$$

$$+\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]\vec{j}$$

$$+\rho\left[u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]\vec{k}$$

$$=$$

$$-\left[\frac{\partial p}{\partial x}\vec{i}+\frac{\partial p}{\partial y}\vec{j}+\frac{\partial p}{\partial z}\vec{k}\right]-\rho g\vec{k}$$

X MOMENTUM

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x}$$

Y MOMENTUM

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y}$$

Z MOMENTUM

$$\rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} - \rho g$$

When viscous traction is accounted for in Conservation of Momentum one gets the equations given below. Hydrodynamic flows are governed by the equations for the Incompressible Case.

MOMENTUM : COMPRESSIBLE CASE

X MOMENTUM

$$\rho \partial U/\partial t + \rho \ (U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z) = - \partial P/\partial x$$

$$+ \partial/\partial x \ (-2/3\mu[\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x \ (\mu[\partial U/\partial x + \partial U/\partial x])$$

$$+ \partial/\partial y \ (\mu[\partial V/\partial x + \partial U/\partial y]) + \partial/\partial z \ (\mu[\partial W/\partial x + \partial U/\partial z])$$

Y MOMENTUM

$$\rho \partial V/\partial t + \rho \ (U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z) = - \partial P/\partial y$$

$$+ \partial/\partial y \ (-2/3\mu[\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x \ (\mu[\partial V/\partial x + \partial U/\partial y])$$

$$+ \partial/\partial y \ (\mu[\partial V/\partial y + \partial V/\partial y]) + \partial/\partial z \ (\mu[\partial W/\partial y + \partial V/\partial z])$$

Z MOMENTUM

$$\rho \partial W/\partial t + \rho \ (U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z) = -\rho g - \partial P/\partial z$$

$$+ \partial/\partial z \ (-2/3\mu[\partial U/\partial x + \partial V/\partial y + \partial W/\partial z]) + \partial/\partial x \ (\mu[\partial W/\partial x + \partial U/\partial z])$$

$$+ \partial/\partial y \ (\mu[\partial V/\partial z + \partial W/\partial y]) + \partial/\partial z \ (\mu[\partial W/\partial z + \partial W/\partial z])$$

MOMENTUM : INCOMPRESSIBLE CASE

X MOMENTUM

$$\rho \partial U/\partial t + \rho \; (U\partial U/\partial x + V\partial U/\partial y + W\partial U/\partial z) = - \partial P/\partial x$$
$$+ \; \mu \; (\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 + \partial^2 U/\partial z^2)$$

Y MOMENTUM

$$\rho \partial V/\partial t + \rho \; (U\partial V/\partial x + V\partial V/\partial y + W\partial V/\partial z) = - \partial P/\partial y$$
$$+ \; \mu \; (\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2)$$

Z MOMENTUM

$$\rho \partial W/\partial t + \rho \; (U\partial W/\partial x + V\partial W/\partial y + W\partial W/\partial z) = -\rho g - \partial P/\partial z$$
$$+ \; \mu \; (\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 + \partial^2 W/\partial z^2)$$

In vector form Conservation of Momentum for Hydrodynamic Flows is

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \rho g \nabla z + \mu \nabla^2 \vec{v}$$

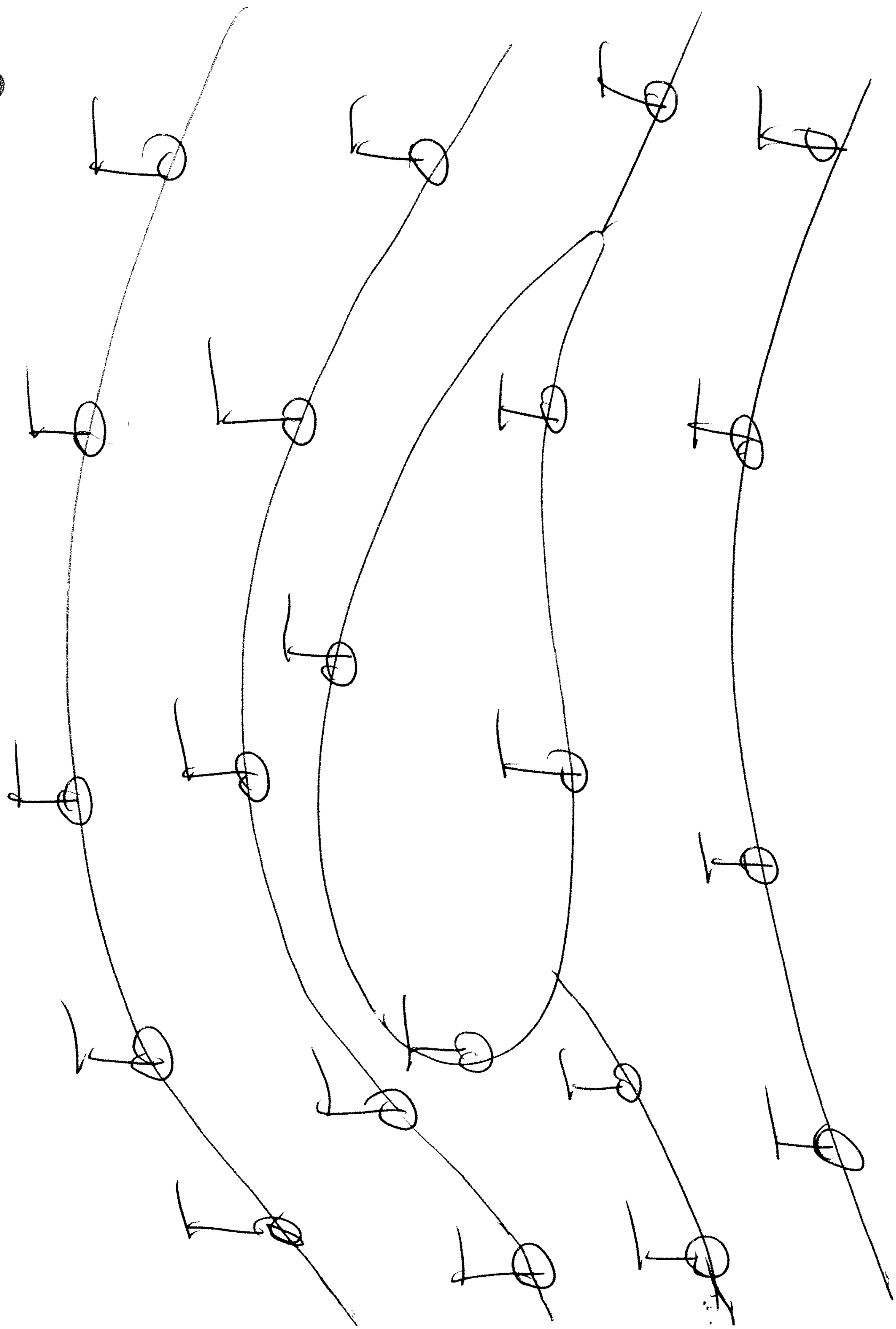
IDEAL FLUID CASE

For many hydrodynamics flows viscosity does not significantly affect the flow. Examples include the propagation of deep water waves and the scattering of waves by large structures.

In such flows the motion of the fluid is approximately irrotational.

In a real fluid each fluid particle can spin on an internal axis.

Irrotational means spin is zero.



By definition the vorticity at any point in a flow is

$$\vec{\omega} = \nabla \times \vec{v},$$

Let $\vec{\Omega}$ be the spin vector of a particle.

It turns out that

$$\vec{\Omega} = \frac{\vec{\omega}}{2}.$$

So vorticity is a measure of local spin.

Zero spin implies

$$\vec{\omega} = \nabla \times \vec{v} = 0,$$

For any scalar ϕ

$$\nabla \times \nabla \phi = 0.$$

This suggests that

$$\vec{v} = \nabla \phi$$

$$= \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$= U\vec{i} + V\vec{j} + W\vec{k}$$

Conservation of Mass ??

$$\nabla \cdot \vec{v} = 0$$

$$\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi = 0$$

Laplace's Equation $\nabla^2 \phi = 0$

Conservation of Momentum ∂_t

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla\left(\frac{P}{\rho}\right) - \nabla(gz)$$

For any vector $\vec{v}$

$$\vec{v} \cdot \nabla \vec{v} = \nabla\left(\frac{1}{2}\vec{v} \cdot \vec{v}\right) - \vec{v} \times \vec{\omega}$$

When $\vec{\omega}=0$ this reduces to

$$\vec{v}\cdot\nabla\vec{v} = \nabla\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)$$

With this momentum becomes

$$\frac{\partial\vec{v}}{\partial t}+\nabla\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right) = -\nabla\left(\frac{P}{\rho}\right)-\nabla(gz)$$

Substitution of $\vec{v} = \nabla\phi$ into this gives

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi\right) = -\nabla\left(\frac{p}{\rho}\right) - \nabla(gz)$$

Integration of this gives

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{p}{\rho} + gz = C$$

Unsteady Bernoulli Equation