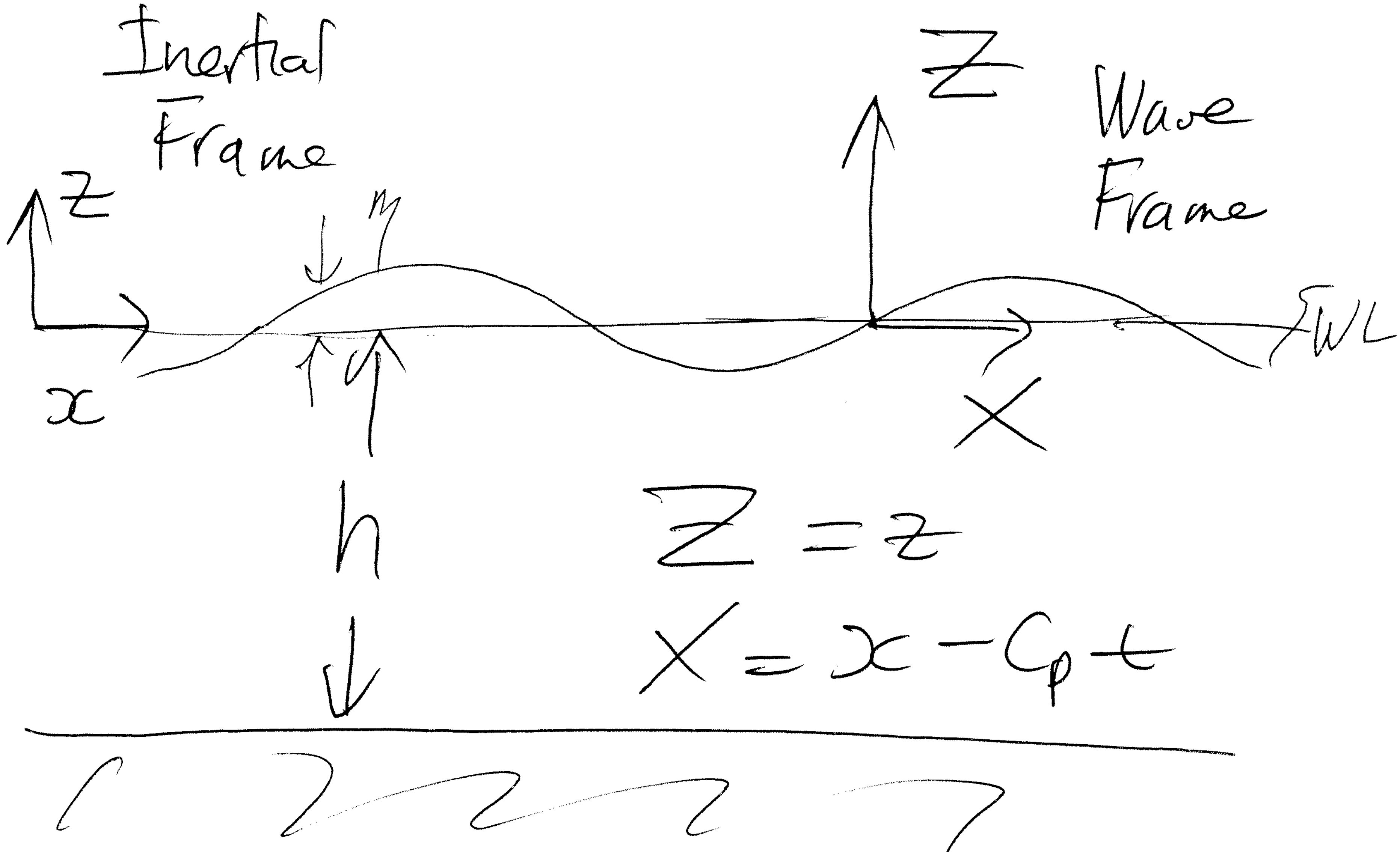

$$Z = z$$
$$X = x - c_p t$$

WATER WAVE THEORY

GOVERNING EQUATIONS

Within the water Conservation of Mass requires that

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

At the seabed the

Normal component of water motion must match the normal component of bed motion

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -h$$

This is known as a kinematic constraint.

What about Tangential Motion of Water at Seabed ??

A wave propagating through an inviscid fluid is like a wave in jelly.

Particles on the water surface remain on the surface. Let $\eta(x,t)$ be the vertical deflection of the water surface from the still water line (SWL). Each particle on the water surface will have an η and a vertical coordinate z,

For particles which remain on surface

$$\frac{D\eta}{Dt} = \frac{Dz}{Dt}$$

Expansion yields

$$\frac{\partial m}{\partial t} + U\frac{\partial m}{\partial x} + W\frac{\partial m}{\partial z}$$

$$=$$

$$\frac{\partial z}{\partial t} + U\frac{\partial z}{\partial x} + W\frac{\partial z}{\partial z}$$

$$\frac{\partial m}{\partial t} + U\frac{\partial m}{\partial x} = W$$

$$\frac{\partial m}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial m}{\partial x} = \frac{\partial \phi}{\partial z}$$

This is the Kinematic Constraint at the water Surface. For small waveheight waves this reduces to

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \text{ on SWL}$$

The Unsteady Bernoulli

equation gives at the water surface

$$\frac{\partial \phi}{\partial t} + \frac{\nabla\phi \cdot \nabla\phi}{2} + \frac{P}{\rho} + g\eta = \frac{P_A}{\rho}$$

For a constant pressure atmosphere $P = P_A$ and Bernoulli reduces to

$$\frac{\partial \phi}{\partial t} + \frac{\nabla\phi \cdot \nabla\phi}{2} + g\eta = 0$$

This is known as a Dynamic Constraint. For small waveheight waves it reduces to

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on SWL}$$

Time differentiation of

The Dynamic Constraint gives

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \eta}{\partial t} = 0$$

Substitution of the Kinematic Constraint into this gives

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{on SWL}$$

Summary of Equations

Mass Conservation

$\nabla^2 \phi = 0$ within water

Constraints

$\frac{\partial \phi}{\partial z} = 0$ on Seabed

$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0$ on SWL

Separation of Variables

Solution procedure gives

$$\phi = \phi_0 \frac{\cosh k(z+h)}{\cosh kh} \cos kx$$

The Wave Profile is

$$\eta = \eta_0 \sin kx$$

Steps in Procedure

Wave Frame Equations

$$X = x - C_p t$$

$$Z = z$$

Differentiation gives

$$\frac{\partial X}{\partial x} = 1 \Rightarrow \partial X = \partial x$$

$$\frac{\partial X}{\partial t} = -C_p \Rightarrow \partial X = -C_p \partial t$$

Laplaces' Equation becomes

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Constraint Equation becomes

$$C_p^2 \frac{\partial^2 \phi}{\partial X^2} + g \frac{\partial \phi}{\partial z} = 0$$

Assume Solution

$$\phi(x,z) = \mathcal{X}(x)\,\mathcal{Z}_1(z)$$

Substitute into Mass

$$\mathcal{Z}_1(z)\frac{d^2\mathcal{X}(x)}{dx^2} + \mathcal{X}(x)\frac{d^2\mathcal{Z}_1(z)}{dz^2} = 0$$

Manipulation gives

$$\frac{Z}{XZ}\frac{d^2X}{dx^2}+\frac{X}{XZ}\frac{d^2Z}{dz^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

ODEs are

$$\frac{d^2X}{dx^2}=-k^2X$$

$$\frac{d^2 Z_i}{dz^2} = +k^2 Z_i$$

General Solutions

$$X = A \sin kx + B \cos kx$$

$$Z_i = M \sinh kz + N \cosh kz$$

$$\left(G \sinh kz + H \cosh kz\right) \cos kx$$

Constants

Seabed Constraint

$$\frac{\partial \phi}{\partial z} = 0 = k\left(G \cosh -kh + H \sinh -kh\right) \cos kx$$

$$\left(-H \sinh kz \frac{\sinh -kh}{\cosh -kh} + H \cosh kz\right) \cos kx$$

$$\left(H \sinh kz \frac{\sinh kh}{\cosh kh} + H \cosh kz\right) \cos kx$$

$$H \left(\frac{\sinh kz \sinh kh + \cosh kz \cosh kh}{\cosh kh}\right) \cos kx$$

$$H \frac{\cosh k(z+h)}{\cosh kh} \cos kx$$

$$\phi_0 \frac{\cosh k(z+h)}{\cosh kh} \cos kx$$