Assignment 1 — 2012

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6892 Due Oct 4 2012

Q0 [10] For integers X and Y design an efficient ($\log Y$ time) iterative algorithm to compute X^Y. (a) Complete the following proof outline

$$\{x = X \wedge y = Y \geq 0\} \ ? \ \left\{z = X^Y\right\}$$

You should assume that variables x, y, and z hold mathematical integers —thus there is no overflow– and that the operations assignment, +, -, ×, ÷, 'odd', and 'even' each take 1 unit of time, as do any comparisons. Be sure to state the invariant of your loop.

(b) List the boolean expressions that must be shown universally true (according to the rules discussed in class)

(c) For each boolean expression from part (b), give a concise argument that it is universally true.

Q1[10] An array a of n marbles (indexed from 0) potentially has marbles of 3 colours, green, white and pink. Unfortunately they are originally in an arbitrary order. Design an iterative algorithm (with a proof outline) that sorts the marbles so that, when it is done, all green marbles are to the left of any white and pink marbles, and all pink marbles are to the right of any white or green marbles. The only operation allowed on the array is to swap two items $a(i) :=: a(j)$.[0]

(a) Specify the problem by writing a precondition and a postcondition.

(b) State the loop invariant that you will use.

(c) State the variant that you will use.

(d) Write out a proof outline.

(e) Give an informal argument for why your initialization code establishes the loop invariant, why the loop preserves the invariant, why the postcondition is implied by the negation of the guard and the invariant, and why the loop body decreases the variant (though never below 0).

[0] Because I am not asking for a detailed proof of correctness, you shouldn't need a rule for showing that

$$\{P\}\ a(i) :=: a(j)\ \{Q\}$$

is correct. However, if you want one, here it is: We consider that $a(i) :=: a(j)$ is equivalent to $a := \mathsf{swap}(a, i, j)$ where swap is a function so that $\mathsf{swap}(a, i, j)$ is a sequence the same length as a and so that

$$\mathrm{swap}(a,i,j)(k) = \begin{cases} a(j) & \text{if } k = i \\ a(i) & \text{if } k = j \\ a(k) & \text{otherwise} \end{cases}$$

Now

$$\{P\}\ a(i) :=: a(j)\ \{Q\}$$

is correct if

$$P \Rightarrow \begin{array}{ll} & Q[a : \mathrm{swap}(a,i,j)] \\ \wedge & 0 \leq i < a.\mathrm{length} \\ \wedge & 0 \leq j < a.\mathrm{length} \end{array}$$

is universally true.