Demystifying complexity

Measuring and uncertainty

3 things to measure

- Worst case time complexity
- Average case time complexity
- Best case time complexity

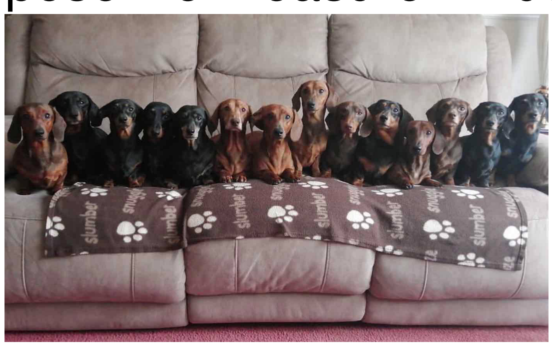
3 ways to answer

- Exact (or tight) bound (Θ)
- Upper bound (O)
- Lower bound (Ω)

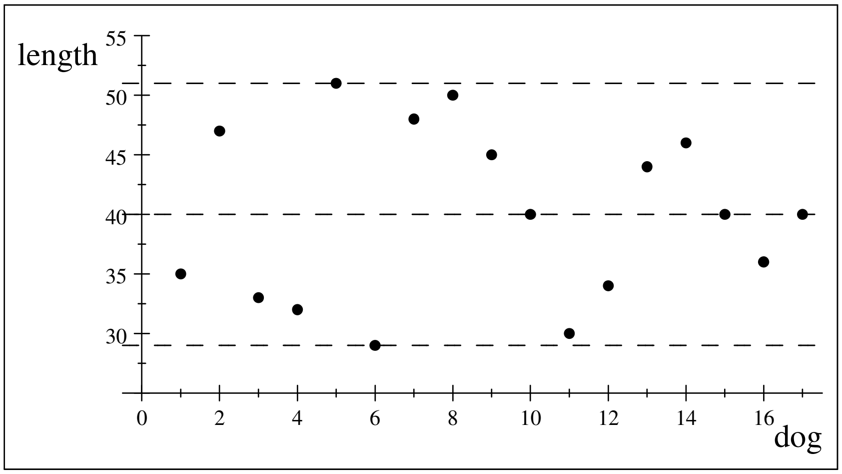
3 and 3: it's a coincidence! Do not read anything into it.

9 combinations.

An analogy: Suppose we measure 17 dogs.



The data is as follows (with max, min, and mean)



Let’s define:

- The worst case to be the length of the longest dog.
- The best case to be the length of the shortest dog.
- The average case to be the length of the most typical dog.

9 questions we might ask about the data

	. . . worst case?	. . . best case?	. . . average case?
What is an upper bound on the . . .	100 or 51.1 or 51	100 or 29.1 or 29	100 or 40.1 or 40
What is (a tight bound on) the . . .	51	29	40
What is a lower bound the . . .	0 or 50.9 or 51	0 or 28.9 or 29	0 or 39.9 or 40

- To say that 100 is an upper bound for the worst case is to say that the worst case lies within $\{x \mid x \leq 100\}$.
- To say that 50.9 is a lower bound for the worst case is to say that the worst case lies within the set

$\{x \mid x \geq 50.9\}$.

- If we know that both these are true, then we know that the worst case lies within
$$\{x \mid 50.9 \leq x \leq 100\}$$
- To say that 51 is (a tight bound on) the worst case is to say that the worst case lies within the set $\{x \mid x = 51\}$.

In complexity, it is the same, except we are comparing functions rather than numbers.

The analogy

- The room is analogous to a set of functions.
- Each length is analogous to a function.

Now considering just functions

- To say that n^3 is an upper bound on a function f is to say that it is in the set
$$\left\{f \mid f \preceq \left(\lambda n \cdot n^4\right)\right\}$$
- To say that n^2 is a lower bound on a function f is to say that it is in the set
$$\left\{f \mid f \succeq \left(\lambda n \cdot n^2\right)\right\}$$
- If we know that n^2 is a lower bound and that n^4 is an upper bound, then we know
$$\left\{f \mid \left(\lambda n \cdot n^2\right) \preceq f \preceq \left(\lambda n \cdot n^4\right)\right\}$$
i.e., that
$$f \in \Omega(n^2) \cap O(n^4)$$
- If we later learn that in fact $f \asymp \left(\lambda n \cdot n^3\right)$, then we know
$$f \in \Theta(n^3)$$

Building complexity and problem complexity

Now consider a whole building: In each room there are 17 dogs.

The (worst-case) complexity of a room is the length of its longest dog.

Define the *best room* to be the room with least (worst-case) complexity.

Call the (worst-case) complexity of the best room *the building's (worst case) complexity*.

If I examine one room, I can determine an upper bound on the complexity of the building.

If I examine 2 rooms, I can use the minimum of the two longest dogs to get a (possibly tighter) upper-bound.

But to find a lower bound on the complexity of the building I need to either exam every single room or use some reasoning about the nature of dogs and of rooms full of dogs.

The analogy

- The Building is analogous to a problem.
- Different rooms are analogous to different algorithms.
- The length the longest dog in a room is analogous to the worst-case time function of the algorithm.