Problem set 0 Discrete math.

Advanced Computing Concepts for Engineering (T.S. Norvell)

2020

Q0 Set notation. Use the filter and/or map notations to concisely express the following sets

(a) The set of all composite natural numbers.

$$\{0, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, ...\}$$

(b) The set of all positive real numbers less than then square root of 2. Don't use the square-root sign in your answer.

(c) The set of all straight lines in the Cartesian plane. (Consider a straight line to be a suitable subset of $\mathbb{R} \times \mathbb{R}$.)

(d) The set of all linear total functions from real to real numbers.

Q1 Paradoxical sets.

In class is was stated that a set is a collection of mathematical objects and that each set is itself a mathematical objects. However it does not follow that any collection of mathematical objects can be used to make up a set.

For this question we will temporarily suppose that there is a set of all sets. We will then see that this concept leads to a contradiction.

Let S be the set of all sets.

Presumably some sets contain themselves. E.g., since S is a set and S contains all sets, we can conclude that $S \in S$. On the other hand, it is clear that some sets do not contain themselves. For example $\emptyset \notin \emptyset$. So some sets contain themselves and some do not.

Let R be the set of all sets that do not contain themselves. I.e. $R = \{x \in S \mid x \notin x\}$.

(a) Prove that if $R \notin R$ then $R \in R$. Can you conclude that it is not the case that $R \notin R$?

(b) Prove that if $R \in R$ then $R \notin R$. Can you conclude that it is not the case that $R \in R$?

The usual way out of this conundrum is to say that there is no set of all sets.

Q2 Counting

Let S and T be finite sets. Let $|S| = m$ and $|T| = n$.

(a) What is the size of $|S \times T|$.

(b) How many binary relations are there with S as source and T as target?

(c) How many total functions are there with S as source and T as target?

(d) How many partial functions are there with S as source and T as target?

Q3 Propositional logic.

Using the laws that appear above them in the notes, prove the following distributivity laws

(a) $(p \wedge q \Rightarrow r) = ((p \Rightarrow r) \vee (q \Rightarrow r))$

(b) $(p \Rightarrow q \wedge r) = ((p \Rightarrow q) \wedge (p \Rightarrow r))$

Q4 Substitutions

(a) Underline all bound occurrences of variables in the following formulae. Circle all free occurrences of variables.

$$\{i \in \mathbb{N} \mid i < f(i) \cdot g(i)\}$$

$$(\forall x \in \mathbb{R} \cdot g(x) < f(y))$$

(b) Make the following substitutions

$$\{i \in \mathbb{N} \mid i < f(i) \cdot g(i)\}\,[x, i, f : y, j, g]$$

$$(\forall x \in \mathbb{R} \cdot g(x) < f(y))\,[y : y + 1]$$

$$(\forall x \in \mathbb{R} \cdot g(x) < f(y))\,[y : x + 1]$$

Q5 Quantifiers and sets

Ranter is a social network in which users publish short messages called rants. Let U be the set of all users on the social network and let $\text{follows} : U \times U \xrightarrow{\text{tot}} \mathbb{B}$ be a boolean function expressing that the first user follows the second.

(a) Use quantifiers to say that following is irreflexive, i.e., that no one follows themselves.

(b) If, for some users a and b, $\text{follows}(a, b)$, we say that a is a degree 1 follower of b. If, for some users a, b,and c, a follows c and c follows b, we say that a is a degree 2 follower of b. And so on. Use quantifiers to say that a is a degree k follower of b. You may assume $k > 0$. [Hint: You may need to quantify over a function. Hint: check that the free variables of your expression are a, b, and k.]

(c) Explain the meaning of the following expression in clear English

$$\forall x \in R \cdot \exists y \in S \cdot \text{follows}(x, y)$$

[Hint: check that the free variables of your English sentence are the same as the free variables of the expression.]

(d) Explain the meaning of the following expression in clear English

$$\exists x \in R \cdot \forall y \in S \cdot \text{follows}(x, y)$$

[Hint: check that the free variables of your English sentence are the same as the free variables of the expression.]

Q6 Scheduling

Let S be a set of course sections. Let T be a set of times. Let R be a set of rooms.

Let at be a binary function $at : S \times T \overset{\text{tot}}{\to} \mathbb{B}$. The intended meaning of $at(s, t)$ is that section s is scheduled at time t.

Let in be a binary function $in : S \times T \times R \overset{\text{tot}}{\to} \mathbb{B}$. The intended meaning of $in(s, t, r)$ is that section s is scheduled to be in room r at time t.

Express the following statements using quantifiers

(a) All sections are scheduled at exactly 3 times, according to the at function.

(b) All sections are scheduled at times (according to at) only when they are in a room.

(c) All sections are in a room, only at times they are scheduled to be at (according to at).

(d) All sections are only scheduled to be in one room at a time.

(e) No room is in use by two (or more) sections at the same time.

Express the following sets using set notation and quantifiers.

(f) The set of all room/time pairs when 2 sections are both scheduled for the room at the time.

(g) The set of all section/time pairs which need a room.